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MECHANICS OF ENGINEERING
AND OF
MACHINERY.

BY DR. JULIUS WEISBACH.

VOLUME III.

PART I., SECTION I.

THE MECHANICS OF THE MACHINERY
OF
TRANSMISSION.

SECOND EDITION, THOROUGHLY REVISED AND GREATLY ENLARGED,
BY
GUSTAV HERRMANN,
PROFESSOR AT THE ROYAL POLYTECHNIC SCHOOL, AACHEN, GERMANY.

DESIGNED AS A TEXT-BOOK FOR TECHNICAL SCHOOLS AND COLLEGES, AND FOR THE USE OF ENGINEERS, DRAUGHTSMEN, ETC.

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AUTHOR'S PREFACE.

ALL machines—even those most diverse in arrangement and mode of action—when grouped according to their *object* will easily come under the three large divisions of

a. Prime Movers.
b. Operators.
c. Transmitters.

Prime Movers are so arranged that, under the influence of certain natural forces, for instance, the weight of water, elasticity of steam, etc., they are set into suitable motion, which motion they are designed to communicate to the machines of the other two groups. The object of Prime Movers is therefore to drive machinery. The special discussion of this first group, whose most important representatives are the water-wheels and steam-engines, is already contained in the second volume of this work.

The object of *Operators* is to effect certain useful operations or processes, which otherwise could only be performed by animal labor, particularly the manual labor of human beings. From the variety of these requirements the diversity of Operators is extraordinarily great, and their number is continually increasing in proportion as the inventive spirit of a developing industry devises new means of substituting machinery for the work of the human hand. In spite of this extraordinary diversity of Operators, their final object can only be of two kinds: it either consists in a *change of position* of certain bodies which are subjected to the action of the machine, or in a *change of form* of these bodies, though a machine may simultaneously accomplish both objects. To place-changing machines belong, for example, hoisting and pumping apparatus; to form-changing machines, mills, stamping presses, metal- and wood-working machines, in fact, most of the machines used in the arts for the manufacture of special objects. In forging apparatus, the raising of a weight, causing a change of position, is followed by a change in the form of the forging produced by the weight of the falling hammer; while in a dredging machine, the change of form, occurring when the bucket seizes its load, precedes the raising of the load. In a spinning machine the change of form suffered by the thread in the drawing frame takes place simultaneously with the change of position occurring during winding and spooling. These examples must suffice to show that the action of all *Operators*, no matter how complicated the manufacture to which they belong, can be reduced to changes of position and form.

Transmitters serve to unite Prime Movers with Operators, by receiving motion from the former and transmitting it to the latter. It is in rare cases that Prime Mover and Operator are so near together, and the motion of one so conforms to that of the other, that direct union becomes practicable, as is the case, for example, with a steam hammer. Here, a direct union of hammer and steam piston is possible by means of the piston rod, all transmitting mechanism disappearing, unless the piston rod and its keys be regarded as such. In most cases the Operator is set up at a certain distance from the Prime Mover, so that the motion of the latter must be transmitted to the former by intermediate parts. To these parts, or transmitters, belong, for example, rods, shafting, belts, wheels, etc., which are also known as *simple machine parts.* It is often desirable not only to transmit motion, but to vary it, either in direction or in velocity, or in both these ways at once. The Transmitters must then be arranged to meet these conditions, and the individual parts then form mechanisms. Now, a knowledge of the simple machine parts, and of the mechanisms composed of them, is not only of importance as regards the Transmitters themselves, but is also necessary for the understanding of the other groups of machinery, particularly that containing the Operators; for these latter are composed almost entirely of mechanisms, which, while having for their object only suitable changes of motion, are nevertheless not to be regarded as Transmitters proper, but rather as integral parts of the Operator.

For this last reason, and in accordance with the practical rule that in explanations the simpler shall precede the complex, an arrangement has been chosen whereby the material before us is discussed in three divisions, of which the first treats of Transmitters, the second of Operators for changing the position of objects, and the third of Operators for changing the form of objects.

TRANSLATOR'S PREFACE.

THE volume now offered to the engineering public is the first installment of the second edition of what was formerly known as Vol. III. of " Weisbach's Mechanics of Engineering." In the hands of the editor, Prof. Gustav Herrmann, the material of the first edition has been so thoroughly reclassified and revised, the methods of discussion so much improved, and so much new material added as to practically constitute this second edition a new work, of three volumes, each as large as the original Vol. III.

As revised by Prof. Herrmann, the work divides itself into three parts:

Part I. The Mechanics of Machinery of Transmission.

Part II. The Mechanics of Machinery, for lifting and transporting (solid and fluid) materials.

Part III. The Mechanics of Machinery for changing the form and size of materials.

The references to Vol. I. and Vol. II. in the text refer to the English translations, unless special mention is made to the contrary. Quantities and formulas inclosed in brackets refer to the English units of measurement.

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INTRODUCTION.

PRINCIPLES OF KINEMATICS.

§ 1. **Kinematics.**—In the first part of this work, which treats of theoretical mechanics, the laws of the motions of bodies were developed under the supposition that these bodies, regarded as material points, were free to move in obedience to the forces acting upon them ; from which follows, that the paths described by these points were the result of the action of the forces. For example, the path of a ball from a cannon is determined by gravity and the resistance of the air. In machines, however, the individual parts can have no such freedom of motion, but are compelled by outside forces, fixed guides for example, to have determinate motions. In Part I., § 299, Fifth Ger. Ed., the effect of such fixed guides has already been explained and prominence given to the proposition, that a guide which controls the course of a body exerts upon it at every moment exactly the same forces as would cause the body to describe the same course were the body free to move under the influence of the forces only. In this case, therefore, as in that of free motion, the described path is the result of forces, but with this difference, that while a body moving freely changes its motion, both as regards form and velocity, with every change of force, the path of a body with constrained motion is a determinate one and independent of forces external to the mechanism. These external forces can indeed increase or diminish the intensity of the motion of all the parts of the mechanism in question, but the velocity ratio of the individual parts to one another, and the nature of their paths, remain in every case unchanged. This velocity ratio and these paths depend only upon the geometrical connection of the guided parts with one another, and with their guides.

An investigation of this dependence will therefore necessarily be of a geometrical character, and need not take into account either the driving forces, time, or *absolute* velocity. It can only be a question of ascertaining the *reciprocal paths* of the moved elements and the *ratio* of their *relative* velocities. The science which relates to this investigation—whose importance corresponds to that of machine theory itself—is called Kinematics* or the science of Pure Mechanism.

---

* Frequently Kinematics or Phoronomy is understood to mean the study of motion, when its nature only is considered, and no regard is paid to its causes or to forces. Kinematics is thus understood in Part I., § 47 (see also Schell, Theorie der Bewegung, p. 5). In the following we understand by Kinematics, "The study of those arrangements of the machine by which the mutual motions of its parts, considered as changes of position, are determined." See Reuleaux's Kinematics of Machinery, p. 40.

For a better understanding of the theory of Transmitters given in this volume, the fundamental laws of the science will now be given.

§ 2. **Motion of Bodies in General.**—In the following we always assume rigid bodies, that is, such as have and keep their individual points at invariable distances from one another. Though there are in reality no absolutely rigid bodies, since all known materials assume under the influence of external forces certain elastic changes of form, nevertheless, for substances used in the construction of machines, these changes are so insignificant, that for the purposes of the following discussions they may be neglected.

When a free rigid body has made any movement whatever, its new position will in general be completely determined, when the positions of three points not in the same straight line are known. This immediately follows from the fact that the motion of a body is completely prevented when any three of its points not in the same straight line are held fast. If in the above-mentioned movement, one point of the body had retained its position, for instance, by being firmly held, it is evident that for determining the new location of the body, a knowledge of the position of only two other points would be needed, provided of course that these two and the fixed one were not in the same straight line. Likewise, were the body held at two points, a knowledge of the position of any other, *not* in line with the two fixed ones, would be sufficient to establish indubitably the new location of the body. As already stated in Part I., § 129, every element of a body rigidly held at a single point, can only move in a spherical surface, whose centre is the rigidly held point, and whose radius is the distance of this point from the element in question ; it was also stated that any element of a body held at two points could only move in a circle perpendicular to the line connecting them, and having a radius equal to the distance of the element from the connecting line. This latter case is represented in practice by an axle held in two bearings, while the former case corresponds to a compass turning about a ball-and-socket joint.

When the motion of a body is such that the paths described by its particles lie in parallel planes, it is evident that a knowledge of the positions of two points will suffice for determining the corresponding location of the body. If in this case one point of the body were fixed, the motion of the body would be confined to rotation about an axis passing through the fixed point and perpendicular to the parallel planes in which motion occurs ; here, therefore, the location of the body will be known from the position of one point.

If in the case in which all the elements move in parallel planes we observe the path of a point in its plane, it will be readily seen that this path, whatever its nature, will agree in every respect with those described by the elements lying on the same perpendicular to the parallel planes as the point observed. If, therefore, for this case we know the motions of the points of a single one of the parallel planes, the motions of the entire system of points will be determined. In what follows this special case—of particular importance to practice—will first be discussed, while the more general case of entirely arbitrary motion, will be discussed subsequently.

§ 3. **Simple Motions.**—All motions of bodies can be reduced to two simple or elementary motions. These consist in a *translation* or *shifting* in a definite direction, and in a *turning* about a certain axis of rotation. In shifting, the different points of a body describe equal, similar, and parallel paths, so that for determining the shifting, the motion of a single point will suffice. By direction of shifting, is here meant the direction, at every moment, of an element of the path of any point. This path may be straight, of simple or double curvature, according as the successive directions lie in a straight line, in a plane, or in space. Since the distances passed over by all the points in a given time are always equal, it follows that the velocities and accelerations of the points are also equal both in magnitude and direction. By means therefore of a distance of definite *length* and *direction*, and by means of an arrow, we can clearly represent the motion of translation of a body, as well as its velocity and acceleration at a particular instant.

At any instant during the rotation of a body about an axis, all its points move with the same angular velocity and describe circular arcs, which are concentric with the axis, and whose lengths are proportional to their radii, that is to the perpendicular distances of the points from the axis. The velocities and accelerations of the individual points, as well as their paths, are therefore proportional to the perpendicular distances from the axis. The rotation of a body is represented graphically by a distance laid off in the direction of the axis, with a length proportional to the angle of rotation, and with an arrow-head at the end for indicating the direction of the rotation. The arrow-head is so placed, that, to an observer looking at its point, the rotation appears to be right-handed, like the hands of a watch.

Shifting along a straight line, can also be regarded as a rotation about an axis at an infinite distance, so that shifting may be regarded as really only a particular case of rotation.

These two simple motions form manifold combinations, of which those most important for the purposes of this book will all be given.

§ 4. **Combination of Simple Motions.**—In the following all shifting or motion of translation is assumed to be rectilinear. The general character of the deductions will suffer no detriment thereby, since each infinitely small element of a curvilinear path can always be regarded as rectilinear. How two or more rectilinear motions can be combined has been sufficiently explained in the propositions concerning the parallelogram of motions, velocities and accelerations. It is immaterial, so far as the result—change of position—is concerned, whether the individual motions follow one another, or take place simultaneously. The latter case occurs, for instance, when a body shifts upon or within another that is itself undergoing translation, for example, the hoisting gear of a traveling crane.

In combining two rotations of a body about the same axis, it is likewise immaterial whether the two rotations take place in succession or simultaneously, as would be the case, for example, with a man, who while on a moving carousel had an independent motion of his own concentric with the axis. The absolute motion of the body is in every case a rotation about the common axis, equal in amount to the algebraic sum of the two individual rotations.

When a body is subjected to both shifting and turning, it is, so far as the *changes of position* of its points are concerned, immaterial in what order the two motions take place; they may, consequently, be assumed as simultaneous. If *AB*, Fig. 1, is the horizontal projection of the jib of a crane, whose post

A is carried by a truck, the end B will reach C by a rotation equal in amount to the angle α, and a subsequent shifting of the traveling crane equal to the length s. But B could also reach C by the path $BB'C'$, first shifting the crane by an amount equal to s, and then turning it out of its position $A'B'$ by an amount equal to the angle α. If both motions are undertaken simultaneously, the end B moves directly to C', nearly in the direction of the diagonal BC'.

FIG. 1.

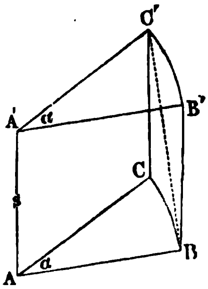
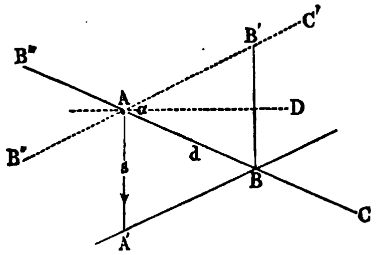


FIG. 2.

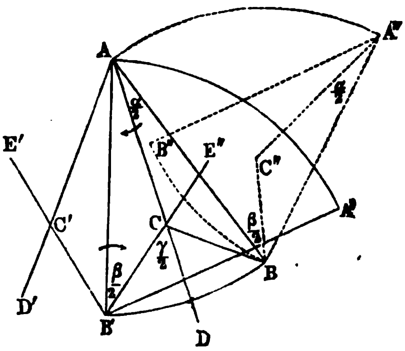


When a body is subjected to a rotation about an axis, and to a translation at right angles to this axis, the two motions can always be combined into a single rotation, whose angular motion equals that of the given rotation. In Fig. 2, let A be the intersection of the plane of the paper with an axis perpendicular to it, and let a rotation be given to a body about this axis equal to the angle α, also a translation $AA' = s$, then make AD perpendicular to AA', and lay off on each side of AD the angle $\frac{\alpha}{2}$, that is, make $DAC = DAC' = \frac{\alpha}{2}$. By turning the body about A, the line AC evidently reaches the dotted intermediate position AC', from which, in consequence of shifting, it passes to the final position $A'B$, which is found by drawing the line $A'B$, parallel to AC'. If now BB' is drawn parallel to AA', we get in B' a point of the line AC', which after shifting will be at B. Since, from the construction, it is evident that $AB' = AB$, it follows, that before rotation, this same point B' was situated at B, that is, the shifting subsequent to rotation returned the point B' to its original position B. This point B, therefore, and the line through it perpendicular to the figure, have not altered their original position, from which at once follows that the motion of the body can only be that of a rotation about the normal axis at B. The figure also shows that the rotation about B is in the same direction and of the same amount as that first undertaken about A. Had the turning about A, or the shifting, taken place in directions opposite to the ones here assumed, the axis of the resulting rotation, it is easy to see, would have been at B'' and B''' respectively instead of at B.

Conversely, it is evident that the rotation of a body about an axis (B) can be resolved into a rotation and a translation. This rotation is alike in amount and direction to the one from which it was derived, but will take place around any axis (A) parallel to the first. The corresponding motion of translation may be found from $AA' = B'B = 2\,d\,sin.\,\frac{\alpha}{2}$, when d signifies the distance AB of the two axes.

§ 5. **Two Rotations.**—*When a body is subjected in succession to two rotations about parallel axes, they can always be replaced by a single rotation parallel to the former, and having an angular motion equal to the algebraic sum of the angles described by the individual rotations.* Let, for example, A and B, Fig. 3, be the points of a body through which pass the axes perpendicular to the plane of the figure, and let α and β represent the angles of rotation belonging to them respectively. If the body is first turned about the axis A, the point B will, in consequence, move to the new position B', which is to be regarded as its final position, since it is not altered by the rotation equal to β, which now takes place. That for these rotations a single one can be substituted, is shown in the following manner: Bisect $BAB' = \alpha$ by AD, then $DAB' = DAB = \frac{\alpha}{2}$. By the rotation about A, the bisecting line AD will reach the position AD', if $D'AB'$ be made equal to $\frac{\alpha}{2}$. Now on each side of AB' lay off the angle $\frac{\beta}{2}$, so that $AB'E' = AB'E'' = \frac{\beta}{2}$. By the rotation about B', the straight line $B'E'$ evidently reaches the position $B'E''$, and it is easy to see that C', the intersection of AD' and $B'E'$, must fall at C, for the construction gives us $B'C' = B'C$. But the construction also gives $AC = AC'$, which shows that the point brought by the rotation about B', from C' to C, originally occupied the position C, from which it was carried to C' by the rotation about A. This point C, therefore, has not altered its position in consequence of the two rotations, and we can therefore assume that for the resultant motion of the body, we may substitute a rotation about the axis C. Since in this rotation the point B must come into its final position B', the amount of its angular motion must be $BCB' = \gamma$. From the figure we readily get for this angle $\gamma = BCB' = 2\,DCB' = \alpha + \beta$.

FIG. 3.



That the manner in which the individual rotations succeed one another is not a matter of indifference, will be seen from the following considerations. Each of the two axes of rotation, A and B, owes its change of position solely to rotation about the other axis, since rotation about its own axis could not cause alteration in its position. Thus, for example, B reaches its definite position B', by the rotation α about A, provided, as was assumed before, that this rotation takes place first. But if we were now to allow the rotation β about B to take place first, A would in consequence be brought to the new final position A'', and the rotation α taking place about A'' would not conduct the axis B as before to B', but to B''. While the line AB was led to $A'B'$, when the rotations followed the order α, β, now that this order is reversed, AB will be led to the final position $A''B''$. It is moreover quite evident that the final positions $A'B'$ and $A''B''$ are parallel to each other, for both can be obtained by rotation in the same direction through an angle equal to $\alpha + \beta$, with the difference, however, that the first must turn about C, and the second about C'' as an axis. C'' can be easily found by making $ABC'' = \frac{\beta}{2}$ and $BA''C' = \frac{\alpha}{2}$. It is also evident from the figure that C and C'' are symmetrically situated with reference to AB, the original position of the line joining the axes.

If we imagine the angles of rotation α and β growing smaller and smaller, the two positions C and C'', for the axes of resultant rotations will continually approach the line AB. Near the limit where α and β are infinitely small, the distances of these axes C and C'' from AB will also be infinitely small; in other words, at the limit. the two axes coincide with a point C_0 of the straight line AB. As regards the position of this point C_0, we can say that for the positions C and C'', the following equations are always true:

$$CA : CB = C'A : C'B = \sin. \frac{\beta}{2} : \sin. \frac{\alpha}{2}.$$

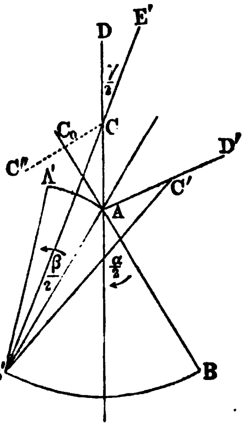
For the case in which α and β are infinitely small, C and C'' coinciding with C_0, this equation becomes

$$C_0A : C_0B = \beta : \alpha;$$

that is, the axis of the resultant of two infinitely small rotations about parallel axes lies in the plane of these two axes, at distances from them inversely proportional to the angles of rotation.

We have hitherto assumed that the two rotations about the axes A and B take place in the same direction (right-handed, like the hands of a watch). But the proposition just demonstrated will also apply when the rotations take place in opposite directions, provided we understand the sum $\alpha + \beta$ to mean the algebraic sum, one of the angles of rotation having a positive sign and the other a negative one. The direction of the resultant rotation will agree with that corresponding to the essential sign of the algebraic sum of α and β, *i.e.*, with the direction of the absolutely greater rotation.

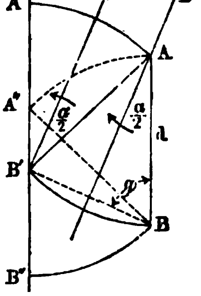
FIG. 4.



In Fig. 4, let the body be subjected to a right-handed rotation α about A, and to a left-handed rotation β about the axis B after it has been removed to B'. Bisect the angle of rotation $BAB' = \alpha$ by AD, and construct the angle $AB'E' = \frac{\beta}{2}$. At the intersection C we get a point which is first taken to C' by the rotation α about A, and then brought back to C by the opposite rotation β about B'. This point represents a normal to the plane of motion, which is also the axis of the resultant rotation, whose amount equals $2\ DCE' = \gamma = \alpha - \beta$. From this we have the rule given above.

As regards the order in which the two rotations shall take place, the remarks made when the directions of the rotations were alike, will now apply also; we accordingly find in the point C', which with C is symmetrical to AB, the axis of the resultant rotation equivalent to the inverted order of rotations β, α. Likewise the axial point C_0, which corresponds to the infinitely small rotations α and β, is so situated on the prolongation of the line AB joining the axes, that as before we have $C_0A : C_0B = \beta : \alpha$.

FIG. 5.



§ 6. **Rotation-Pair.**—When the two opposite rotations around the parallel axes A and B, Fig. 5, have the same magnitude α, the axis of the resultant rotation is at an infinite distance, because in this case the bisecting lines employed in determining a point of the axis become parallel. Rotation about a point infinitely distant, is, however, equivalent to a rectilinear shifting of the body. Now, the fact is, that the final position $A'B'$ due to the two rotations, is parallel to the original AB, and consequently a shifting has taken place which is expressed by the distance AA' or BB'. The amount of this shifting or motion of translation is: $AA' = BB' = 2\, d\, sin.\ \frac{\alpha}{2}$. Here d again represents the distance apart of the axes. The angle φ which the direction of shifting makes with the line AB is

$$\varphi = 90^\circ - \frac{\alpha}{2}.$$

That a pair of opposite and equal rotations should really only produce a simple motion of translation is evident from the following: Let us assume a rotation-pair, and, acccording to § 4, substitute for the rotation α about the axis A its equivalent, namely an equally large rotation about the axis B and a motion of translation equal to $2\, d\, sin.\ \frac{\alpha}{2}$; we shall then have as the result of the rotation-pair only the shifting $2\, d\, sin.\ \frac{\alpha}{2}$, since the rotation α about B neutralizes the rotation $-\ \alpha$ about the same axis.

Such a pair of equal and opposite rotations about parallel axes is called a *Rotation-pair; it may always be replaced by a corresponding motion of translation.* The *converse* of this proposition is *also true.*

That the order in which turning takes place is not immaterial, even in the rotation-pair, an examination of the figure shows. When the rotation takes place first about A and then about B', the line AB connecting the axes passes to the position $A'B'$, but this line takes the position $A''B''$, when the rotations occur in the opposite order. The shifting is therefore in the one case represented by BB' and in the other case by BB''; each of these two distances is equal to $2\, d\, sin.\ \frac{\alpha}{2}$, and has the same inclination $\varphi = 90^\circ - \frac{\alpha}{2}$ to the plane AB of the axes, but the angle φ must be laid off in different directions in these two cases.

The two positions $A'B'$ and $A''B''$ fall therefore into one and the same straight line. The difference due to the order of rotations about A and B evidently ceases when BB' coincides with BB'', *i.e.* when $\varphi = 90°$. For this it is necessary that α be infinitely small; for the case of differential motions, therefore, the remarks concerning unequal rotations also hold good. The expression $2\,d\,sin.\,\frac{\alpha}{2}$ for amount of translation becomes αd for the case when α is infinitesimal.

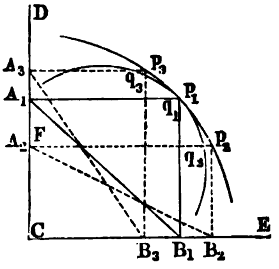
The propositions advanced in § 4 to § 6 concerning the composition and resolution of motions of translation and rotation, apply also to the spaces described in a *unit of time, i. e.*, to *velocities of rotation and translation.*

§ 7. **Instantaneous Centre.**—When all the points of a system move in parallel planes, it easily follows from what has preceded that the motion of the system at every instant may be regarded as an infinitely small rotation about a certain axis perpendicular to the parallel planes. For whatever the character of the motions may be, they can always be resolved into shiftings in the parallel planes, and into rotations about axes perpendicular to the planes. By means of the parallelogram of motions all shiftings can be united into a resultant shifting or translation, while all the rotations can be combined according to § 5 and § 6 into a single resultant rotation. This last can then be combined, according to § 4, with the resultant translation at right angles to the axis, so as to form a single rotation about an axis normal to the parallel planes. If instead of a resultant rotation, a rotation-pair should put in its appearance, we must remember that the latter is equivalent to a simple motion of translation, which can likewise be regarded as a rotation about an axis infinitely distant and perpendicular to the direction of the translation.

The axis of this resultant rotation will in general have no fixed position in space, but will change its place at every instant. In order to determine its position at any particular instant, we have only to remember that the various points of a body rotating about an axis describe paths at right angles to the radii drawn from the points in question perpendicular to the path. If, therefore, in the plane of a path described by an element of the body, we erect a normal at any point of the path, this normal will cut the particular axis about which the body is turning at the instant when the element in question reaches that point of its path from which the normal was drawn. From this follows, that the position of the axis of rotation can be exactly determined at any moment when the directions of the motions of any two points of the body are known. This is done by erecting at each of the two points a normal to the direction of motion, and the intersection of these normals will then give a point through which the required axis of rotation can be drawn perpendicular to the parallel planes. For a system of points moving in parallel planes this axis is called the *instantaneous axis*, and the intersection of the normals, the instantaneous centre.

§ 8. **Centroids.**—The important part played by the instantaneous centre during the motion of a system in a plane is most easily explained by an example. For this purpose let the motion of a rod or rigid straight line, *AB*, Fig. 6, be chosen, whose ends A and B are compelled to remain on the two straight lines CD and CE at right angles to one another. This kind of motion occurs in a known elliptograph, and will be discussed more fully hereafter. We can get the instantaneous centre for any position whatever of the straight line AB by finding the intersection of the two normals A_1p_1 and B_1p_1 erected to the paths CD and CE of the points A_1 and B_1 respectively. In exactly the same way, the instantaneous centres p_2, p_3. . . for any other positions A_2B_2, A_3B_3. . . . of the moving body. If this construction is made for every possible position of the rod AB, there will be obtained a series of points which constitute a curve $p_2p_1p_3$. . . . depending on the nature of the motion. This curve can be regarded as the path traversed by the instantaneous centre during the motion of the body, by imagining the curve as a fixed guide in the plane of motion along which the instantaneous centre is led. This curve has therefore received the name of *centroid*, and because of its fixed position in the plane of motion it may well be called the *fixed* centroid.

FIG. 6.



If to the movable rod AB we imagine a plane rigidly attached in such a manner that while moving, it will always coincide with the plane of motion DCE, we can readily get the following result. At every instant all the points of the rod AB and all points but one of the moving plane turn about the instantaneous centre. The point constituting the exception has for that instant no motion of its own, since it coincides with or covers the instantaneous centre. It must therefore be regarded as the centre of rotation for the instant in question. Let q_1 represent that point of the movable plane when the instantaneous centre is at p_1, and the rod in the position A_1B_1. At the very next instant, this centre will have changed its position from p_1 to p_2, by moving along the fixed centroid, the distance p_1p_2 being regarded as infinitely small. The rod AB has now moved from A_1B_1 to A_2B_2 and that point of the movable plane which coincides with the instantaneous centre of the moment is no longer q_1 but q_2. To determine the position of the point q_2 when the system is at A_1B_1 we have only to construct upon the line A_1B_1 as a base the triangle $A_1B_1q_2$ similar to $A_2B_2q_2$, the vertex q_2 will be the required point. In the same manner by making the triangle $A_1B_1q_3$ similar to $A_3B_3q_3$, we shall get at q_3 that point of the movable plane which coincides with the instantaneous centre p_3 when the rod is at A_3B_3. If we imagine the points q to be thus found for all possible positions of the system, *i. e.*, corresponding to all points of the curve p we shall get a certain curve $q_2q_1q_3$. . . . in the movable plane. This curve also possesses the property of always containing the instantaneous centre and being traversed by it during the motion of the system. This line is consequently also called a *centroid*, the term *moving centroid* being used to distinguish it from the fixed centroid $p_2p_1p_3$. . . .

§ 9. According to what has preceded, for every motion of a system in a plane there are two definite curves or centroids, one fixed in the plane of motion and the other in the moving plane, both of which contain the instantaneous centre. These two curves must therefore always have one point in common, namely, the instantaneous centre of the moment, and indeed they must be tangent to one another at this point, for an intersection of centroids is inconceivable, because else we should have the case of two or more instantaneous centres existing simultaneously, which is of course impossible. During the motion of the system these two curves or centroids are always in contact, the point of tangency changing constantly, and this motion can be regarded as a rolling of the moving centroid $q_2q_1q_3$. . . . on the fixed centroid $p_2p_1p_3$. . . . It is therefore evident from what has been said that we may put the two centroids in place of the external means of constraint which determine the character of the intended motion. If in the example chosen, the two fixed guides CD and CE, Fig. 6, were omitted and in their place were substituted two material forms, the one outlined like the curve $p_2p_1p_3$. . . . rolling on that rigidly connected with the rod AB and outlined like the curve $q_2q_1q_3$. . . . we should get by the rolling of the curves on one another the same motion of the system as would be caused by the guides CD and CE. Accordingly the motion is alike in both cases, and not only for the points A and B, but the agreement holds for any point F whatever, since the motions of two points completely determine that of a system whose points all move in parallel planes. When therefore the centroids are known for a moving system, we can easily determine for any instant the motion of any point whatever of the system. For example, if at the required instant we draw from the point of contact (p_1) of the two centroids to the point (F) whose motion is to be determined, an instantaneous radius, it will be normal to the motion of the point, and the velocity of its motion will be proportional to the length of the instantaneous radius. The correctness of this assertion follows immediately from the remark that the point of contact of the centroids is the instantaneous centre of the movement. If therefore from the centre p_1 with the radius q_1F we describe an arc, it must touch the path of the point F. But the same is true of the arc described about p_2 with the radius q_2F, and about the point p_3 with the radius q_3F, from which follows that *if arcs be described about the various points of the centroid p, with corresponding radii drawn from the points q to the point F of the system, these arcs will envelop the path of the assumed point F.* We can easily see from the figure that in our example the arcs described from the points p with instantaneous radii obtained by joining the points q with A and B, will touch respectively the straight lines CD and CE, in other words the paths of A and B.

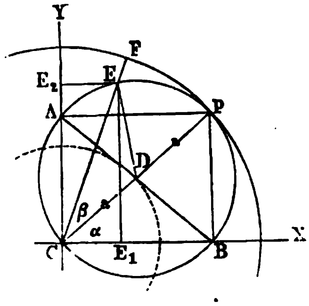
§ 10. **Reciprocal Character of Centroids.**—We have hitherto assumed that the curve q, in the capacity of moving centroid, moves on the fixed centroid p, and we saw how the path or curve described in the fixed plane by any point A, B, or F of the system could be drawn. It is now easy to so reverse the conditions that the fixed centroid p will become the moving one and the moving centroid q become fixed, the curve p rolling on q. First of all it is clear that nothing about the motion of the body AB in the plane DCE will be changed if an additional motion be imparted to AB, provided that exactly the same motion be simultaneously imparted to the plane in which the body moves. The relative motion of the latter in the plane of motion will evidently not be altered, and the paths of the various points will retain their geometrical character unchanged. Let us now imagine that the additional motion thus imparted to both the body and the plane of motion be at every moment equal and opposite to that motion which the body possessed originally. The latter will then come to rest, and the plane of motion originally considered fixed will assume a motion at every moment equal and exactly opposite to the original motion of the system. But as the relative paths of the different points, as already explained, do not suffer any change on this account, we can deduce the proposition that the nature of the motion remains the same whichever of the two centroids is assumed as the moving one. In our example, these two cases of motion could be represented something after this manner: when the curve q is the moving centroid, the motion of the system is such that two of its points are guided along two fixed straight lines at right angles to one another, while in the case when p is the moving centroid, the motion is such that the two straight lines CD and CE, at right angles to one another and fixed in the moved system, constantly pass through the two fixed points A and B. There lies in this a certain principle of reciprocity which has received the name of *Dualism* from *Chasles*, who first called attention to this peculiarity of motion.

A clear example, frequently adduced in illustration of this property, is the ordinary lathe. If a plane disk be fastened at right angles to the mandrel, a tool point fixed at a distance a from the axis will mark on the disk a circle with radius a and concentric with the axis of rotation. But this circle can also be obtained by holding fast the disk and allowing the tool to rotate with the mandrel at a constant distance a from the axis, as often happens, for example, in milling and drilling.

§ 11. It is evident from the preceding of what importance centroids are in determining the motion of any system in a plane; for with their help the path of any point whatever of the system can be determined. In every case the centroids can be constructed point by point, as shown above, when the paths of two points are known. This procedure always attains its object, but in many cases the centroids can more conveniently be drawn directly, when their geometrical character is readily ascertained. For instance, in our assumed example of the motion of a straight line whose ends are guided along two fixed straight lines at right angles, it is easy to see that both centroids must be circles. We see immediately from Fig. 6 that the moved line AB in each of its positions is the diagonal of a rectangle $ApBC$, whose two other vertices are respectively the instantaneous centre p and the point of intersection C of the straight CD and CE. Now since the diagonals of a rectangle are equal, it follows that the distance Cp of the instantaneous centre p from the point of intersection C is constant and equal to the moved line AB, consequently $Cp_2 = Cp_1 = Cp_3 \ldots$, hence the centroid $p_2p_1p_3 \ldots$ is a circle described about C with a radius equal to AB.

Likewise the points $q_2, q_1, q_3 \ldots$ lie in the circumference of a circle whose diameter is the straight line A_1B_1, for when the points $q_2, q_1, q_3 \ldots$ are determined as shown above by constructing right-angled triangles $A_1B_1q_2$, $A_1B_1q_3 \ldots$ on A_1B_1 equal respectively to $A_2B_2p_2$, $A_3B_3p_3 \ldots$ then the locus of these points q will be a circle with diameter A_1B_1, because the angles at $q_2, q_3 \ldots$ are right angles. This circle of course passes through C whatever the position of A_1B_1 may be.

FIG. 7.



For this motion therefore of a straight line AB with its ends guided by the sides of a right angle, we can get identical motions by allowing a circle with diameter equal to AB to roll inside of another circle with diameter equal to twice AB and with centre at the vertex of the right angle. This may easily be realized practically by means of two toothed wheels of the given dimensions. In this motion every point of a circle described on AB, that is, every point of the moving centroid, generates a diameter of the fixed centroid. Let E, Fig. 7, be any point whatever, and let the two guiding lines CA and CB be taken as the axis of x and of y, then the co-ordinates of the point E will be obtained from the equations:

$$x = CE_1 = CE \cos(\alpha + \beta) = 2a \cos\beta \,.\, \cos(\alpha + \beta).$$

$$y = CE_2 = CE \sin(\alpha + \beta) = 2a \cos\beta \,.\, \sin(\alpha + \beta).$$

The diameter of the rolling circle $Cp = AB = 2a$, the angle $BCp = \alpha$, and $ECp = \beta$.

From the above equations we have:

$$\frac{y}{x} = \tan(\alpha + \beta).$$

Since $\alpha + \beta$ is constant for every position of the rolling circle, being equal to half the central angle BDE which measures the arc joining the points E and B, the above equations become

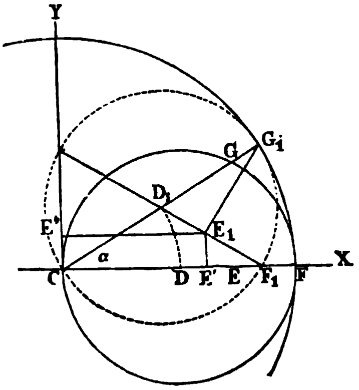
$$\frac{y}{x} = \tan \tfrac{1}{2} BDE = \tan \frac{\gamma}{2},$$

which represents a straight line EC whose inclination to the axis of x is equal to half the central angle.

The figure shows moreover that the point E of the rolling circle will touch the fixed circle at F, for the arc Ep is equal to Fp because the central angle of the latter is one-half and its radius twice as large as the corresponding functions of the arc Ep. When the moving circle has rolled on the entire circumference of the fixed circle, the first will have made two revolutions, and every one of its points will have twice traversed a diameter of the fixed circle. The centre D of the moving circle, on account of its constant distance from C, will of course describe a circle.

In order to determine the path of any point E, Fig. 8, at a distance DE from the centre D of the moving circle, draw through D and E a diameter, and assume as the axis of X that position of this diameter which passes through the centre C

FIG. 8.



of the fixed circle, so that C, D and E lie in the same straight line. From what has preceded we see that for any second position assumed by the rolling circle after rolling over the arc FG_1 and through the angle FCG_1, the points F and G will have moved in a radial direction to F_1 and G_1 respectively. The present position of E is found at E_1 by laying off on the diameter D_1F_1, the distance $D_1E_1 = DE = e$. If $2a$ again represents the diameter CF of the rolling circle, we have for the co-ordinates of E:

$$x = CE' = a \cos \alpha + e \cos \alpha$$
$$= (a + e) \cos \alpha.$$
$$y = CE'' = a \sin \alpha - e \sin \alpha = (a - e) \sin \alpha.$$

Consequently

$$\left(\frac{x}{a+e}\right)^2 + \left(\frac{y}{a-e}\right)^2 = 1.$$

This is the equation of an ellipse whose semi-axes are $a + e$ and $a - e$. This equation also applies to the case when E lies without the rolling circle. The various points of the rolling circle therefore describe ellipses, which become straight lines for points on the circumference, and pass into a circle for the centre.

We can now easily show what takes place when the moving and fixed centroids are interchanged. For this purpose imagine the moving system or the rod AB Fig. 7, fixed in its plane, and the originally fixed plane with the two right lines CA and CB (at right angles to one another) so moved that these two right lines continually pass through the two points A and B, now assumed as fixed. That part of the moving system represented by a circle described from C as a centre and radius $CD = a$ will pass through the fixed centre D of the straight line AB, and in the now fixed system every point like E which is at a distance a from the centre D of the rod possesses the property of being traversed by a straight line of the movable system, the straight line passing through the vertex C of the right angle. Finally, every other point of the fixed system, as E Fig. 8, remains during the motion of the plane of the right angle in an ellipse lying within the latter, and having its centre at the intersection of CA and CB.

Moreover, since every point of the circumference of the rolling circle, as E Fig. 7, moves on a straight line passing through C, it is evident that we can obtain the same motion when the two straight lines inclose any oblique angle whatever, for in place of the system in which the points A and B traverse CA and CB we can evidently put a system in which the points E and B are guided along CE and CB. The correctness of this assertion is immediately seen when we remember that the paths of two points of a plane system completely determine its motion.

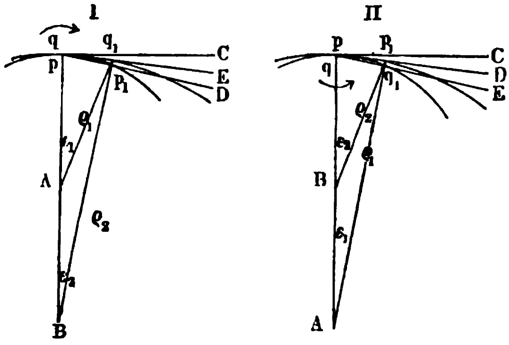
§ 12. **Velocity of Change of the Instantaneous Axis.**—For determining the instantaneous axis or centre of a system moving in a plane at a given moment, it is only necessary, according to the foregoing, to know the *direction of motion* of two points. But for determining the centroids, *i.e.*, all the positions of the instantaneous centres, a knowledge of the *paths* of the two points is necessary. From this, as has been shown, the *path* of any point of the system can be determined. The motion of any point of the system is, however, only known when not only its position and the direction of its path are known, but also the magnitude of its velocity. For determining this latter element for *any point whatever* of the body, it suffices to know the velocity of a *single* point of the body. If A, for example, is such a point, and its distance pA from the instantaneous centre at a particular instant is expressed by r, which distance is to be regarded as the radius for the rotation of A about p, we have for the velocity v of the point A:

$$v = r\omega.$$

ω here represents the angular velocity of the whole system at the moment in question. If we know the velocity v of the point A for every moment, and from the centroids can obtain r for every instant, the angular velocity $\omega = \frac{v}{r}$ will likewise be known for every instant. From this we can get the velocity $v = \omega r_1$, of any other point A_1 which is at the distance r_1 from the instantaneous centre.

If the instantaneous axis of a system moving in a plane had an invariable position while the motion lasted, it would only be necessary to know the angular velocity about the instantaneous axis at every instant in order to completely determine the motion of the body. But since the instantaneous axis is continually changing its position on the fixed centroid, in order to determine the motion of the body we must know the velocity with which the instantaneous centre changes its position. This velocity u is called the *velocity of change of the instantaneous centre or axis.* At a given instant let pq, Fig. 9, I. and II. be the point of contact of the two centroids. Let pp_1 and qq_1 be infinitely small elements of the fixed and moving centroids respectively. If at p, p_1, q, and q_1, we erect normals, their respective intersections, A and B, will give centres of curvature; and therefore $Ap = Ap_1 = \rho_1$ will be the radius of curvature of the fixed, and $Bq = Bq_1 = \rho_2$ that of the moving centroid for the moment when the instantaneous axis is at pq. The infinitely small

FIG. 9.



angle ε_1 inclosed by the normals is evidently equal to the angle CpD formed at the instantaneous centre by the infinitely small element pp_1—regarded as a straight line—with Cp the common tangent of the centroids; likewise

$$\varepsilon_2 = qBq_1 = CqE,$$

where qE represents the directions of the element qq_1. If we now assume that the moving centroid qq_1 rolls on the element qq_2 of the fixed centroid pp_1, q_1 will fall on p_1 and qq_1 will coincide with pp_1. In order that this may occur the moving system and the centroid qq_1 must turn about q an amount equal to $EqD = \varepsilon_1 - \varepsilon_2$. In Fig. I. this amount is positive, but in Fig. II., when $\varepsilon_2 > \varepsilon_1$ it is negative.

The difference in signs serves to indicate that the directions of rotation of the body about the instantaneous centre p is opposite in the two cases, being toward the right in I., and toward the left in II.; the amount of the rotation is, however, equal to the absolute value of $\varepsilon_1 - \varepsilon_2$. If ω again represents the angular velocity, we have

$$\omega = \frac{\varepsilon_1 - \varepsilon_2}{\delta t};$$

and if the length $pp_1 = qq_1$ of the element of the centroids coming into contact with one another during the element of time δt be designated by δs, we can, in like manner, obtain the velocity of change u of the instantaneous centre from

$$u = \frac{\delta s}{\delta t}.$$

Dividing one equation by the other, we get

$$\frac{\omega}{u} = \frac{\varepsilon_1 - \varepsilon_2}{\delta s}.$$

Since ρ_1 and ρ_2 are radii of curvature,
we have

$$\delta s = \rho_1 \varepsilon_1 = \rho_2 \varepsilon_2;$$

therefore

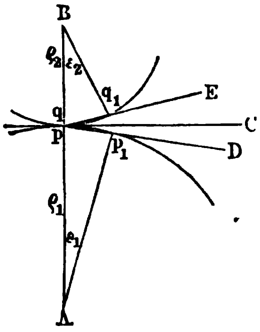
$$\frac{\varepsilon_1}{\delta s} = \frac{1}{\rho_1} \text{ and } \frac{\varepsilon_2}{\delta s} = \frac{1}{\rho_2};$$

hence the above equation may be written

$$\frac{\omega}{u} = \frac{1}{\rho_1} - \frac{1}{\rho_2}.$$

When the two centroids, Fig. 10, have their centres of curvature on opposite sides of the common tangent, we have evidently from the figure

FIG. 10.



$$\omega = \frac{\varepsilon_1 + \varepsilon_2}{\delta t},$$

and therefore

$$\frac{\omega}{u} = \frac{1}{\rho_1} + \frac{1}{\rho_2};$$

but we can retain for this case the equation

$$\frac{\omega}{u} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

by giving to the radii of curvature contrary signs, whenever they lie on opposite sides of the common tangent of the centroids.

If the reciprocal of the radius of a curvature of a curve $\frac{1}{\rho} = \frac{\varepsilon}{\delta s}$ be called the curvature, we can designate the value $\frac{\varepsilon_1 - \varepsilon_2}{\delta s} = \frac{1}{\rho_1} - \frac{1}{\rho_2}$ as "relative curvature" of the two centroids. This, according to the preceding, is equal to the ratio of the angular velocity ω to the velocity of change u.

§ 13. **Example.**—If we choose for illustration the motion of a body which has two points, *A* and *B*, at a distance from one another equal to $2a$, and guided along the fixed sides of a right angle, *ACB* (Fig. 7), the centroids in this case will be two circles whose radii are $2a$ and a. We have, therefore,

$$\frac{\omega}{u} = \frac{1}{\rho_1} - \frac{1}{\rho_2} = \frac{1}{2a} - \frac{1}{a} = -\frac{1}{2a}.$$

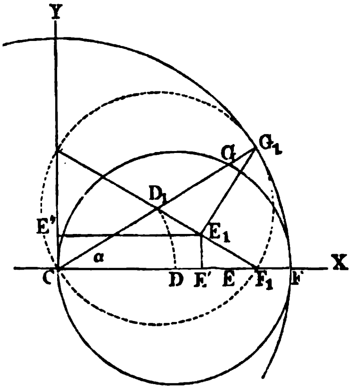
If we now suppose the small circle to roll in a particular direction and with constant velocity on the larger one, the angular velocity ω about the instantaneous centre on the fixed centroid corresponds to a right handed rotation about the centre of curvature.

If we know the angular velocity $\omega = -\frac{u}{2a}$ of the rotation about the instantaneous centre, we also know the velocity of any point whose distance from the instantaneous centre is r, namely $v = \omega r = -r\frac{u}{2a}$. Of all points of the moved body only the centre of the rolling circle *D*, Fig. 11, remains at an invariable distance from the instantaneous centre, whence follows that this point alone traverses its path with a uniform velocity when the velocity of change u is constant. The velocity $v = r\varphi$ of the point *D* is therefore

$$v = a\omega = a\frac{u}{2a} = \frac{u}{2},$$

i.e. half as great as the velocity of change of the instantaneous centre, which also follows from this, that the point D traverses its circular path with radius a, and with constant velocity in the same time that the instantaneous centre traverses the fixed centroid with radius $2a$.

FIG. 11.



In order to determine the velocity of any point of the body moved, let E be any point of the moved system at the distance e from the centre D of the rolling circle. If the latter has moved from the position D (E, C, and D lying in the same straight line CX) to the position D_1 by rolling through an angle $FCG = \alpha$, the points E and G having moved to E_1 and G_1, we get for this position from the triangle $G_1D_1E_1$ whose angle $D_1 = 2\alpha$, the distance G_1E_1 of the point E from the instantaneous centre:

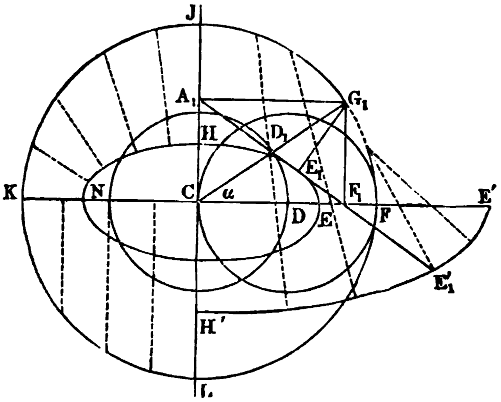
$$G_1E_1 = r = \sqrt{a^2 + e^2 - 2ae\cos 2\alpha}$$

hence the velocity of the point in the position E_1 is found to be

$$v = \omega\sqrt{a^2 + e^2 - 2ae \,.\, \cos 2\alpha}$$

$$= \frac{u}{2a}\sqrt{a^2 + e^2 - 2ae \,.\, \cos 2\alpha}$$

FIG. 12.



A clear picture of the velocities of different points can easily be obtained by means of a graphical representation. Let C, Fig. 12, be the centre of the fixed and D that of the centroid which moves in the circle DD_1 with radius a and centre C. To find at any moment the position of E, which is at a distance from D equal to $DE = e$, the instantaneous centre having moved from F to G_1, we have only to project G_1 on to CF and CJ, then join the projected points F_1 and A_1 by the diagonal A_1F_1, and finally lay off from the middle of A_1F_1 the distance $D_1E_1 = e$.

In this way we find the locus of the point E to be an ellipse CEE_1, whose centre is C, and whose semi-axes are

$$CE = a + e \text{ and } CH = a - e.$$

The line G_1E_1 is evidently the radius of rotation of the point E at the moment when G_1 is the instantaneous centre, and as above, we have:

$$G_1E_1 = \sqrt{a^2 + e^2 - 2ae\ cos.\ 2\alpha.}$$

The velocity v is proportional to this radius. The line G_1E_1, being normal to the path of the point E, *i.e.*, to the ellipse at the point E_1, we can say that the velocity at any moment of any point E is proportional to the normal drawn from the corresponding point of the fixed centroid to the path of the point E. The construction of normals to the ellipse is easily accomplished by means of the rectangle $CA_1G_1F_1$ and its diagonal A_1F_1 in the manner described above.

Let us take for example the angle $\alpha = 0$; for this value E lies in CF, and we have $cos.\ 2\alpha = 1$, hence:

$$v = \frac{u}{2a}\sqrt{a^2 + e^2 - 2\,ae} = \frac{u}{2a}(a - e),$$

and therefore v is proportional to FE. Likewise when $\alpha = 90°$, *i.e.* when the instantaneous centre is at J, $cos.\ 2\alpha = -1$, and

$$v = \frac{u}{2a}\sqrt{a^2 + e^2 + 2ae} = \frac{u}{2a}(a + e),$$

and v is proportional to the distance JH.

The normals to the ellipse drawn in dotted lines within the quadrant $NKJH$ represent therefore the different velocities of the point E while it traverses the elliptical quadrant NH. If in the formula for v we put $e = a$, *i.e.*, take the given point on the circumference of the rolling circle, for instance at F, then from what has preceded we know that the path of this point will be the diameter FK, and the normals in this path drawn from the various positions of the instantaneous centres will be ordinates of the circle perpendicular to FK. The dotted normals in the quadrant CKL represent therefore the velocities of the point F. These velocities are always proportionalto the sine of the angle of rolling α, as can also be seen from the expression for v when we put $e = a$:

$$v = \frac{u}{2a}\sqrt{a^2 + a^2 - 2aa\ cos.\ 2\alpha.}$$

$$= \frac{u}{2a}\sqrt{2a^2(1 - cos.\ 2\alpha)} = \frac{u}{2a}\sqrt{2a^2 \times 2\ sin.^2\ \alpha} = u\ sin.\ \alpha.$$

These velocities are therefore the same as those corresponding to the simple law of oscillation; when $\alpha = 0$, $v = 0$; when $\alpha = 90°$, $v = u$, *i.e.*, equal to the velocity of change of the instantaneous centre.

Finally let us make $e = 0$; we shall then get, for the velocity with which the centre D of the moving centroid traverses the circle DD_1, the constant value:

$$v = \frac{u}{2a}\sqrt{a^2 + 0} = \frac{u}{2}$$

already found. When $e > a$ the given point lies at E' outside of the moving circle, and describes while the latter is rolling in the circular quadrant FG_1J, the elliptical quadrant $E'E'_1H'$, whose semi-axes are also $CE' = a + e$ and $CH' = a - e = -(e - a)$. We easily see that here also the normals, drawn from the quadrant FG_1J to the path $E'E'_1H'$, as FE', G_1E' and JH' represent the velocities of the moved point E'.

§ 14. **Acceleration of a Point of a System.**—When a body has a simple motion of translation, the paths of all the points, in any element of time, are always equal and have the same direction, and from this follows the equality of the *velocities* of all the points for the same instant. If in the course of time these velocities change in magnitude or direction, this *change* must be the same for all points, from which simply follows that the acceleration p must at every moment be the same in amount for all the points of the moved body. This holds good not only for the tangential acceleration p_t which changes the magnitude of the velocity, but also for the normal acceleration p_n which affects a change of direction. The accelerations of all the points of the system are therefore known when we know that of a single point.

If, on the other hand, a body be subjected for a finite interval of time to a simple rotation about an invariable axis, the velocities of the different points will be proportional to their distances from the axis of rotation, and therefore expressed by $v = r\omega$, when ω represents the angular velocity. If we decompose the acceleration p of each point into the tangential acceleration p_t and the normal acceleration p_n, we shall then have according to Vol. I., § 44, for every point the general values:

$$p_t = \frac{\delta v}{\delta t} = r\,\frac{\delta \omega}{\delta t},$$

$$p_n = \frac{v^2}{r} = r\omega^2,$$

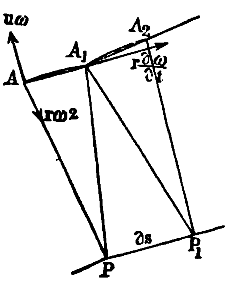
and therefore,

$$p = p_t^2 + p_n^2 = r\sqrt{\left(\frac{\delta \omega}{\delta t}\right)^2 + \omega^4}.$$

In a simple motion of rotation therefore the total acceleration as well as its tangential and normal components are proportional to the distance of the point from the axis of rotation. As already stated, the whole acceleration is reduced to the normal component $r\omega^2$ when the rotation takes place with uniform velocity.

We can now regard the general motion at every instant, of a system whose points move in parallel planes as a rotation about an instantaneous centre. But in addition to this rotation about the instantaneous centre of the moment, the centre itself changes its position on the fixed centroid with a certain velocity of change u. There results, therefore, for every point of the moved body, not only the acceleration corresponding to the rotation about the instantaneous centre, but also another acceleration due to the changing motion of the instantaneous centre. In order to find this acceleration, let $PP_1 = \delta s$, Fig. 13, be an element of the rigid centroid of a system whose points move in parallel planes, and let A be any one of these points. If at a particular instant the instantaneous centre be at P_1, in the following element of time δt the point A will describe about the centre P the arc AA_1, with the radius $AP = r$ and the angle $\omega\,\delta t$. The length of this arc will therefore be

FIG. 13.



$$AA_1 = r\omega\,\delta t.$$

After the lapse of an element of time δt, the instantaneous centre has passed from P to P_1, by traversing the space $\delta s = \omega\,\delta t$, with its velocity of change u. The system now turns about the new instantaneous centre P_1 with an angular velocity which has changed from ω to $\omega + \delta\omega$; the rotation during the new interval of time δt therefore amounts to $(\omega + \delta\omega)\,\delta t$. According to § 4 we can put in place of a rotation about an axis, another equally great rotation about a parallel axis and a corresponding shifting of definite magnitude which is perpendicular to the plane of the axes. We can, consequently, substitute for the rotation $(\omega + \delta\omega)\,\delta t$ about the instantaneous centre P_1, an equally great rotation about P_1 and in addition a normal shifting whose amount is

$$PP_1(\omega + \delta\omega) = \omega\,\delta s,$$

since the infinitesimal magnitude $\delta\omega$ disappears in comparison with ω. If we substitute for δs its value $u\,\delta t$, we obtain for the shifting the value $u\omega\,\delta t$. We thus find, that the space described by the point A during the first interval of time δt has the value $r\omega\,\delta t$, on the other hand that the space described during the second interval is represented by $r\,(\omega + \delta\omega)\,\delta t + u\omega\,\delta t$. If we divide these amounts by δt, we get the velocity of the point A during the first interval equal to $r\omega$, and during the second interval equal to $r\,(\omega + \delta\omega)$ perpendicular to PA and $u\omega$ normal to PP_1. The total acceleration imparted to the point A during the element of time consists therefore of two parts, of which the first is that acceleration p, which is necessary in order that the point A, rotating about P with the angular velocity

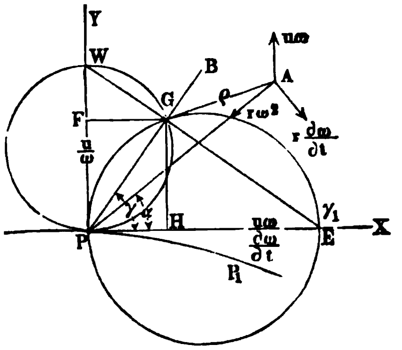
ω, may rotate about the same instantaneous centre P with the velocity of rotation $\omega + \delta\omega$, and the second is the acceleration $u\omega$ due to the velocity of change u. The first acceleration p, which would alone act on A if the instantaneous centre did not change, can, as we already know, be resolved into a tangential acceleration $p_t = r\frac{\delta\omega}{\delta t}$ and a normal acceleration $p_n = r\omega^2$. For every point of the moved body, therefore, these two components, p_t and p_n are proportional to the distance of the point from the instantaneous centre of the moment, while the acceleration $u\omega$ due to the velocity of change is the same for all the moved points.

§ **15. Centre of Acceleration.**—After finding, according to the preceding articles, the acceleration of any point of a system whose points move in parallel planes, several interesting relations between the accelerations of different points of such a system can be developed.

For this purpose we choose the centre P, Fig. 14, of a system, at a particular moment, as the origin of co-ordinates of the rectangular axes PX and PY, of which PX coincides with the tangent, and PY with the normal to the fixed centroid PP_1 at the point P. From what has already been said, we see that at any point A at the distance $PA = r$ from the instantaneous centre, of the three components of acceleration the first, $r\omega^2$, acts in the direction of the instantaneous centre, the second, $r\frac{\delta\omega}{\delta t}$, perpendicular to the radius of rotation PA, and the third, $u\omega$, parallel to PY, the normal to the centroid.

If we resolve these accelerations into components parallel to the co-ordinate axes, and designate the sums of these components respectively by X and Y, we will get, it is easy to see, when we put $XPA = \alpha$

FIG. 14.



$$X = r\frac{\delta\omega}{\delta t}\,sin.\ \alpha - r\omega^2\,cos.\ \alpha$$

$$Y = u\omega - r\frac{\delta\omega}{\delta t}\,cos.\ \alpha - r\omega^2\,sin.\ \alpha\,;$$

or since

$$r\,sin.\ \alpha = y \quad \text{and} \quad r\,cos.\ \alpha = x,$$

$$X = \frac{\delta\omega}{\delta t}\,r - \omega^2 x,$$

$$Y = u\omega - \frac{\delta\omega}{\delta t}\,x - \omega^2 y.$$

If we now make $X = 0$, we get,

$$\frac{\delta\omega}{\delta t}\,y - \omega^2 x = 0,$$

a condition fulfilled by the co-ordinates of these points which have no acceleration in the direction of the tangent to the centroid, in other words, whose accelerations are in the direction PY of the normal to the centroid at the instantaneous centre. But the equation

$$\frac{\delta\omega}{\delta t}y - \omega^2 x = 0$$

is evidently that of a straight line PB, passing through the origin P, and making an angle γ with the axis of X; the inclination of γ is determined from

$$tan.\, \gamma = \frac{\omega^2}{\frac{\delta\omega}{\delta t}}.$$

In the same manner we obtain the locus of all those points which have no acceleration in the direction of PY, the normal to the centroid, but are accelerated only in a direction parallel to the tangent PX:

$$Y = u\omega - \frac{\delta\omega}{\delta t}x - \omega^2 y = 0.$$

This equation, likewise, corresponds to a straight line WE, which cuts the axis of X at the distance

$$PE = \frac{u\omega}{\frac{\delta\omega}{\delta t}}$$

and the axis of Y at the distance $PW = \frac{u}{\omega}$ from the origin. For the angle of inclination γ_1 of this straight line to the axis of x, we have

$$tan.\, \gamma_1 = -\frac{\frac{\delta\omega}{\delta t}}{\omega^2} = -cos.\, \gamma;$$

from which follows that the two straight lines PB and WE are normal to one another.

The intersection G of these two lines evidently has no acceleration, for, as a point of the right line PB its acceleration parallel to the axis of X is equal to zero, and as a point of the right line WE, it has no acceleration parallel to the axis of Y. This point, as the discussion will show, is of especial importance for the motion of the system, and, for reasons to be hereafter given, is called the *centre of acceleration*. Moreover, since at any given instant of the motion there can be only one point whose acceleration is zero, it follows that the two straight lines PB and WE, though continually changing their positions during the motion, can only have one point of intersection.

Furthermore, let the acceleration of the point A be resolved into components in still another way, in the direction of the tangent and normal to the path of the point A itself. From what has been said, we know that the normal to the path coincides with the instantaneous radius AP, and the components produced by this resolution are the same as the so-called tangential and normal accelerations p_t and p_n of the curvilinear motion.

By this resolution we evidently get

$$p_t = r\frac{\delta\omega}{\delta t} - u\omega \cos.\, \alpha,$$

and

$$p_n = u\omega \sin.\, \alpha - r\omega^2.$$

If we again make

$$\cos.\, \alpha = \frac{x}{r} \qquad \sin.\, \alpha = \frac{y}{r} \qquad r^2 = x^2 + y^2$$

we have

$$p_t = \frac{x^2 + y^2}{r}\frac{\delta\omega}{\delta t} - u\omega\,\frac{x}{r}$$

and

$$p_n = u\omega\,\frac{y}{r} - \frac{x^2 + y^2}{r}\,\omega^2$$

If, again, we make $p_t = 0$, the equation gives

$$0 = (x^2 + y^2)\frac{\delta\omega}{\delta t} - u\omega\,.\,x,$$

the locus of all those points which have normal accelerations but no tangential accelerations. This equation represents a circle which passes through the origin P, and cuts the axis of X at a distance from the origin equal to $PE = \frac{u\omega}{\frac{\delta\omega}{\delta t}}$, because for $y = 0$ this equation gives $X = 0$ and $X = \frac{u\omega}{\frac{\delta\omega}{\delta t}}$. If, in like manner, p_n be made equal to zero, the equation

$$0 = u\omega\,.\,y - \omega^2\,(x^2 + y^2)$$

gives for the locus of all the points without normal acceleration a circle which also passes through the origin P, and which cuts the axis of Y or the normal to the centroid at the instantaneous centre at a distance from the origin equal to $PW = \frac{u}{\omega}$. These two circles, or the loci of the points without tangential and without normal accelerations, cut the axes at the same points, W, P, and E, through which pass the two straight lines that were found to be the loci of all the points which have no acceleration in the direction of the tangent and normal to the centroid. The point of intersection *G* of these straight lines has no acceleration either in the direction of the axis of *X* or in that of the axis of *Y*; in other words, is without any acceleration, it consequently does not possess either tangential or normal acceleration, and must lie at the intersection of the two circles just determined. These latter must, therefore, cut one another in the same point *G* in which the straight lines *PB* and *WE* meet, and to which the designation, centre of acceleration, was given. This follows, moreover, from the fact that the straight line *PB* is perpendicular to *WE*, and therefore the vertex *G* of the right angle must lie on the circles described on *PE* and *PW*.

The co-ordinates x_0 and y_0 of the centre of acceleration *G*, are easily obtained from the figure, and equal

$$GF = x_0 = PW\ .\ sin.\ \gamma\ cos.\ \gamma = \frac{u}{\omega}\, sin.\ \gamma\ cos.\ \gamma\ ;$$

$$PF = y_0 = PW.\ sin.^2\ \gamma = \frac{u}{\omega}\, sin.^2\ \gamma.$$

In order to express the acceleration of any point *A* by means of its distance $GA = \rho$ from the centre of acceleration *G*, we subtract from the accelerations *X* and *Y* of the point *A*, the expressions X_0 and Y_0 for these accelerations when referred to the point *G*, *i.e.*, when X_0 and Y_0 equal zero. We have namely.

$$X = \frac{\delta\omega}{\delta t}\, y - \omega^2 x,$$

$$0 = \frac{\delta\omega}{\delta t}\, y_0 - \omega^2 x_0,$$

consequently, by subtraction,

$$X = \frac{\delta\omega}{\delta t}\,(y - y_0) - \omega^2\,(x - x_0)\ ;$$

and likewise

$$Y = u\omega - \frac{\delta\omega}{\delta t}\, x - \omega^2 y,$$

$$0 = u\omega - \frac{\delta\omega}{\delta t}\, x_0 - \omega^2 y_0,$$

consequently

$$Y = -\frac{\delta\omega}{\delta t}\,(x - x_0) - \omega^2\,(y - y_0)\ ;$$

but $x - x = \xi$ and $y_0 - y_0 = \eta$, which are the co-ordinates of the point *A* corresponding to a system of co-ordinates whose origin is at the centre of acceleration G, and whose axes are parallel to the original ones PX and PY. With reference to this system of co-ordinates we have, therefore, the accelerations for A parallel to the axes:

$$X = \frac{\delta\omega}{\delta t}\eta - \omega^2\xi.$$

$$Y = \frac{\delta\omega}{\delta t}\xi - \omega^2\eta.$$

Uniting these components of acceleration by pairs according to the parallelogram of accelerations, we get first an acceleration

$$p_n = -\omega^2\sqrt{\xi^2+\eta^2} = -\omega^2\rho$$

directed toward the origin G, and secondly an acceleration

$$p_t = \frac{\delta\omega}{\delta t}\sqrt{\xi^2+\eta^2} = \frac{\delta\omega}{\delta t}\rho,$$

which is *normal to the first* p_n. We see, therefore, from the formulas that the acceleration of the point A is the same as that which it would assume were the whole system rotating about the point G, the point G having no velocity of change, but acting for two consecutive intervals of time as centre of rotation. For in this case we would have, according to a previous article, the normal and tangential accelerations of a point A at the distance ρ from G, respectively equal to $p_n = \rho\omega^2$ and $p_t = \rho\frac{\delta\omega}{\delta t}$, which expressions are equal to those just calculated.

The total acceleration of any point at the distance ρ *from the centre of acceleration is* $p = \rho\sqrt{\omega^4 + \left(\frac{\delta\omega}{\delta t}\right)^2}$, *that is,* p *is proportional to the distance* ρ. The angle α, made by the acceleration p with the direction of the line joining the point with the centre of acceleration, is given by the equation:

$$tan.\ \alpha = \frac{\frac{\delta\omega}{\delta t}}{\omega^2},$$

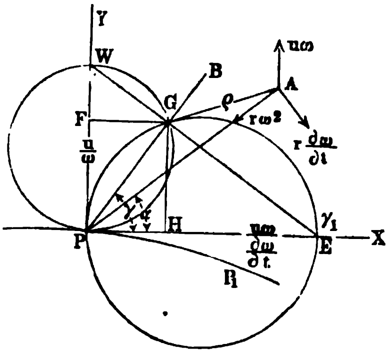
consequently this angle is constant for all points of the system.

It immediately follows from this that all those points must have equal accelerations whose distance from G is equal to ρ.

Such points may of course be found on the circumference of a circle concentric with the centre of acceleration, or on the surface of a cylinder whose axis passes through this centre and is normal to the plane of motion. It is this relation which justifies giving the name of *centre of acceleration* to the intersection G of the two circles.

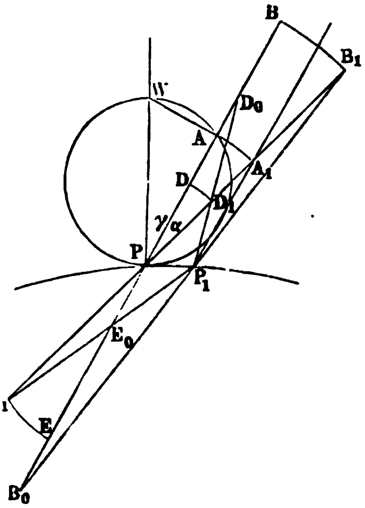
§ 16. **Inflexion Circle.**—The points of the circle PGE, Fig. 15, for which the tangential acceleration at the given instant equals zero, move with their maximum or minimum *velocity* at this instant, because the tangential acceleration or differential coefficient of the velocity then equals zero and changes its sign.

Fig. 15.


Y
W
B
$u\omega$
A
G
ρ
F
$r\omega^2$
$r\frac{\partial\omega}{\partial t}$
$\frac{u}{\omega}$
γ
α
H
γ_1
P
$\frac{u\omega}{\frac{\partial\omega}{\partial t}}$
E
X
P_1

The points of the other circle PGW, for which the normal acceleration equals zero, describe at the given instant the *points of inflexion* of their paths. The radii of curvature are then infinitely great, for it is only when $\rho = \infty$ that the expression $\frac{v^2}{\rho}$ for normal acceleration equals zero, v having a definite, finite value. The circle PGW may therefore be called the *inflexion circle*, and its intersection with the normal to the centroid at P, the *inflexion centre*.

Fig. 16.



Moreover, it may also be directly shown from the following discussion that all points of the system which simultaneously occupy the points of inflexion of their paths, lie on the circumference of a circle which touches the fixed centroid at the instantaneous centre P.

Let PP_1, Fig. 16, be the fixed centroid of a system whose points move in parallel planes, and let P be the instantaneous centre at a given moment, around which the moved body turns with the angular velocity ω, during the element of time δt. Any straight line EPB drawn through P will be brought by this rotation into the position E_1PB_1, the points $A_1B_1D_1$ and E describing respectively the paths AA_1, BB_1, DD_1, and EE_1. If, during the same interval, δt the instantaneous centre moves along the fixed centroid with the velocity of change u, it will reach P_1 if $PP_1 = u\,\delta t$. If we now connect the instantaneous centre P_1 with the present position A_1, B_1, D_1, and E_1 of the moved points, the lines P_1A_1, P_1B_1, P_1D_1, and P_1E_1 will evidently be normal to the paths of these points at A_1, B_1, D_1, and E_1. In like manner the line EPB is normal to the paths of these same points at A, B, D and E. Now since the intersection of the normals to a curve at two consecutive points gives the centre of curvature of the element to which they belong, we get at B_0, D_0, and E_0 the centre of curvature respectively of the paths BB_1, DD_1, and EE_1. But the centre of curvature of the path AA_1 is at an infinite distance so soon as the two normals PA and PA_1 become parallel, and this is a sign that at this moment the point A traverses a point of inflexion of its path. Now, in order that the two normals PA and P_1A_1 may be parallel, the angle APA_1 or $\omega\,\delta t$ must equal the angle $PA_1P_1 = PAP_1$. But if we imagine this angle $\omega\,\delta t$ to be constructed in all possible positions on the element PP_1, we shall evidently get for the locus of its vertex A a circle PAW, which will have an infinitely small element PP_1 in common with the centroid at P, in other words which touches the centroid at P, and whose centre is therefore situated on the normal PW to the centroid. For the diameter PW of this circle we have the relation $PW \, . \, \omega\,\delta t = PP_1$ and since $PP_1 = u\,\delta t$ we have for the diameter PW of the inflexion circle :

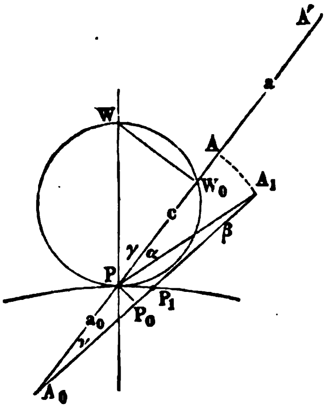
$$PW = \frac{u}{\omega},$$

the same result as was obtained above. From the figure we also immediately see that a point *within* this circle, as D, has its centre of curvature at D_0, *i.e.*, upon the *same* side of the centroid as the inflexion circle, while a point *outside* of the inflexion circle, as B or E, has its centre of curvature at B_0 or E_0, *i.e.*, upon that side of the centroid *opposite* to the inflexion circle. The paths of all the points *within the inflexion circle* are therefore convex to the instantaneous centre, while those outside of the inflexion circle are concave to this centre ; the points on the circumference of the inflexion circle describe paths whose radii of curvature at the given instant are infinitely great because of the change of sign of the curvature.

§ 17. **Curvature of the Paths of Points.**—Since a knowledge of the radius of curvature of the path of a point is of importance because both the tangential and normal acceleration of a point depend upon it, we will now discuss the principal relations with respect to curvature which exist among the paths of a system whose points move in parallel planes.

Let P, Fig. 17, be the instantaneous centre at a given moment, PW the normal to the centroid, W the inflexion centre, and therefore PW_0W the inflexion circle ; moreover, let A be any point of the moved body whose distance $AP = a$ from the instantaneous centre makes an angle γ with the normal WP, and which during the rotation about P describes the small arc AA_1. According to the preceding, the centre of curvature of the element AA_1 of the path lies at the intersection A_0 of the two instantaneous radii AP and A_1P_1, provided the instantaneous centre passes from P to P_1 during the given interval. If for the

FIG. 17.



sake of brevity we designate the infinitely small angle of rotation APA_1 or $\omega\,\delta t$ by α, the contiguous angle PA_1P_1 by ν, and the small angle PA_1P_1 by β, we shall have, since α is an exterior angle, $\alpha = \nu + \beta$. In order to express these angles by linear measure, we make use of the inflexion circle, because it contains the inscribed angle α subtended by the element PP_1 (see § 16), and get

$$PP_1 = WP \,.\, \alpha \quad \text{or} \quad \alpha = \frac{PP_1}{WP}.$$

Let P_0 be the projection of P on A_0A_1, then will $PP_0 = PP_1 \; cos.\; \gamma$, and we see from the figures that

$$A_0P \,.\, \nu = PP_1 \,.\, cos.\; \gamma = AP \,.\, \beta.$$

By substituting the following values of

$$\nu = \frac{PP_1}{A_0P}\, cos.\; \gamma = \frac{PP_1}{a_0}\, cos.\; \gamma,$$

$$\beta = \frac{PP_1}{AP}\, cos.\; \gamma = \frac{PP_1}{a}\, cos.\; \gamma,$$

in the equation $\alpha = \nu + \beta$ we get

$$\alpha = \frac{PP_1}{WP} = \frac{PP_1}{a_0}\, cos.\; \gamma + \frac{PP_1}{a}\, cos.\; \gamma$$

or

$$\frac{1}{WP \,.\, cos.\; \gamma} = \frac{1}{a_0} + \frac{1}{a}.$$

Now if W_0 is the projection of the inflexion centre W on the instantaneous radius AP, $WP \,.\, cos.\; \gamma = W_0P = c$, so that the above equation can also be written

$$\frac{1}{c} = \frac{1}{a_0} + \frac{1}{a},$$

or

$$a_0a = ac + a_0c.$$

Hence $a_0a - a_0c = ac$; adding this to the last equation, we get

$$a_0(2a - c) = c\,(2a + a_0),$$

or

$$a_0 : c :: 2a + a_0 : 2a - c.$$

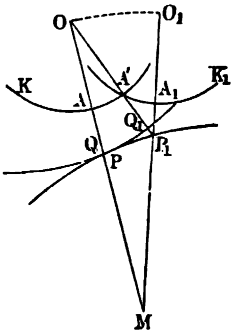
This proportion means that A' is the fourth point of the harmonic system to which the three points A_0, P, and W_0 belong, and that it is conjugate to P; this point A' is at a distance from A equal to $AA' = a$, which equals the distance AP of the instantaneous centre. If we make $AA' = a$, A_0A' will equal $2a + a_0$, and $W_0A' = 2a - c$, we may write the above proportion:

$$A_0P : W_0P = A_0A' : W_0A.$$

The centre of curvature A_0 of the element of the path of a point A and the projection W_0 of the inflexion centre on the instantaneous radius AP, divide in harmonic ratio the distance PA′, the instantaneous centre P being at one end of this distance and the point A at its middle. This important relation can in many cases be employed to determine the centre of curvature A_0 and consequently the radius of curvature of the path of a point of the system.

§ 18. **Curvature of Envelopes.**—Before we make any deductions from the proposition just demonstrated, let us consider the case in which any curve whatever is attached to the moved system so as to take part in its motion. The position of this curve at any instant is of course known when not only the position of any two points of the system for this instant are known but the position of the curve with reference to these two points is also known, for it has been previously shown that this is sufficient to determine completely the motion of the system. If we imagine all the positions occupied by this curve in succession, we can easily conceive of a line which will be touched by the curve in its various positions. This line is called the *envelope* of the moved curve, and we can get an idea of its appearance by supposing the moved curve constructed in the form of a wire and attached to the moved system; it will, while moving, trace out its envelope on a sand-covered surface parallel to the plane of motion. In practice, it is often of interest to be well informed of the nature of these envelopes.

FIG. 18.



At a given time t, let P, Fig. 18, be the instantaneous centre of a system whose points move in parallel planes, and let K be a given curve attached to this system. During the element of time δt, let the instantaneous centre move along the centroid to P_1, K will then be in the position K_1. This curve K continually touches a certain line E which was called above an envelope. If we join the instantaneous centre P with the point of contact A of this envelope and the curve K, this instantaneous radius will be normal to the common tangent of K and E. This follows because the instantaneous radius is perpendicular to the paths of all its points, and consequently also to the path of the point of contact A, and because the point of contact of the curves K and E evidently has a motion in the direction of the tangent to the envelope. The radius PA therefore coincides with the normal common both to the curve K and to the envelope, and consequently passes through the centre of curvature of both. The same is true of the instantaneous radius P_1A_1 which is drawn from the instantaneous centre P_1, a position consecutive to P, to the corresponding point of contact A_1 of the curve K_1 with the envelope. It follows from this that a point M, the point of intersection of these two instantaneous radii, is the centre of curvature of the envelope for the element AA_1. During the slight motion of the system the element QQ_1 of the moving centroid has rolled on the fixed centroid till the point Q_1 reached the position P_1. The normal drawn from Q_1, through A', to the curve K, likewise passes through the centre of curvature of the latter. This centre may be found at O, the intersection of QA and Q_1A'. After the rotation by which Q_1 reached P_1, the normal Q_1A' must coincide with the normal P_1A_1, and the centre of curvature passes from O to O_1. Now, since the instantaneous radii which are normal to the envelope are also normal to the path of each of its points, and therefore normal to OO_1, it follows that *their intersection M is not only the centre of curvature of the element AA_1 of the envelope E but also the centre of curvature of the element OO_1 of the path described by the centre of curvature of the moved curve.* The centre of curvature of the envelope of any curve whatever is therefore known as soon as we can determine the centre of curvature of that path which is described by the centre of curvature of the moved curve. The determination of the latter is, however, easily accomplished by means of the proposition developed in § 17.

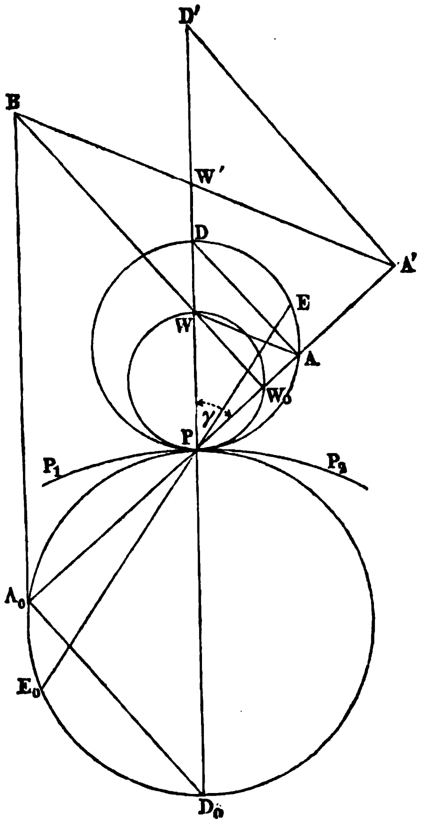
Since the moving centroid, in every one of its positions, touches the fixed centroid, the latter can be regarded as the envelope of the moving centroid, and it follows from the proposition just proved, that the centre of curvature of the fixed centroid at every instant coincides with the centre of curvature of that element of the path described by the particular centre of curvature of the moving centroid belonging to the point of contact.

§ 19. **Determination of the Centre of Curvature.**—If in a system (whose points move in parallel planes) we know the instantaneous centre and inflexion centre at a given instant, we can easily determine, from what has preceded, the path of any point whatever for the given instant.

Let P, Fig. 19, be the instantaneous centre and W the inflexion centre of the system at a given instant. Furthermore, let A be any point of the moved body for which we wish to determine the centre of curvature of an element of its path. Draw the instantaneous radius PA and make $AA' = PA$, then the desired centre of curvature will be found when we find a point A_0 which with the three points P, A' and W_0 will constitute an harmonic system and be conjugate to W_0. This construction can always be made by well-known methods, but it can be greatly simplified if we join the point A with the inflexion centre W and draw $A'W'$

parallel to AW, prolonging it till it intersects the line of projection W_0W in B. The three points P, W and W', of which W lies midway between P and W', and a fourth which is at an infinite distance because it is the centre of curvature of the path of W, constitute an harmonic system. Therefore the fourth ray BA_0 belonging to the harmonic system of BW', BP, and BW becomes parallel to the normal WP of the centroid. Now these four rays divide the straight line AP in harmonic ratio at the points A', P, W_0, and A_0, consequently A_0 is the centre of curvature of that element of the path described at the instant when the instantaneous centre is at P. The rule for finding A_0 when P, W, and A are given is now easily obtained. At the points A_0, A, and A' imagine the perpendiculars A_0D_0, AD, and $A'D'$ erected; it will then be evident that the four points D_0, W, P, and D' form an harmonic series, since all the segments on D_0D' are greater in the ratio of $\frac{1}{\cos \gamma}$, than the segments on A_0A'. Consequently since D lies midway between P and D', D_0 is the centre of curvature of the element of the path described by D. If a circle be now drawn through P, A, and D, and another through P, A_0, and D_0, their centres will lie on the normal D_0D' to the centroid, because the angles at A and A_0 are right angles; moreover it will readily be seen that the remarks just made concerning A apply to every point of the circle PAD. If, therefore E is any point of this circle, the instantaneous radius EPE_0 will intersect the circle PA_0D_0 at a point E_0 which will be the centre of curvature of the element of the path described by E.

FIG. 19.



From this follows the relation that *the points of a circle tangent to the centroid at the instantaneous centre*, describe elements of paths whose centres of curvature lie on a second circle likewise tangent to the centroid at the instantaneous centre.

In order that we may determine the relative position and magnitude of these circles, let us assume first that the circle through A gradually approaches the inflexion circle till it coincides with it. The circle A_0 will at the same time gradually increase and finally become infinite when A reaches the inflexion circle. This happens because the projection W_0 of the inflexion centre lies midway between P and A', and therefore the fourth point of the harmonic series is at an infinite distance. This corresponds also with the property of the inflexion circle already developed, according to which all points of this circle describe elements of paths having infinite radii.

Suppose the reverse to take place, the point A gradually receding from the instantaneous centre P, then the circle A_0 for centres of curvature will gradually decrease, and finally at the limit—when A is at an infinite distance—will become equal in magnitude to the inflexion circle, and will lie on that side of the tangent to the centroid opposite this circle; moreover, the two circles are symmetrical with reference to the tangent to the centroid. This follows because A and therefore A' are at an infinite distance, wherefore the instantaneous centre P conjugate to A' must bisect the distance $A_0 W_0$.

When the describing point A moves toward the instantaneous centre P, the centre of curvature, according to § 16, lies on the same side of the tangent to the centroid as the inflexion circle, and the circle on which it lies gradually approaches the inflexion circle as A gets nearer to the instantaneous centre. If the point A gets on to a circle which is tangent to the centroid, and with a diameter half that of the inflexion circle, the corresponding centre of curvature will lie on the inflexion circle, for since two of the points A' and W_0 of the harmonic series P, A', W_0 and A_0 coincide on the inflexion circle, A_0 will also fall on this double point. If A approaches still nearer to the instantaneous centre, A_0 will do likewise, till finally A and A_0 simultaneously reach the instantaneous centre P.

Let us assume that the moving point A passes through the instantaneous centre, to that side of the tangent to the centroid opposite to the inflexion circle; the centre of curvature of the path of the point will then lie on a circle which touches the centroid on A's side of the tangent. This circle is smaller than the circle touching the centroid and passing through A and the instantaneous centre. The greater the circle on which A lies the greater the circle containing the centre of curvature A_0, and the latter will have a magnitude equal to the inflexion circle, and will lie on the opposite side of the tangent to the centroid as soon as the describing point A is infinitely distant.

§ **20. Determination of the Inflexion Centre.**—The previous investigation supposed the inflexion centre of the moved system given. Generally this point can be obtained from the conditions characterizing the motion in question. If, for example, the two centroids be given, the radii of curvature ρ_1 and ρ_2 for any point will be known, and we can get the inflexion centre by laying off from the instantaneous centre, on the normal common to the centroids, a distance $\frac{\omega}{u}$ equal to the diameter of the inflexion circle. This value, according to § 12, is obtained from the equation $\frac{\omega}{u} = \frac{1}{\rho_1} - \frac{1}{\rho_2}$. If the motion of the body is given by the paths of two points A and B, we can, according to a former article (§ 7), determine, at any moment, the instantaneous centre by finding the intersection of normals to these curves at A and B respectively. If, now, we lay off on these instantaneous radii PA and PB (upon which lie also the centres of curvature A_0 and B_0 of the known paths of A and B) the distances AA' and BB', respectively equal to PA and PB, we shall have three points in each of the two harmonic series, namely, P, A', A_0 and P, B', B_0 ; the fourth points will be conjugate to the centres of curvature and will be found at the two projections W_0 of the inflexion centre on the two instantaneous radii PA and PB. The inflexion centre itself is therefore found at the intersection of the normals erected to PA and PB at the two points W_0.

In particular cases often occurring in practice, definite rules can be given for determining the inflexion centre. The principal rules only will be given here. If a point A of the moving body describes a rectilinear path, its centre of curvature A_0 is at an infinite distance, and consequently its cönjugate W_0, the projection of the inflexion centre W must lie midway between the instantaneous centre and the point A', *i.e.*, must coincide with the point A at the middle of PA'. Since the inflexion centre lies on the normal to the instantaneous radius PA at A, *i.e.*, in the direction of the path of the point A, it follows *that if a point moves in a straight line, this line must pass through the inflexion centre.*

In mechanisms the case often occurs that the moving body and the curve K attached to it move along a fixed curve E, Fig. 18, in such a manner that the fixed curve E is continually touched by the moving curve K. In this case E is the envelope of K, and since, according to § 18, the centre of curvature of the envelope coincides with that of the path of OO_1 described by the centre of curvature O of the curve K, the four points, namely the instantaneous centre P, the point O' lying on the prolongation of PO at the distance $OO' = PO$, the centre of curvature M of the fixed curve E, and the projection W_0 of the inflexion centre on the radius MO, must constitute an harmonic system. If in this case the moved curve K becomes a straight line continually touching a fixed curve E, two of the points of the harmonic system, O' and O, will be at an infinite distance, consequently *the instantaneous centre P will lie midway between the centre of curvature M of the fixed curve E and the projection W_0 of the inflexion centre on the instantaneous radius.*

If in this case the fixed curve E be reduced to a single point, through which the moved curve K continually passes, then of the four points of the harmonic series, the centre of curvature M of the fixed curve coincides with the fixed point E itself, and the projection W_0 of the inflexion centre is conjugate to E and is the fourth point of an harmonic system containing the fixed point E, the instantaneous centre P and the point O' at a distance $OO' = PO$. If in this case the moved curve becomes a straight line, O' will be at an infinite distance, and therefore *the instantaneous centre lies on the instantaneous radius through the fixed point and midway between the fixed point E and the projection W_0 of the inflexion centre.*

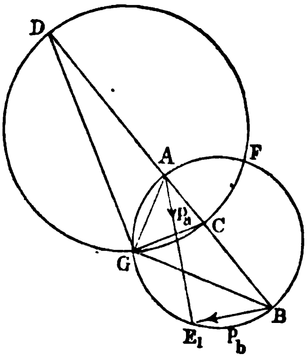
If, finally, the fixed curve E becomes a straight line continually touched by the moved curve K, the centre of curvature M will be at an infinite distance, and as a consequence its conjugate W_0—the projection of the inflexion centre—will lie midway between the instantaneous centre and the point O', *i.e.*, *it will coincide with the centre of curvature O of the moved curve K.* If in this case also the curve K be reduced to a single point, which describes the rectilinear path, then the centre of curvature O and the projection W_0 will fall into the straight line E, from which follows a property once before enunciated, namely, that the straight line E which guides the point K passes through the inflexion centre.

Finally, as regards the velocity of the inflexion centre, we know, from what has been already established, that it is equal to $PW.\ \omega$, when ω is the angular velocity of the whole system about the instantaneous centre P, and PW the distance of the inflexion centre from P. But this distance PW is the diameter of the inflexion circle, which in § 15 was found equal to $\frac{u}{\omega}$. From this follows that the velocity of the inflexion centre must equal $\frac{u}{\omega}.\ \omega = u$, that is, it must equal the velocity of change of the instantaneous centre. Consequently, we have for the velocity of any point W_0 of the inflexion circle, Fig. 19, whose instantaneous radius makes the angle γ with the normal PW to the centroid, the formula:

$$v = PW_0.\ \omega = \frac{u}{\omega} \cos\gamma\ .\ \omega = u \cos\gamma.$$

§ 21. **Determination of the Centre of Acceleration.**—In § 15 we found the centre of acceleration to be a point having no acceleration, its position being determined by the intersection of two straight lines which were at right angles to each other, and of which one contained all those points having no acceleration in the direction of the tangent to the centroid (at the instantaneous centre), while the other contained all the points having no acceleration in the direction of the normal to the centroid (at the instantaneous centre). This instantaneous centre of acceleration also lies at the intersection of the inflexion circle and the circle containing all those points which have no acceleration tangential to their paths. A knowledge of the position of this point being of considerable importance in investigating the motions of mechanisms, we will now proceed to explain how this determination of the instantaneous centre may be effected.

Fig. 20.



According to § 15 the centre of acceleration G, Fig. 20, has the property that the acceleration of any point A of the system is proportional to the distance GA of the point from the centre of acceleration G, and that the angle made by the acceleration p_a of A with the radius GA, is the same for all points of the system, being equal to

$$\tan\alpha = \frac{\frac{\delta\omega}{\delta t}}{\omega^2}.$$

If, therefore, the acceleration of p_a and p_b of any two points A and B are known, it follows that the centre of acceleration must be so situated that $GA : GB = p_a : p_b$. Consequently, if we divide the distance AB at C and D in an harmonic ratio, so that $CA : CB = DA : DB = p_a : p_b$, the centre of acceleration must lie on the circumference of the circle described on DC as a diameter, this circle being the locus of all those points whose distances from A and B are in the ratio of $p_a : p_b$. For if we join any point G with A, B, C, and D, of the four harmonic rays thus obtained, the two, GD and GC, which are at right angles, will each bisect the angle AGB formed by the other two rays, and we shall, consequently, have $GA : GB = CA : CB = p_a : p_b$.

If the acceleration p_a and p_b of the points A and B are known in direction, and are laid off accordingly from A and B, these directions prolonged till they intersect at E will form an angle AEB equal to that included by the rays drawn from A and B to the instantaneous centre of acceleration. This follows very readily from the property that the direction of acceleration of any point of the system makes a constant angle with the ray drawn from the point to the instantaneous centre of acceleration, for two of these latter rays will enclose an angle equal to that enclosed by the accelerations of the corresponding points. Any point on the circle passed through A, B and E will, therefore, have the property, that the rays from it to A and B will enclose an angle equal to AEB. The centre of acceleration must, therefore, also lie on this circle, and will, consequently, be found at the intersection G of the two circles. Which one of the two intersections is to be taken for the centre of acceleration may easily be decided as follows: Resolve each of the two accelerations p_a and p_b into two components, of which one coincides in direction with the ray drawn from A or B to the centre of acceleration, while the other is at right angles to the ray. The components which coincide with the rays will be directed toward the *centre of acceleration.* In the present case, therefore, the second point of intersection F does not act as centre of acceleration, because the directions of the aforesaid (coincident) components are pointed not toward but away from F, which would not agree with known properties of the centre of acceleration.

If we also know the magnitude of the acceleration p_a of any point A we can easily determine the acceleration of any other point C of the system, for the latter must equal $p_c = \frac{p_a}{GA} . GC$. For example, if the instantaneous centre P has been obtained by erecting normals to the paths of A and B, the acceleration of the instantaneous centre will be given by $\frac{p_a}{GA} . GP$. But according to § 14 the instantaneous centre has an acceleration $u\omega$ in the direction of the normal to the centroid. If we therefore put

$$\frac{p_a}{GA} . GP = u\omega,$$

we can get the velocity of change u, when the angular velocity ω is known. If, finally, the acceleration p_a makes the angle α with the ray GA, the component of the acceleration which is at right angles to GA will be $p_a \sin \alpha$, and since this component must also equal $GA . \frac{\delta\omega}{\delta t}$ we shall have

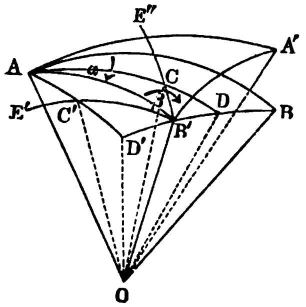
$$\frac{\delta\omega}{\delta t} = \frac{p_a}{GA} \sin \alpha.$$

§ **22. Parallelogram of Rotations.**—In all of the foregoing investigations the moved system was assumed to be such that the paths of all of its points lay in a series of parallel planes, each point continually remaining in its own plane, the motion being briefly designated as con-plane motion. In this kind of motion, in which the axes of rotation were all parallel, being at right angles to the parallel planes, the investigation of the motion in one of these planes sufficed to determine the motion of the whole system. Now although by far the greater part of the mechanisms met in practice can be reduced to this simple case of motion in a single plane, mechanisms do, nevertheless, occur in which the motions of the individual points have a more general and less simple character. Particularly well known cases are those in which the axes, about which rotation takes place, are not parallel.

The most general form of motion is that in which the system is subjected to any translations or rotations whatever, the former taking place in any direction in space, and the latter rotating about axes crossing one another in any manner. Before we discuss this most general case, let us investigate more closely the special case in which the system is subjected to two rotations about two axes which intersect.

Let OA and OB, Fig. 21, be two straight lines intersecting at O and lying in a body which is to receive in succession a rotation about each of these lines as an axis. Let the amount of the rotation about OA equal α, and that about OB equal β. It may be easily shown in a manner analogous to § 5 that the same result can be attained by a single rotation about a certain axis OC. For this purpose imagine O to be fixed in space as the centre of a sphere whose radius equals unity, and that the axes OA and OB meet the surface of this sphere in the points A and B respectively. If the body now receives a rotation α about the axis OA, equal to $B(OA)B'$; the straight line OB (to be used hereafter as an axis) will move into position OB', which is to be regarded as its final position, because in the following rotation about itself the position of the axis is invariable. When the second rotation $A(OB')A' = \beta$ takes place, the axis OA will be brought into its final position OA'. Now it is evident that because the point O is the intersection of the two axes, its position in space remains unchanged; moreover the new positions A' and B' of the points A and B are also known from the angles α and β; the three points, therefore, O, A' and B', not in the same straight line, determine the position of the whole system.

Fig. 21.



If we imagine a plane OAD passed through OA and bisecting the angle of rotation $B(OA)B' = \alpha$, so that $D(OA)B' = \frac{\alpha}{2}$, this plane OAD will reach the position OAD' in consequence of the first rotation about OA, provided $B'(OA)D' = \frac{\alpha}{2}$ and therefore $D(OA)D' = \alpha$. In like manner, imagine, after the second rotation has taken place about OB', a plane $OB'E''$ so united to the system that it not only passes through the second axis of rotation OB' but bisects the angle $A(OB')A' = \beta$ making $A(OB')E'' = \frac{\beta}{2}$. We can easily see that before the second rotation takes place this plane must have the position $OB'E'$, provided the angle $E'(OB')A$ is likewise equal to $\frac{\beta}{2}$. This plane $OB'E'$ intersects the plane OAD' in the radius OC of the sphere, and it follows immediately from the construction that this line of intersection OC', by means of the rotation about OB' is brought to the position OC, so that it coincides with the line of intersection made by the bisecting planes OAD and $OB'E''$. This follows because the spherical triangles $AB'C$ and $AB'C'$ are symmetrical, the line AB' being common and the adjacent angles equal. OC is therefore a straight line which is carried by the rotation α about OA to OC' and is then carried back to its original position by the rotation β about OB'. From this follows that the result of these two rotations can likewise be attained by means of a single rotation about OC.

The position of the resultant axis is obtained as shown by the construction just given, in a manner analogous to that given in § 5 for a system whose points move in parallel planes. We have only to bisect the angles α and β after they have been properly laid off from OA and OB'. The amount γ of the resulting rotation about OC may be obtained from the figure by remembering that in this rotation the axis OA must be transferred to OA' and OB to OB'. The external angle $D(OC)B' = E''(OC)A$ of the spherical triangle $AB'C$ is therefore equal to half the angle of rotation about the resultant axis OC.

This half angle of rotation $\frac{\gamma}{2}$ may be determined from the well known property of spherical triangles:

$$\cos A(OC)B' = \cos\left(180^\circ - \frac{\gamma}{2}\right) = -\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$$

$$+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos AOB'.$$

or,

$$\cos\frac{\gamma}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos AOB'.$$

In like manner for the position of the resultant axis we have the relation:

$$\frac{\sin AOC}{\sin\frac{\beta}{2}} = \frac{\sin B'OC}{\sin\frac{\alpha}{2}} = \frac{\sin AOB'}{\sin\frac{\gamma}{2}}.$$

That the angles α and β must be laid off to correspond to different directions of rotation is sufficiently evident. In this case, as in that for a system with points moving in parallel planes, and for reasons wholly analogous to those given in § 5 and § 6, a change in the order of rotations is only allowable when the angles of rotation α and β, and consequently γ, are infinitely small magnitudes. When this occurs, the axis OC of the resultant rotation lies in the plane AOB' of the two axes, and since :

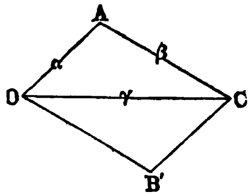
$$sin\,\frac{\alpha}{2} = \frac{\alpha}{2}\,;\; sin\,\frac{\beta}{2} = \frac{\beta}{2},\text{ and } sin\,\frac{\gamma}{2} = \frac{\gamma}{2},$$

we have,

$$sin\,\frac{AOC}{\beta} = sin\,\frac{B'OC}{\alpha} = sin\,\frac{AOB'}{\gamma}.$$

This equation is evidently satisfied by the sides of a triangle and the angles opposite them, and we can employ the following construction for determining the resultant rotation γ. Lay off on the directions of the axes OA and OB', Fig. 22, the distance OA proportional to α and OB' proportional to β; then will the diagonal OC of the parallelogram constructed on OA and OB' give the direction of the axis of the resultant rotation, and its length OC will be proportional to the angle of rotation of the resultant rotation, the unit employed being the same as that used in representing α by OA and β by OB'. That this construction is correct is evident because the parallelogram satisfies the equation:

FIG. 22.



$$sin\,\frac{AOC}{\beta} = sin\,\frac{B'OC}{\alpha} = sin\,\frac{AOB'}{\gamma},$$

which is also true for the resultant rotation.

The law herein contained, designated by the term *Parallelogram of Rotations*, can be thus expressed: *The resultant of two infinitely small rotations about two axes which intersect is a rotation about a third axis lying in the plane of the other two and passing through their intersection ; and the diagonal of a parallelogram whose sides are parallel to the axes and proportional to their angles of rotation gives by its direction the position of the resultant axis, and by its length the magnitude of the resultant rotation.*

By repeatedly applying this proposition, the composition of any number of *infinitely small* rotations about axes which intersect at the same point becomes possible, and we can, therefore, in accordance with the previously established propositions of the parallelogram of (rectilinear) motions, of velocities, etc., speak of a *polygon* and a *parallelopipedon* of *rotations* according as the axes intersecting in the same point lie in the *same* or in *different* planes.

The preceding proposition likewise always allows an infinitely small rotation to be resolved into two or more which will rotate about axes intersecting the principal axis in the same point. For the resolution of rotations we have rules analogous to those applied in the propositions concerning the parallelogram of motions, velocities, etc., already mentioned in different places.

§ 23. **Motion of a Body about a Fixed Point.**—In the previous article the remark was made that when the body is subjected to rotations about two axes which intersect, their point of intersection does not change its position in space. This, of course, also holds good when there are any number of rotations, provided the axes all intersect in a common point. When, therefore, we have a case, frequently occurring in practice, in which one point of the body is fixed, we can make the converse statement that all the motions of which the body is still capable can be reduced to rotations about axes passing through the fixed point. For not only every translation, but every rotation about an axis that does not pass through the fixed point O is excluded by the supposition that the point O has a fixed invariable position. Evidently in this kind of motion any point A, at a distance $OA = r$ from the fixed point, must remain on the surface of a sphere concentric with O and having a radius equal to r; in other words, all the points of a body must describe spherical paths. Now according to the preceding article we can always substitute for two rotations of any magnitude whatever, and rotating about two axes which intersect, another rotation about an axis passing through the intersection of the other two. Since this rotation can be combined with any third or fourth rotation about an axis passing through the same point, it follows, that for *every motion* of a body rotating about a point, though it be composed of ever so many different rotations, we can substitute *a single rotation* about a certain axis passing through the fixed centre. Let us now assume a body in which a point O is fixed, and about this point imagine a sphere to be described with O as a centre and a radius equal to unity. The position of the body will then be completely known when the positions of two points of the body lying on the surface of the sphere are given, for the position of a body is always determined when the positions of three points as O, A, and B not in the same straight line are given. Let us now assume that after the lapse of a certain time the body will have passed from its original position given by A and B into another position given by A_1 and B_1, the new positions respectively of A and B. This new position of the body may be designated by A_1B_1. Then it follows from what has been said that the body can be brought from its original position AB into its new position A_1B_1 by a rotation about a certain axis which may be determined in the manner already mentioned. Let P_1 represent the point in which this axis meets the assumed spherical surface. Then after another interval of time has elapsed, imagine the body to have moved to a third position A_2B_2, an axis OP_2 can again be found, of such a character that a rotation around it will transfer the body from A_1B_1 to A_2B_2. If this procedure be continued for any number of successive positions of the body, we shall get an equal number of axes OP_1, OP_2, OP_3. . . . . which will all pass through the fixed point O and pierce the spherical surface in the points P_1, P_2, P_3. . . . . These axes then form the edges of a certain pyramid, which meet the spherical surface in the spherical polygon $P_1P_2P_3$. . . . . and each one of them is to be regarded as an axis of rotation about which the body must turn in order that the body may be transferred from the position $A_{n-1}B_{n-1}$ to the following one A_nB_n. If we now diminish the distances between the successive positions of the body, till they become infinitely small, the changes of position of the body, instead of taking place by bounds, will take place continuously, as is really the case in the actual motion. The pyramid will then change into a conical surface whose vertex is at the fixed point and which will intersect the spherical surface in a curve $P_1P_2P_3$ . . . into which the spherical polygon passed at the limit. The elements OP of the cone are to be regarded as axes of rotation around which infinitely small rotations must by degrees take place if the actual motion of the body is to be produced.

That element of the cone about which the body must turn at a particular moment is called the *instantaneous axis* for that moment; it evidently corresponds to the instantaneous centre of a system whose points move in parallel planes; and strictly speaking, to an axis which is perpendicular to the parallel planes at the instantaneous centre. The fixed conical surface whose vertex is at O corresponds to the fixed centroid of a plane system which likewise when taken strictly does not mean a curve but a fixed cylindrical surface whose generatrix is the instantaneous axis. Just as the instantaneous axis in a plane system moves along the cylindrical surface or centroid, the instantaneous axis in this case will move along the conical surface $OP_1P_2P_3$. It is evident from this last that the motion of a plane system is only a particular case of a system rotating about a fixed point obtained by supposing the fixed point at an infinite distance.

It will be remembered that in the plane system we had, besides the fixed centroid, another attached to the moved body and rolling on the former. We have likewise in the present case a second conical surface, which is attached to the moved body and partakes of its motion. The vertex of this cone coincides with the fixed point O or vertex of the fixed conical surface, upon which last the moving cone rolls while the system is in motion. Let us also suppose a spherical surface attached to the moved body with centre O and radius equal to unity. It will of course coincide with the *fixed* spherical surface already assumed, and will be pierced at Q_1 at the moment when OP_1 is the instantaneous axis. At the next instant the new instantaneous axis OP_2 will be that element of the conical surface consecutive to OP_1. But the moved body will now no longer touch OP_2 along the straight line OQ_1, but along another, OQ_2, which meets the spherical surface in Q_2. In like manner the body will successively touch the instantaneous axes OP_3, OP_4 . . . of the fixed conical surface by means of an equal number of consecutive straight lines OQ_3, OQ_4. . . . All these straight lines through O now form a conical surface attached to the moved body, which, in a manner analogous to the moving centroid of the plane system, rolls on the fixed conical surface $OP_1P_2P_3$. . . . This surface meets the moving sphere in a spherical curve $Q_1Q_2Q_3$ . . . which touches the other curve $P_1P_2P_3$ . . . in a point (P_1) and rolls on it when the cone OQ rolls on the cone OP. The analogy to the moving centroid of the plane system is unmistakable.

According to the preceding, therefore, every possible motion of a body about a fixed point can be generated by rolling a cone attached to a body, with its vertex at the fixed point, on a fixed cone with the same vertex, the element of contact of the two surfaces being the instantaneous axis of the moment. The motion of a body for every instant will therefore be given as soon as we know the two conical surfaces; in other words, when we know the two spherical curves

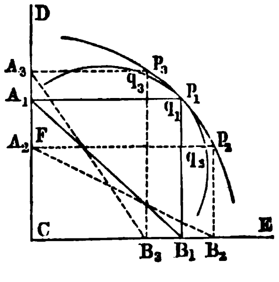
$P_1P_2P_3$ . . . and $Q_1Q_2Q_3$ . . . The same deductions can be made concerning the individual points of the body as were made in § 9 for the plane system, because the centroids are analogous to the cones or spherical curves.

§ 24. **Example.**—The proposition demonstrated in § 22, concerning the parallelogram of rotations holds good for infinitely small rotations; it also holds good for velocities of rotation, since these velocities must be regarded as the infinitely small angles through which the body rotates in the time δt about the corresponding axes. Let us designate by ω_1, ω_2, ω_3 . . . the angular velocities of the moved body at a given instant, and let *OA*, *OB*, *OC* . . . be the corresponding axes, then the axis of the resultant rotation, *i.e.*, the instantaneous axis, and the resultant velocity of rotation ω about it, can be determined according to the parallelopipedon of the velocities of rotation in exactly the same way as rectilinear motions or velocities are combined to form a resultant. For this purpose, it is only necessary to lay off on the axes, beginning at *O*, distances proportional to the velocities of rotation ω_1, ω_2, ω_3 . . . The diagonal of the parallelogram finally obtained will by its direction determine that of the instantaneous axis, and by its length the resultant angular velocity ω. Moreover, as formerly, the velocity v of any point *A* of a body is obtained from $v = r\omega$. The resulting angular velocity ω is therefore known at every instant from $\omega = \frac{v}{r}$, when we know the velocity v of a point *A* and its distance r from the instantaneous axis of the moment. The determination of the latter is accomplished by a construction analogous to that employed for the plane system. It consists in finding the intersection of two planes which pass through the fixed point *O*, and are respectively perpendicular to the simultaneously generated elements of the spherical paths described by two of the points *A*, B_1, *C* . . . From considerations similar to those employed in § 12 for the plane system, we can find for the velocity of change u of the instantaneous axis the relation

$$\frac{\omega}{u} = \frac{1}{\rho_1} - \frac{1}{\rho_2}.$$

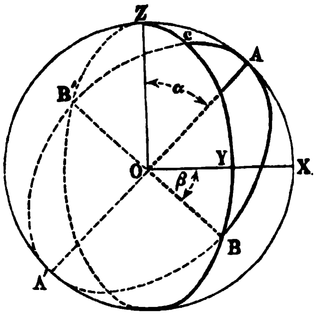
Here ρ_1 and ρ_2 are to be understood as the radii of curvature at the point of contact of two lines of intersection obtained by cutting the two conical surfaces already mentioned, by a plane perpendicular to the instantaneous axis or element of these cones, at a distance from the centre equal to unity.

FIG. 23.



Let us take the form of motion discussed in § 8 and § 13, first modifying it so as to become more general, and then examine it as an example of the rotation of a body about a fixed point. In § 8 and § 13, the two points *A* and *B*, Fig. 23, of a straight line were compelled to remain on two rectilinear guides *CD* and *CE*; in reality there were at *A* and *B* two straight lines perpendicular to the parallel planes of motion, which were compelled to remain in two planes, likewise perpendicular at *CD* and *CE* to the system of planes of motion. We can suppose these three planes—the guided plane *AB*, and the guiding ones *CD* and *CE*—to intersect at a point infinitely distant : from this supposition immediately follows that this kind of motion is only a particular case of the one in which the three planes intersect in a point at a finite distance. As a consequence of this last supposition, the two lines at *A* and *B*, assumed as parallel in § 8, intersect, and we now assume that they intersect at right angles in order that the example may apply to the mechanism known as the universal joint. The kind of motion to be discussed can be described as follows : a body is so moved that two intersecting straight lines *OA* and *OB* within it, Fig. 24, are compelled to remain in the two intersecting fixed planes *OXZ* and *OYZ* respectively, and that the point of intersection of the two lines continually coincides with a particular point of the line of intersection of the two planes. (As will be seen hereafter, in the universal joint, the plane of the two rectilinear straight lines is represented by the so-called cross, and the two guiding planes by planes which pass through the forks and are respectively perpendicular to the two shafts thus joined.)

FIG. 24.



Let us imagine about the fixed point *O*, a sphere to be generated with radius equal to $OA = OB =$ unity, this sphere will be intersected by the two guiding planes, *OXZ* and *OYZ* in the two great circles *ZAX* and *ZYB*, and by the moved plane *AOB* in the great circle *ABA'B'*. When in accordance with the supposed motion, the straight line *OA* coincides with the intersection *OZ* of the two guiding planes, the straight line *OB* will coincide with the diameter *OY* perpendicular to *OZ* because the angle *AOB* is a right angle. If we assume these corresponding positions *OZ* and *OY* as the original positions of *OA* and *OB* from which are estimated the angles of rotation $ZOA = \alpha$ and $YOB = \beta$, α and β representing the amounts that the guided straight lines are rotated in their planes, we shall get from the spherical triangle *OABZ*, when the angle $A(OZ)B$ between the fixed planes is designated by c:

$$\cos AOB = \cos AOZ \,.\, \cos BOZ + \sin AOZ \,.\, \sin BOZ \,.\, \cos A(OZ)B,$$

or,

$$\cos 90^\circ = 0 = \cos\alpha \cos(90^\circ + \beta) + \sin\alpha \sin(90^\circ + \beta) \cos c\,;$$

for this we can write,

$$0 = -\cos\alpha \sin\beta + \sin\alpha \cos\beta \cos c,$$

and after dividing by $\cos\alpha \cos\beta$, we can get

$$\cos c = \frac{\tan\beta}{\tan\alpha}.$$

The angles of rotation of the two straight lines, estimated from the assumed original positions, are therefore of such a nature that the ratio of their tangents is equal to the constant $\cos c$.

In order to find also the ratio of the angular velocities ω_1 and ω_2 of the straight lines OA and OB respectively, make

$$\omega_1 = \frac{\delta\alpha}{\delta t}, \text{ and } \omega_2 = \frac{\delta\beta}{\delta t};$$

consequently

$$\frac{\omega_1}{\omega_2} = \frac{\delta\alpha}{\delta\beta}.$$

To determine the value of this expression we differentiate

$$\cos c \tan \alpha = \tan \beta.$$

We shall then get

$$\cos c \frac{\delta\alpha}{\cos^2 \alpha} = \frac{\delta\beta}{\cos^2 \beta};$$

from which follows

$$\frac{\delta\alpha}{\delta\beta} = \frac{\cos^2 \alpha}{\cos^2 \beta} \frac{1}{\cos c} = \frac{\omega_1}{\omega_2}.$$

Now since

$$\cos^2 \alpha = \frac{1}{1 + \tan^2 \alpha},$$

we can write for this expression:

$$\frac{\omega_2}{\omega_1} = \frac{1 + \tan^2 \alpha}{1 + \tan^2 \beta} \cos c = \frac{1 + \tan^2 \alpha}{1 + \tan^2 \alpha \,.\, \cos^2 c} \cos c$$

$$= \frac{1 + \tan^2 \alpha}{1 + \tan^2 \alpha (1 - \sin^2 c)} \cos c = \frac{\cos c}{1 - \sin^2 \alpha \,.\, \sin^2 c}.$$

This last expression is obtained by substituting in the numerator

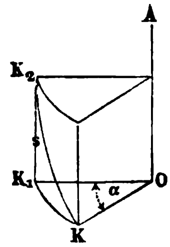
$$\tan^2 \alpha = \sin^2 \alpha (1 + \tan^2 \alpha).$$

§ 25. **General Motion of a Body.**—In the foregoing we have investigated the motion of a body for special cases only; we will now suppose the motion to be of any kind whatever, and will consider it from its most general point of view. The characteristics of this most general form of motion evidently are, that none of its points are fixed, and that no exterior force compels them to remain on a given straight line or in a given surface; in other words, it is perfectly free to move, and may be subjected to any number of wholly arbitrary rotations and translations. However complicated the latter may be it will, under all circumstances still be possible to so unite them that they will form only one translation. This can be easily seen from the following considerations.

Let the shiftings to which a body is subjected during a given time be represented in direction and magnitude by the distances s_1, s_2, s_3 . . . then all the shiftings can be reduced to a single resultant one s by means of the parallelopipedon of motions.

In order also to unite the separate rotations about the axes A_1, A_2, A_3 . . . which are assumed as distributed in any manner in space, we have, according to § 4, only to transfer the axes to any point O, each transfer of an axis being accompanied by a shifting σ whose magnitude is expressed by $\sigma = 2d \sin \frac{\alpha}{2}$, where d is the distance through which an axis is transferred, and α the corresponding angle of rotation. By means of this construction we have substituted for a number of rotations about axes crossing one another in space an equal number of equally large rotations α_1, α_2, α_3 . . . about axes which are parallel to the former and pass through a point O, and an equally large number of shiftings. Now according to § 22 all these rotations can be combined into a single one whose axis OA likewise passes through the point O; all the shiftings σ_1, σ_2, σ_3 . . . can likewise be combined, by the parallelopipedon of motions, into a single shifting σ which can be united with the formerly obtained resultant s so as to form a total resultant S. In this manner the sum of all the motions is reduced to a rotation equal to a about the axis OA, and a shifting represented in direction and magnitude by S. The directions of the axis OA and of S will generally intersect; let γ designate the angle which they form with one another. The resultant shifting S can then be resolved into two components $S \cos \gamma$ and $S \sin \gamma$, of which the former will be parallel to the axis OA and the latter perpendicular to it. For this last shifting $S \sin \gamma$ together with the rotation α about the axis OA we can put, according to § 4, a single rotation of equal amount α about an axis parallel to OA and at a distance from it equal to $d = \dfrac{S \sin \gamma}{2 \sin \frac{\alpha}{2}}$; besides this rotation there now remains only a shifting $S \cos \gamma$ parallel to the axis of rotation. From the foregoing considerations it follows that it is always possible to substitute for the change of place occurring in a body perfectly free to move, in consequence of any rotations and shiftings whatever, a single rotation about an axis OA, Fig. 25, through an angle $KOK_1 = \alpha$ combined with a shifting $K_1K_2 = s$ parallel to that axis.

FIG. 25.



When a point is turned a certain amount about a given axis and at the same time is shifted parallel to this axis a certain distance the motion is said to be *helical;* this can also be expressed by means of the following proposition: *Every change of place to which a freely moving body is subjected, can also be produced by a definite helical motion about a certain axis.* If we assume a succession of infinitely small changes of position, the corresponding helical motions will of course also be infinitely small, and it further follows that the whole motion of a body for a given time can be replaced by *a continuous succession of helical motions.*

In the foregoing a proper combination of any shiftings and rotations whatever, resulted in a helical motion, it is therefore very evident that a reversal of the process would lead to a resolution of the helical motion into any shiftings and rotations. We can now remark more particularly that every *helical motion can always be resolved, in an infinite variety of ways, into two rotations about two axes crossing each other, and conversely.*

If we imagine a body which is subjected to two rotations about axes A and B, which cross each other but do not intersect, it is possible to shift the axis A parallel to itself (see § 4) till it intersects the other axis at a point B; the two rotations can then be combined according to § 22 into a single one about a resultant axis C. The shifting due to the transfer of A is oblique to this axis C because the shifting is perpendicular to A, while C, as the resultant axis of A and B, differs in direction from A. If therefore this shifting be resolved into two components parallel and perpendicular to C, the perpendicular component and the rotation about C can be combined into a rotation about an axis C_1 parallel to C, which can be regarded as the axis of a helical motion whose motion of translation is the parallel component of the shifting just mentioned. Conversely, we can of course resolve each helical motion into two rotations about axes crossing each other, one of which can be assumed as lying in any given direction. Every pair of these axes which cross one another without intersecting are called *conjugate axes.* As was already stated in § 22, the order of the rotations is a matter of indifference so long as the rotations are infinitely small.

The proposition according to which any motion whatever of a system can be reduced to helical motion was discovered by *Chasles*[*] and the axis of the helical motion is also called the *central axis* of the motion.

§ 26. **Axoids.**—If a body K, free to move, successively occupies the different positions K_1, K_2, K_3 . . . we can, according to the preceding, always substitute certain helical motions $\alpha_1 s_1$, $\alpha_2 s_2$, $\alpha_3 s_3$ . . . about certain axes A_1, A_2, A_3 . . . which will bring about the same result of causing the body to successively occupy the positions K_1, K_2, K_3. . . . The body therefore assumes in consequence of these helical motions the same final positions K_1, K_2, K_3 . . . which it would occupy during the actual motions; but an agreement must not be expected for the positions intermediate between K_1 and K_2, K_2 and K_3 . . . when the motions under consideration are of finite magnitude. It is only when the changes of place between the positions K_1, K_2, K_3 . . . are infinitely small that the continuous succession of infinitely small helical motions will be identical with the actual motion of the body. The helical motions succeeding each other in infinitely small intervals of time, generally take place about different axes in space, and it is easy to see that these axes, whose distances from each other must likewise be infinitely small, taken together constitute a certain fixed surface in space.

Let A_1, A_2, A_3 . . . be these axes or the successive positions of the generatrix of this fixed surface A, then in this case, as formerly in the plane system, and in that which rotates about a fixed point, we have a second surface united to the body and always in contact with the first surface during the motion of the body.

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[*] *Bulletin des sciences mathem.*, by Ferusac, 1831. Shell, Theorie der Bewegung und der Kräfte, 1870. Thl. II. Kap. V.

Suppose, for example, that at a given instant t the instantaneous helical axis is at A_1, then a certain straight line B_1 of the moved body will coincide with A_1, about which the rotation α_1 takes place, and simultaneously the body and its straight line shift on the fixed axis A_1 a distance equal to s_1, in consequence of the helical motion. After completing this infinitely small helical motion about the axis A_1 there follows a second rotation about A_2—a position of the generatrix of the fixed surface consecutive to A_1—and another line B_2 of the body coincides with A_2. The body then not only turns about this second axis A_2 an amount equal to α_2, but it shifts with its straight line B_2 on the fixed axis A_2 an amount equal to s_2, whereupon a third straight line B_3 of the body coincides with the following helical axis A_3, etc. All the straight lines B_1, B_2, B_3 . . . which follow one another at infinitely small distances apart, form therefore a given rectilinear surface B united with the body. Any motion whatever of a body can therefore always be understood as the simultaneous rolling and sliding of a moving ruled surface B united with the body and rolling on another ruled fixed surface A.

The results of the investigations hitherto made can now be summarized as follows. *Whatever the nature of the relative motion of one rigid body with reference to another, its equivalent can always be found by firmly attaching to these bodies two ruled surfaces A and B of such a character that they will roll on one another, by continually touching along the instantaneous axis of the moment, the rolling being accompanied by a shifting of these surfaces along the line of contact.* The surfaces containing these instantaneous axes may be called Axoids—a name chosen for them by *Reuleaux.* An hyperboloid may serve as an example of an axoid.

It of course follows that the axoids corresponding to this most general form of motion will also contain those corresponding to certain special motions, like the motion of a plane system, or the rotation of a body around a fixed point, cases which were specially discussed above. The motion of the body might, for example, be such that it could be produced by a simple rolling of the axoids on one another without any sliding along the line of contact. The two axoids must then satisfy the geometrical condition of *development* as when an hyperboloid of revolution rolls on a helical surface. A particular case is when the two bodies have one point O in common, whose relative motion is consequently equal to zero. This condition not only renders relative shifting impossible, but causes the instantaneous axes of both axoids to pass through this point, *i.e.*, the two axoids change into two conical surfaces, whose common vertex is O. This case evidently corresponds to that of a body rotating about a point (§ 23).

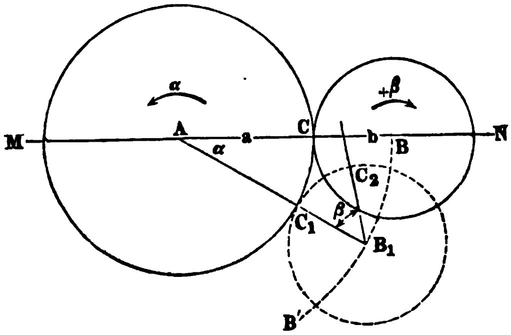
Moreover when the vertex O of these conical surfaces is at an infinite distance, the axoids change into cylindrical surfaces and their intersections by a plane perpendicular to the axis form the centroids characteristic of the plane system.

We must now remark that the axoids A and B of two bodies moved in any manner by rolling and sliding on one another *determine the relative motions of these bodies with reference to one another.* If therefore we wish to determine the absolute motion of a body in space—for example, that having the axoid B—it will be necessary to hold fast the other body whose axoid is A. This can always be attained by imagining an equal and additional motion to be communicated to both bodies so that the relative motion shall remain unchanged. If we suppose the additional motion communicated to the whole system to be always equal and opposite to the motion of the body A, the latter will attain a condition of absolute rest, and the motion of B will then be its absolute motion in space. In the foregoing investigations this case has always been assumed, the body with the axoid A being regarded as fixed while the absolute motion of the body in space was investigated. Accordingly mention was always made of fixed and moving centroids or axoids. It has also been stated in § 10 that the fixed centroid could be made the moving one, and that the moving centroid could be held fast without changing the character of the motion; this property of interchangeability is not confined simply to the centroids or cylindrical axoids of a plane system, but is characteristic of the axoids representing general motion.

Since axoids determine the relative motion of two bodies with reference to one another, the motion of one body will be completely determined by the axoids, whatever the motion given to the other body. This is of great importance for the theory of mechanisms, for although certain parts of a machine always remain at absolute rest, it is nevertheless just as often necessary to compare two parts of which each has its particular motion. An example occurring very often in practice will now be adduced in illustration.

§ 27. **Example.**—Two parallel axles or shafts A and B, Fig. 26, are to be so connected with one another (either by two toothed wheels or two friction wheels) that both will turn with an invariable angular velocity respectively equal to $-\alpha$ and $+\beta$, and in opposite directions as shown by the arrows at α and β. In order to determine the as yet unknown instantaneous centre at any moment, let first of all an additional motion $+\alpha$ be imparted to both shafts, *i.e.*, a rotation about the axis A of a magnitude equal to α, and therefore equal to the motion already possessed by the shaft, but opposite in direction; we shall then evidently have the same relative motion of the two axes as before. Each of the two shafts will then be subjected to two rotations, the shaft A to a rotation $-\alpha$ and another $+\alpha$ about its own axis, in consequence of which A will come to rest. The shaft B likewise receives a rotation $+\alpha$ about the axis A and a rotation $+\beta$ about the axis B. These infinitely small rotations united according to § 5 give a resultant rotation $\alpha + \beta$ about an axis C perpendicular to the connecting line AB, at such distances a and b from A and B respectively, that

FIG. 26



$$a\alpha = b\beta \text{ or } a : b = \beta : \alpha.$$

Since the motion is to be constant, we immediately see that the axoids in the present case are normal cylindrical surfaces described about A and B by AC and BC respectively. When the two shafts are connected by friction wheels the axoids are realized practically in the cylindrical friction wheels used, and when toothed wheels are employed the axoids are represented by the surfaces of the so-called pitch cylinders of these wheels. A circular motion of the axle B, causing it at the same time to rotate about itself and to be carried around the axis A in the circle BB_1 like a planet, corresponds to the supposition just made that the shaft A is wholly at rest. Such motions actually occur in practice, for example, in the turning arrangements of cranes. The total amount of rotation of the axis B for an element of time of course equals in magnitude $\alpha + \beta$. To show this clearly, imagine a straight line MN fixed in space and coinciding with the line of connection AB, which at the same time is assumed as the initial position of the motion, and let us also imagine a hand or index BC, lying in the direction BA, to be rigidly attached to B. Then if the axis B has swung through the infinitely small angle $BAB_1 = \alpha$, and the instantaneous centre has passed from C to C_1, the index will be at B_1C_2, which deviates from the direction B_1C_1 by an angle $C_1B_1C_2 = \beta$. This index therefore makes with its original position—the fixed direction MN in space—an angle equal to $\alpha + \beta$.

If now the whole system—the shaft A as well as that of B—receives an additional rotation $- \alpha$ about A, there will remain for the shaft B only the rotation β about its own axis, and we shall then have our original case, in which it was assumed that the axes should rotate in opposite directions to an extent represented by $- \alpha$ and $+ \beta$ respectively; this case often occurs in practice where there are fixed parallel shafts. Since relative motion can be expressed as a difference, we have for the relative motion of the shaft B to A the expression $\beta - (- \alpha) = \beta + \alpha$, while the shaft A has to B the relative rotation:

$$- \alpha - (+ \beta) = -(\alpha + \beta).$$

The relative rotation of one shaft to another is in this, as in every case, equal to $\alpha + \beta$.

If we now add to the whole system another rotation $- \beta$ about B, the shaft B will come to rest and the shaft A will be carried around in a circle passing through A and concentric with B, to an extent represented by $- \beta$, at the same time that it is undergoing a rotation $- \alpha$ around its own axis. The whole rotation of the shaft A now likewise amounts to $\alpha + \beta$, but is opposed in direction to that communicated to the shaft B when A was at rest.

In the foregoing, three different forms of motion have been found for the example before us, all of which occur in practice. Kinematically speaking, these motions are exactly alike, because the same pair of centroids or axoids answers for each. The whole difference in these cases consists simply in whether the motion is communicated only to one or the other of the centroids, or to both at once. The relative motion, as was shown, is always expressed by the same amount, $\alpha + \beta$.

But it immediately becomes evident that the variety of the different forms of motion of which the mechanism under consideration is capable, may be infinite. For if we imagine that in each interval of time δt in which the shaft B receives a rotation $+\beta$ about itself the shaft *A* receives a rotation not equal to $-\alpha$ but to $-\alpha_1$, it follows that the shaft *B* must have such a motion β_1 that the relative motion of *B* to *A* preserves the constant value $\alpha+\beta$. β_1 can therefore be obtained from equation:

$$\beta_1 - (-\alpha_1) = \alpha + \beta, \text{ or } \beta_1 = \alpha + \beta - \alpha_1,$$

for which we must put $\beta_1 = \alpha + \beta + \alpha_1$ when a right-handed rotation equal to $+\alpha_1$ is communicated to the shaft *A*.

Since α_1 can be chosen arbitrarily, it follows that the motion of the simple mechanism under consideration can be infinitely varied, but that all its different forms of motion are alike in having the same pair of axoids. The variations of motions last considered often occur in practice in the so-called *differential mechanisms*, which will be specially discussed hereafter. From what has been said it should now be easy to see that an examination of motions by the help of their axoids will greatly aid in making clear and plain many of the difficult problems of mechanism.

§ 28. **Pairs of Elements.**—The laws established in the previous articles are perfectly general for the motion of rigid bodies. But since we are only dealing with mechanisms, we will apply to them only the propositions that have been deduced. At the very beginning of this introduction attention was called to the difference, as regards motion, existing between a body free to move—for example, a stone thrown into the air—and one of the moving pieces of a machine. While the path of a freely moving body is wholly the result of the forces acting upon it from without (these being in the case of a stone, the original impulse, the force of gravity, and the resistance of the air), the path of a moving piece of a machine is independent of these forces. In this latter case there are not only external forces, but, in a certain sense, internal forces, which compel the body to take a determinate motion, by preventing it from assuming any other motion which it may be striving to take in consequence of the action of an external force. These internal forces depend on the capacity for resistance of the materials which surround the body under consideration. These forces of resistance are only called out by the action of external forces, and disappear with them. They have therefore been appropriately designated as *latent* forces in contradistinction to external or *sensible* forces.

The stone thrown into the air, for example, moves in a certain approximately parabolic path under the influence of the forces acting upon it, and every accidentally and externally applied force, as a lateral gust of air, will deviate it from the path which it would have described had this last force not acted. On the other hand, a moving piece of a machine, a shaft for example, is prevented by its close-fitting bearing from assuming any other motion than a rotation about its own axis. An external force which strives to shift the shaft in any direction whatever will cause indeed a pressure of the shaft against the bearing, but will not succeed in producing any real shifting, because the bearing or support will immediately react with a force of resistance equal and opposite to this pressure. It follows from this that the elements of a machine must continually occur *in pairs;* in the case just mentioned the journal and its bearing constitute such a pair. The kind of motion which one or the other parts of such a pair can assume is wholly a question of the form of these parts. If, for example, a journal has a cylindrical form terminated by raised parts or collars, it will, on applying an appropriate external force, be compelled to rotate about its axis, because the properly finished bearing cannot offer any resistance (neglecting friction) to rotary motion. But if the journal were changed to a prismatic form it could be shifted in the direction of its axis, and its properly finished prismatic bearing would oppose rotation.

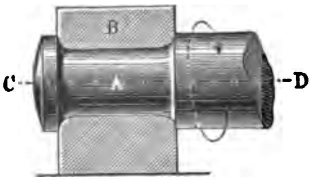
With regard to the above *Reuleaux* defines a machine as *a combination of resistant bodies so arranged that by their means the mechanical forces of nature can be compelled to do work accompanied by certain determinate motions.*

The study of the arrangement of mechanisms will accordingly resolve itself into determining the forms to be given to the separate machine parts in order that a determinate prescribed motion may be realized. Now in the above it was quite fully explained how all motions, be they ever so complicated, could be reduced to certain elementary ones, namely to rotations and shiftings, or if we so prefer, to a helical motion, which is to be regarded as the most general form, since simple rotation and shifting are obtainable from it by respectively causing the shifting and rotation of the helical motion to disappear. Now we have, answering to these three elementary motions out of which all others may be composed, exactly the same number of fundamental forms for the corresponding machine parts, or more properly, pairs of machine parts. We will employ the expression, *pairs of elements*, chosen by Reuleaux* for the corresponding parts.

These pairs of elements are:

1. The *turning pair*, Fig. 27, consists of a solid of revolution *A* and its corresponding envelope *B*, whose meridian line is so formed that all motion in the direction of the axis *CD* is prevented. The relative motion of these two parts to one another is consequently limited to a rotation about the geometrical axis *CD*. If, in accordance with what has been said, we wish to characterize or distinguish this motion by its axoids, we shall immediately find that both axoids in this case reduce to a straight line, namely the geometrical axis *CD*, since every plane perpendicular to an element of the path passes through this axis *CD*. This pair of elements in the form of journal and bearing has a most extended application in machinery. The fundamental form is then usually cylindrical, more rarely conical, but other forms of revolution are not excluded; particularly in pivots we often have a flat surface which can be regarded as a surface of revolution generated by a straight line perpendicular to the axis.

FIG. 27.



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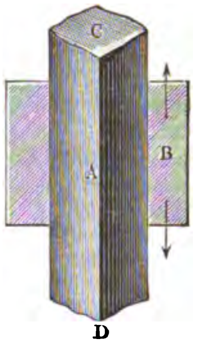
* In the brief explanations here given of the principles of Kinematics and in their applications hereafter, the method followed by Reuleaux—the founder of the real Science of Mechanisms—is adopted.

2. The *sliding pair* consists of a solid straight prism *A*, Fig. 28, and a hollow prism enclosing it, whose cross-section may be any plane figure whatever with the single exception of the circle, because that would not prevent rotation. If, in spite of this, practice often chooses a circular section for the sliding pair, as is the case for example in piston rods and stuffing boxes, it is because accurate cylinders are readily made, and other means are employed for preventing rotation. The relative motion of the two parts of this pair consists in a shifting in the direction of the axis of the prism *CD*. If we conceive such a shifting as a rotation about an infinitely distant axis, we can designate the infinitely distant straight line lying in the plane of a cross-section, as the constant instantaneous axis into which the two axoids have been reduced.

FIG. 28.



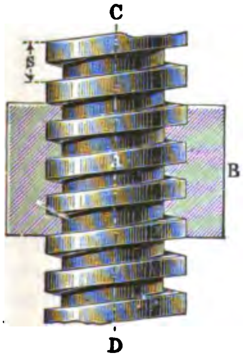
3. The *twisting pair* consists of a cylindrical screw *A* and its nut *B*, Fig. 29. The amount of pitch *s* of this screw and the radius of its cross-section is a matter of indifference, for in order that there may be motion when one part encloses the other, it is only necessary that the intersection of the helical surface by a circular cylinder concentric with the axis of the screw be a helix of constant pitch, and that all the helices thus obtained have the same pitch *s*. If *r* is the radius of such a cylinder, we have for the angle of inclination of the helix lying upon it, the equation:

$$\tan \alpha = \frac{s}{2 r \pi},$$

from which follows that for the same screw the angle of inclination α is smaller according as the distance *r* is greater.

The relative motion of the two parts to each other consists in a rotation about the geometrical axis *CD* and a simultaneous shifting in the direction of *CD* to such an extent that the ratio of the angle of rotation to shifting will always remain constant. In this case also the two axoids have shrunk into the same straight line, namely the axis of the screw, and we can imagine that this straight line is rolling on itself while it is at the same time shifting itself in the direction of its length.

FIG. 29.

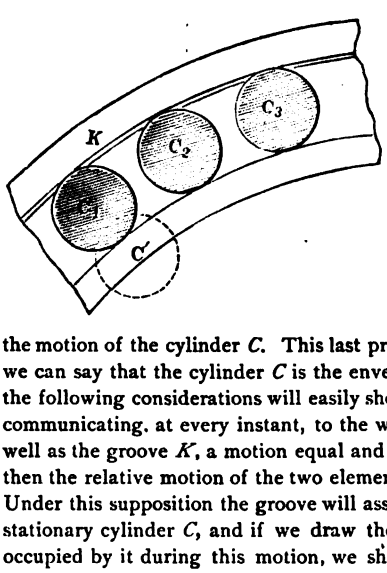


In all of these pairs of elements it is immaterial which of the two parts receives motion, and according to § 27 both parts may move at the same time; thus in the twisting pair the screw may turn and the nut have the shifting or sliding motion, or the reverse may take place. These latter cases occur perhaps oftener in practice than those in which one part is wholly at rest while the other executes the whole motion.

§ 29. **Higher Pairs of Elements.**—The pairs of elements considered in the preceding paragraphs possess the peculiarity, that one element is always completely enclosed by its companion element, because both elements have the same form, one of them representing the hollow body and the other the solid. We will therefore call these pairs *closed pairs*.

On the other hand, there are many pairs of elements in machinery where such a complete enclosing or contact of all points of the surface does not occur, but where contact always takes place only at a few points. But these pairs of bodies must be considered as real pairs of elements, because they satisfy the necessary condition that each of the two bodies assumes a perfectly determinate motion in consequence of the form and the capacity for resistance of the other body which exclude all other motions.

FIG. 30.



Let, for example, C_1, C_2, C_3 . . ., Fig. 30, be different positions which an ordinary cylinder must successively occupy in consequence of a determinate motion which it is to receive, rotations about its own axis not being excluded. Then a channel or groove-shaped piece K can be designed, of such a construction, that it will permit the cylinder C to assume only the motion prescribed, and will exclude every other motion, for instance in the direction C_1C' or in the direction of the axis of the cylinder. These two pieces C and K, according to the above definition, constitute a pair of elements, K containing the *envelope* for the motion of the cylinder C. This last property is moreover a reciprocal one, and we can say that the cylinder C is the envelope of the groove K; that this is true the following considerations will easily show. Imagine the cylinder set at rest, by communicating, at every instant, to the whole system, *i.e.*, to the cylinder C as well as the groove K, a motion equal and opposite to that which the cylinder has, then the relative motion of the two elements to each other will remain unchanged. Under this supposition the groove will assume a certain motion with regard to the stationary cylinder C, and if we draw the groove in all the successive positions occupied by it during this motion, we shall find that all these different positions of the curve of the groove will touch the circle C in its fixed positions ; in other words the circle C is also the envelope of the groove K.

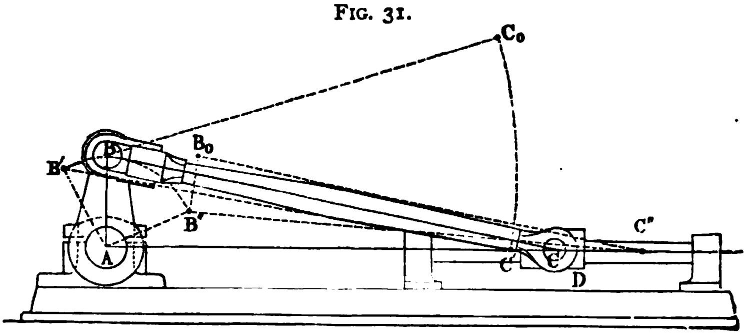
From this follows that besides the three simple pairs of elements discussed in the previous paragraphs, and characterized as *closed pairs*, there must be still a great number of pairs occurring in practice, whose elements possess the property of mutual envelopment. Since the motions which can thus be obtained offer a much greater variety than the simple motions corresponding to the closed pairs, *Reuleaux* has chosen the name "higher pairs of elements," for the bodies that possess forms which are reciprocally envelopes of one another ; in contradistinction to these elements the closed pairs are designated as "lower pairs."

As was already intimated, the number and variety of the higher pairs are very great, an enumeration of them all would be both unnecessary and impracticable ;

in the following investigations therefore only those representatives principally occurring in practice will be discussed.

§ 30. **Kinematic Chains.**—All mechanisms are composed of the pairs of elements discussed in § 28 and § 29, however complicated the motions may be which directly or indirectly are obtained by their means. The method of combination is always very simple, and consists solely in uniting an element A_1 of a pair A with an element B_1 of another pair B so as to form a rigid body, the second element B_2 of this pair B is then connected with an element C_1 of a third pair C, etc. This method of uniting different elements and thereby forming mechanisms is best illustrated by means of an example. For this purpose let us choose the crank and connecting rod, a mechanism of great importance to practice, Fig. 31, and composed of three turning or cylindrical pairs A, B, C, and a sliding pair D. Let A_1, B_1 and C_1 represent the solid elements or cylindrical journals, and A_2, B_2 and C_2 the hollow elements or the bearings belonging to the journals, and in the same manner let D_1 represent the solid or full prism, and D_2 the prismatic socket which acts as slide. We can combine these four pairs of elements in different ways and always so that an element of one pair united to an element of another pair will form a rigid body. We will suppose the combination usual with cranks to obtain here. Accordingly we unite the axis or shaft A_1 with the crank pin B_1 by means of a body called the *crank*, and which may be represented by A_1B_1. In like manner the crank-pin bearing B_2 is united to the bearing C_2, so as to form a rigid *connecting rod.* The journal or pin C_1 of the third cylindrical pair is now connected with the hollow socket D_2 forming the piece C_1D_2, which is ordinarily called the *cross-head;* finally the solid prism D_1 is united to the bearing or support of the shaft by a rigid frame or *bed-plate* A_2D_1. When combined after this fashion, the separate elements are united to each other in a manner similar to the links of a continuous chain, and *Reuleaux* has therefore given to such a combination of pairs of elements the name of *kinematic chain,* and to each of the bodies formed by the union of two elements the name of *link* of a kinematic chain. In the present example there are four links, namely, crank, connecting-rod, cross-head, and bed-plate; it is evident that by following the method of combination above mentioned. we shall get as many links as there are pairs in the combination. But this latter property is not the result of a law that is universally true, since it

FIG. 31.

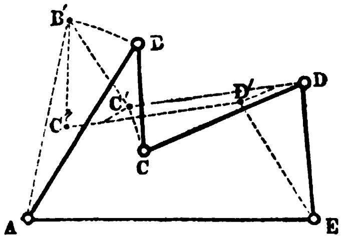


is very easy to conceive, as will be shown hereafter, of kinematic chains in which a link unites more than two elements of different pairs. That the four pairs under consideration can be combined in a different manner will be easily shown by uniting, for example, the shaft A_1 with the bearing B_2 and the crank-pin B_1 with the bearing C_2 as well as with the journal or cross-head pin C_1, etc. ; such arrangements do occur in practice, though less often than that assumed above.

In whatever manner the combination be effected the separate pairs will still in every case retain the general character of the relative motion peculiar to elements of the pairs, since the two elements of a pair are never rigidly connected with one another. The journal of a cylindrical pair, even after combination, retains the property of rotating relatively to its bearing, and the slide (corresponding to the hollow prism), after combination as well as before, admits of a shifting relative to the solid prism in the direction of the axis of the latter. Now although a combination does not change the nature of the motion peculiar to the pairs, it does limit its *amount*, or *extent*, by making the motion of one pair in a certain measure dependent upon its neighboring pairs. It is clear, for example, that in the present case, before combination, the cylindrical pair C permits a relative motion of the two elements C_1 and C_2 to the extent of a complete revolution, yet after the parts are united to form the crank mechanism, the rotation can only take place through a definite angle. In like manner it is evident that the shifting of the slide D_2 on the prism D_1 in the crank mechanism is limited to a perfectly definite amount, twice the length of the crank A_1B_1, while in the sliding or prismatic pair the shifting is limited only by the length of the prism.

If, in a combination of a pair of elements, as for example that represented by Fig. 31, we communicate to any link, as the crank A_1B_1, motion relative to a neighboring link A_2D_1, the remaining links will generally be induced to move also. We must now distinguish two cases according as the motion of one link relatively to its neighbor does or does not cause *perfectly determinate* motions in all the remaining links. The former is evidently the case for the crank mechanism here considered, for if we communicate, for example, to the crank AB, any motion from AB to AB' the corresponding sliding of C to C' is perfectly determined, since the data for drawing the triangle ABC' are known. But in this respect we have a different state of things in the combination sketched in Fig. 32. This has five cylindrical pairs $ABCDE$, and if we impart to the link AB a motion relative to AE, say from AB to AB', the positions of C and D will still be completely undetermined, since the data—five sides and an angle EAB'—given for determining the pentagon $ABCDE$, are insufficient. The pair C, for example, is as likely to be found at C'' as at C', and D is as likely to remain in its original position D as it is to move to D'. Since it is necessary in the construction of mechanisms that the separate parts shall have perfectly determinate motions, we see that only those kinematic chains can be considered in which the motion of one link relatively to its neighbor causes perfectly determinate motions of the

FIG. 32.

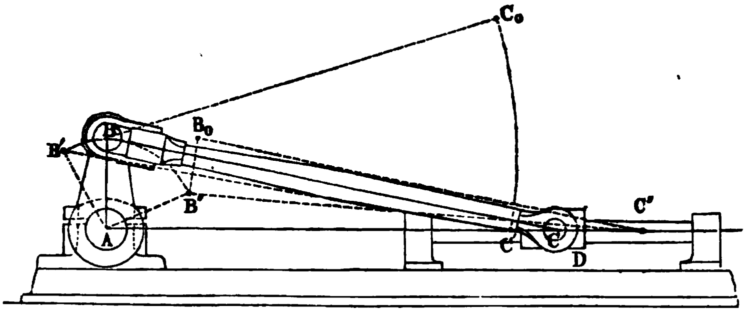


remaining links. Such a combination of links is called a *constrained closed* chain, or simply *closed* chain.

If in a constrained kinematic chain, *i.e.*, a combination of pairs of elements like that discussed above, Fig. 31, we hold fast a certain link, the frame A_2D_1 for example, the relative motions of the remaining links, *viz.*, crank A_1B_1, connecting rod, etc., will become absolute motions in the space connected with the fixed link. Such a closed kinematic chain in which one link is fixed is called by *Reuleaux* a *mechanism*, and it becomes a machine, in accordance with the above definition, when an external force so acts on one of its links that it is obliged to move.

§ 31. **Formation of Mechanisms from a Kinematic Chain.**—If we imagine the link A_2D_1 of the kinematic chain, Fig. 33, fixed, we obtain the known

FIG. 33.



and much-used crank mechanism. As regards the motion of the separate links it is first of all clear that the absolute motion of the links adjacent to the fixed one A_2D_1, is prescribed by those pairs connecting the fixed link with the adjacent ones. Thus, for example, the crank A_1B_1 can only rotate about the axis A_1, and the two centroids which determine the relative motion of the crank A_1B_1 to the bed-plate A_2D_1 are in this case reduced to a single point—the centre of the shaft A_1. In like manner we recognize that the motion of the cross-head C_1D_2 can only shift or move in a straight line as is required by the sliding or prismatic pair D connecting the cross-head C_1D_2 with the fixed link A_2D_1. The motion of a link like the connecting rod B_2C_2, which is not directly united to the fixed link but is connected with the latter by means of intermediate links, is of a more composite nature, for the motion of B_2C_2 depends upon the nature of the pairs B and C connecting it with the adjacent links as well as upon the pairs A and D which connect the intermediate links with the fixed one. In the present case, therefore, the motion of B_2C_2 depends upon all of the four pairs. If we wish to determine the absolute motion of such a link B_2C_2, *i.e.*, its relative motion to the fixed one A_2D_1, it will only be necessary to draw in the manner indicated in § 8 the centroids for the motion. By so doing two centroids will be obtained one P_1 fixed and connected with the link A_2D_1, and a movable one P_2

attached to the connecting rod B_2C_2, the latter centroid is to be regarded as rolling on the former.

Now although the motion of the connecting rod B_2C_2 is dependent on the nature of all the pairs A, B, C, and D, it nevertheless can be shown that this motion may be composed of only two motions, one of which will correspond to the pair uniting the rod with its neighbor, while the other motion belongs to the pair which joins this neighbor with the fixed link. We can imagine the motion of the rod BC to be a combination of a rotation about B with a rotation about A, or on the other hand we can regard it as a rotation about C combined with a sliding along the prism D.

In order to recognize this, we have only to imagine an additional motion communicated to the whole system which shall be equal and opposite to the rotation of the crank AB about A; the relative motion of the various links will remain unchanged. There will result, however, a different arrangement of the mechanism, for the crank AB will now be at rest and the formerly fixed bed-plate AD will have commenced to rotate about A. The absolute motion of the other two links BC and CD is of course also changed by the communication of an additional motion, the rod BC having now only a motion of rotation about B. But this rod, when AD was the fixed link, had a motion which corresponded to the rolling of the centroid P_2 on the fixed centroid P_1, and this latter motion was reduced to a rotation about B by adding a rotation about A; from this it follows that the absolute motion of the rod BC when AD is fixed, must be composed of two rotations about A and B. This may also be illustrated by means of the figure. If we imagine the crank moved out of its position BA into another position AB', C will move to C' and the rod BC will then be in the position $B'C'$. But the rod BC will also be transferred to the same position by rotating it first about A till it comes into the position $B'C_0$, the rod during this rotation being supposed to be rigidly attached to the crank at B and consequently moving through the same angle; the rod can now be brought to the position $B'C'$ by rotating it about B.

If, on the other hand, we communicate to the whole system in which the bed-plate AD is again supposed to be fixed, an additional motion equal and opposite to the sliding of the cross-head CD, the cross-head will be brought to rest while the frame AD will assume the sliding motion. The motion of the connecting rod BC is now reduced to a rotation about C, and in the same manner as above we can infer that the motion of the connecting rod when the frame is held fast, is composed of a rotation about C and a sliding along the prism D. This peculiarity or property of the motion of BC can also be perceived from the figure. For this purpose let the crank AB be moved to AB'', C will move to C'', and therefore BC to its new position $B''C''$. It will then be seen that the rod can also be moved from BC to $B''C''$ by sliding an amount equal to CC'', bringing the rod to B_0C'' and then rotating about C'' till B_0 coincides with B''.

We have therefore proved the assertion made above, that the motion of any link (BC) is composed of two motions of which the first is like that allowed by one of the pairs (B or C) connected by this link, and the second like that which this pair can assume by virtue of its connection (A or D) with the fixed link.

If we consider one of the cases already discussed, for example that in which the crank is stationary because of an additional rotation imparted to the whole system, we will have a mechanism in which the links BC and AD, adjacent to the crank, have only motions of rotations about B and A respectively, their centroids reducing themselves to these axes. The fourth link CD, however, has a compound motion, which according to the above law may be regarded as composed of one rotation about C and another about B, or of a sliding along D and a rotation about A. This motion, like that of the connecting rod, may also be conceived as a rolling motion by bringing in centroids. By means of the construction given in § 8 we get two centroids, one P_3 connected with the fixed crank, the other P_4 united to the cross-head and rolling on the first.

If, on the other hand, we wish to bring the cross-head CD to rest, we start with the original supposition that the frame AD is held fast, and then suppose the whole system to be subjected to a sliding parallel to the prism D. After constructing the centroids representing the motion of the crank with respect to the cross-head we should find that we had the very same curves P_3 and P_4 as when the crank was held fast, only the centroid P_4 belonging to the cross-head would be regarded as fixed, while P_3 would be movable. A perfectly analogous relation exists between the two opposite links AD and BC; if the connecting rod BC be made stationary, the centroids representing the motion of the frame AD with respect to the connecting rod will be the very same centroids P_1 and P_2 that were found when the frame AD was fixed, the only difference being that the centroid P_2 attached to the connecting rod is now the fixed curve and P_1 the moving one, while the reverse obtained when AD was fixed.

The foregoing discussions have shown how it is possible to derive from the same kinematic chain as many mechanisms as there are links, by successively fixing or holding fast the various links of the chain. Different as the motions of these mechanisms frequently seem, they all have for their basis the same law of relative motions which is expressed by the same centroids. This law of the formation of mechanisms by *fastening various members of a kinematic chain* is of the greatest importance in machinery, and was first discovered by *Reuleaux*.

Literature.—The introduction of kinematics as the special science of motion, without considering the forces causing the motion, is due to *Ampère*, see his "Essai sur la Philosophie des Sciences," 1830. The instantaneous centre was first recognized by Descartes (1724) in generating cycloids, and was found by *Johann Bernouilli* for the general motion of a plane system (1742). Of greater importance to theoretical kinematics are the papers of *Chasles* (Comptes rend. de l'Académie T. XVI, 1843, and T. LI, 1860; see also his "Aperçu Historique," translated into German by *Sohnke*, Halle, 1839), also *Poinsot's* "Théorie de la rotation des corps" and "Théorie des cônes circulaires." The following works treat the subject of Kinematics very fully. *Laboulaye*, "Cinématique," 1849; *Girault*, "Elémens de géometrie appl. à la transformation du mouvement," 1858; *Resal*, "Traité de Cinématique pure," 1862; *Belanger*, "Traité de Cinématique pure," 1864; and *Schell*, "Theorie der Bewegung und Kräfte," 1870. For the work of *Bresse*, who first discovered the circles called by his name and mentioned in § 15, see "Journal de l'école polytechnique, T. XX," 1853. The works of

Chelini, *Gilbert*, *Aronhold*, and others are also of importance. *Willis's* "Principles of Mechanism," 1841, is a valuable work full of practical applications; *Giulio's* "Cinematica applicata," also deserves special mention. *Reuleaux's* "Theoretishe Kinematik," 1875, has already been frequently referred to; there is also an excellent English translation of this latter work "Kinematics of Machinery" by *Prof. A. B. W. Kennedy.*

MECHANICS
OF
MACHINERY OF TRANSMISSION.

CHAPTER I.

JOURNALS, SHAFTING, COUPLINGS, AND BEARINGS.

§ 1. **Journals.**—A *journal*, in machinery, is a rigid body bounded by a surface of revolution about whose axis the machine-part attached to the journal may turn. Journals are generally of a cylindrical form, though other forms of revolution do occur, such as cones, spherical surfaces and flat surfaces or pivots; these last are often employed for upright shafts.

In order that the journal and the machine-part to which it is fastened may rotate about the axis, to the exclusion of every other motion, it is necessary, according to § 28 of the Introduction, that another piece be employed, of such a hollow form that it will enclose the journal. To this piece the name *bearing* is given. The journal and its bearing therefore constitute the pair of elements designated above as the turning pair; the hollow interior of the bearing must consequently coincide, either wholly or partially, with the exterior of the journal. In many cases the journal's cylindrical surface is completely enclosed by the bearing, as in Fig. 34, but it frequently happens that there is contact at only a few parts of the surface of the journal, as in Fig. 35. The restraining sur-

FIG. 34.

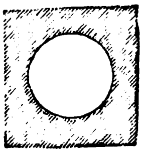
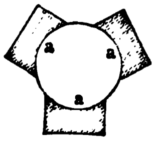
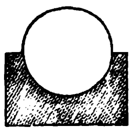


FIG. 35.



faces *aaa* must generally, however, have such an arrangement that the position of the geometrical axis will remain fixed; this requires that a cross section of the journal be restrained at least at three points, and that any two adjacent directions of restraint include an angle of less than 180°. Only in those cases where external forces are either not expected to occur in certain directions or are more than neutralized by other external forces, will a restraint or support suffice which does not fulfill the above conditions. Thus, for example, it is generally sufficient to support the journals of heavy water wheels by means of half cylindrical bearings, as in Fig. 36, since the great weight of the wheel is generally sufficient to prevent the journal from obeying any external influences which tend to raise it.

FIG. 36.



The length of the journal, *i.e.*, the dimension in the direction of the axis, may be made of any desired magnitude if we consider only the possibility of rotation and leave out of account other practical considerations, for during the turning of a body every point in the axis of rotation remains at rest. We can, therefore, select any portions of the solid of revolution surrounding the axis *A*, and form them into separate journals by giving to each portion a bearing. This is done in all those cases in which considerations of strength, etc., require that the rotating piece shall be supported at different points. It is of course a necessary condition of all such constructions that all the journals belonging to a given machine-piece must be solids of revolution having the *same* geometrical axis, otherwise the body would not revolve at all or would only do so accompanied by bends and other strains in its different parts.

§ 2. **Axles, Shafts, and Shafting.**[*]—The machine-piece which has a motion of rotation relatively to its supports and is provided with journals is called an axle or shaft. Owing to the frequency with which rotary motions occur in machines, shafts with their journals and bearings are very important parts of machinery and fully justify a closer examination. Whenever the rotation of a shaft or a journal is spoken of, the relative motion of this piece to its support must be understood; for although the bearing is generally fixed while the journal revolves, the character of the motion is not changed when the journal or shaft is fixed and the bearing revolves, or when, as is often the case with jointed rods, both journal and bearing revolve.

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[*] [These terms are used somewhat indiscriminately, but we may define an axle as a rotating bar subject principally to bending stress, a shaft as a rotating bar which is subject principally to twisting stress, and shafting as a line of connected shafts. When the axles or shafts are small, the terms *arbor* and *spindle* are often substituted. As shafts are often subject to considerable bending as well as twisting stress, we can properly regard the term shaft as the more general one.—TRANS.]

An *axle* serves to support machine-pieces which rotate; for example, we speak of the axles of pulleys used in rope transmissions and of the axle of a water-wheel. When the axle is horizontal, the weight of the parts (pulleys, water-wheels, etc.) which it carries and such external forces as the tension of the rope and the weight of the water all tend to break the axle, and its dimensions must therefore be determined in accordance with those principles of the Strength of Materials which relate to flexure. It often happens, as in knuckle joints, that the weight of the carried parts is insignificant in comparison with the external forces acting on them; on the other hand, the weight of the carried parts is sometimes greatly in excess, as in railway axles. When the shaft is vertical the weight of the carried pieces tends to crush and bend the shaft; this is likewise true of the vertical components of the external forces that act obliquely on the shaft.

The journals which sustain the pressures acting in the direction of the axis of an upright shaft are called *pivots.* The journals belonging to horizontal and inclined shafts and axles are sometimes called gudgeons, but more often journals; we will therefore hereafter confine the term journals to horizontal and inclined shafts and axles.

Shafting is employed in shops for transmitting rotary motion from the prime movers to the operators or special machinery of the shop. On account of their great length they must be supported by bearings at a greater number of points than is necessary for ordinary axles and shafts, for which, as a rule, two supports or journals are sufficient. If we suppose a line of shafting to receive power at a point (A) and to give it off to some machine at another point (B), the torsion, due to the twisting moment transmitted, will be felt by every part of the shaft situated between (A) and (B). All the journals situated between these points, at which the forces enter and leave, are subjected to torsion, while the journals which lie outside these points (A) and (B) are not exposed to this twisting strain, but, like the journals of axles, are strained only or principally by pressures and weights acting transversely. These latter journals may be called *end-journals* in contradistinction to the *neck-journals* which lie between the points at which the force is received and given off.

Strictly speaking the journals of a line of shafting and the shafting itself are subjected not only to twisting stresses, but to bending stresses by the weights of the rotary pieces, by the pressures of gear wheels, by the tension of belts, etc.; but as a rule the bending moment is small in comparison with the twisting moment, and may be neglected. In like manner all axles and the end-journals of shafts are subjected not only to bending stresses but also to a slight twisting stress due to the frictional resistances on the journal. This resistance is, however, almost always insignificant in comparison with the bending stress. Nevertheless cases may occur in which both the bending and twisting stresses are considerable, in which cases the dimensions must be calculated according to the rules established for compound stress. Such a case occurs, for example, in the axle of a water-wheel, which is subjected between its two bearings to the heavy load due to the weight of the wheel, and which transmits its energy to a toothed wheel fastened to the axle, the latter having been prolonged beyond one of the bearings for this purpose. The neck-journal between the water-wheel and toothed wheel is subjected to a considerable twisting and bending stress, while the other or end-journal of the axle is only subjected to a considerable bending stress. It is evident that when the construction is such that the toothed wheel is directly geared to the crown of the water-wheel, the twisting moment on the axle will only be the insignificant amount due to friction on the journals. Such constructions are, therefore, often chosen for hoisting drums as well as for water-wheels, in order that the shaft or axle may not be subjected to torsion.

§ **3. The Dimensions of End-journals.**—The form of these journals is usually cylindrical, motion in the direction of the axis being prevented by shoulders on one side as at B, Fig. 37, or on both sides as at A and B, Fig. 38. The diameter d of such a journal is determined from the load P to which it is subjected, by means of the well known formula for the strength of a beam fixed at one end: $M = k\,\frac{W}{e}$ (see I. § 235). M is here the maximum bending moment given for the point B by

FIG. 37.

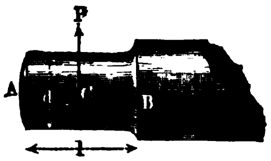
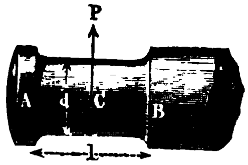


FIG. 38.



$$M = P\frac{l}{2},$$

the pressure P being supposed to act at the middle of C of the journal. W is the moment of inertia of the cross-section of rupture and k the greatest permissible stress occurring in the fibre which is subjected to the maximum strain, and whose distance from the neutral layer is designated by e. Since for a circular cross-section

$$\frac{W}{e} = \frac{\pi}{32}d^3.$$

we have here

$$M = P\frac{l}{2} = \frac{\pi}{32}d^3k.$$

If we put

$$\frac{l}{d} = \lambda$$

we get,

$$P = \frac{2\pi k}{32\lambda} d^2,$$

or,

$$d = 2.26 \sqrt{P \frac{\lambda}{k}}.$$

For a given pressure P, therefore, the diameter of the journal will be the greater the smaller the permissible stress k of the material and the greater the ratio $\lambda = \frac{l}{d}$. The diameter of the journal is directly related to the mechanical work consumed by the friction of the journal, for although the *amount* of this friction μP is nearly independent of the journal's diameter d, the *work* of friction per revolution $\mu P. d\pi$ increases with and is directly proportional to the diameter of the journal. In order therefore that this resistance be reduced as much as possible, the diameter of the journal is to be made as small as is consistent with durability. From the above formula for d it follows that, in order that d be as small as possible, it is necessary to take k as large as possible. For this reason the journals are made of the strongest materials, usually either of wrought iron or of steel, cast iron being seldom, and wood never used. Even when for practical reasons a shaft is made of cast iron it is quite customary to attach to it journals either of wrought iron or steel, in order that minimum diameters may be obtained. This happens especially when for economical reasons the iron shaft is cast hollow, in which case if the journals were likewise to be of cast iron, convenience of construction would require that they should also be hollow, that is of large diameter. Hollow cast iron journals should never be used, indeed the use of cast iron for journals and shafts is becoming less and less frequent.

Moreover, the diameter d of the journal is smaller, the smaller we assume the length l, or the ratio $\lambda = \frac{l}{d}$, consequently the journal would have to be as short as possible if the only question to be considered were the reduction of the loss of work which is due to friction. But in order that the journal may satisfy the important practical condition of sufficient durability, its length cannot be diminished below a certain amount. For the effect of friction is always to cause a wearing away of journal and bearing, which wear, we know from experience, increases with the specific pressure, *i.e.*, with the pressure per unit of bearing surface. We are therefore obliged to prevent the pressure per unit of surface of the journal from exceeding a certain amount, and from this follows a certain ratio λ between length and diameter of journal, below which we cannot go. Now the projection of the bearing surface on a plane at right angles to the direction of pressure, for a cylindrical journal of diameter d and length $l = \lambda d$, is expressed by,

$$f = l d = \lambda d^2;$$

then if p represents the highest permissible pressure per unit of projected area, we must have besides the above formula for the strength of the journal, another involving the condition of durability which is as follows:

$$P = p f = p \lambda d^2$$

or,

$$d = \sqrt{\frac{P}{p\lambda}}.$$

Putting this value of d equal to that found above, we have

$$d = 2.26 \sqrt{P \frac{\lambda}{k}} = \sqrt{\frac{P}{p\lambda}}$$

and for λ the expression,

$$\lambda = 0.442 \sqrt{\frac{k}{p}}.$$

The ratio $\lambda = \frac{l}{d}$ therefore depends not only on the permissible stress k of the material of the journal, but also on that amount of pressure p which experience has shown will give durability with minimum (cubical) wear. Since the wearing away of a bearing depends not only upon the material, but also most markedly upon the speed of the journal, an increase in speed being accompanied by increased work of friction, and as a consequence by increased *molecular work* and *heating* of the journal, it follows that in choosing p or λ, the speed must be taken into account, λ being assumed greater as the speed is greater. In practice it is customary to choose the ratio λ between 1.5 and 3 for continuously running journals, while for journals which are subjected to but a slight motion, as joint pins, the ratio of length to diameter is assumed smaller, and is often as small as 0.5. λ having been chosen in accordance with the circumstances just mentioned, we have for the diameter according to the above formula:

$$d = 2.26 \sqrt{P \frac{\lambda}{k}} = \alpha \sqrt{P}.$$

For ease of calculation the following little table has been prepared containing the values of $\alpha = 2.26 \sqrt{\frac{\lambda}{k}}$ when λ equals 0.5, 1, 1.5, 2, 2.5, and 3, and $k = 3$ [4,270 lbs. per sq. in.] for cast iron, $k = 6$ [8,530 lbs.] for wrought iron, and $k = 10$ [14,220 lbs.] for cast-steel journals.

$\lambda = \frac{l}{d} =$	0.5	1.0	1.5	2.0	2.5	3.0
Cast iron, $k = 3$........	0.92	1.30	1.60	1.84	2.06	2.26
$= 4{,}270$....	0.024	0.035	0.042	0.049	0.055	0.060
Wrought iron, $k = 6$........	0.65	0.93	1.13	1.30	1.46	1.59
$= 8{,}530$....	0.017	0.024	0.030	0.035	0.039	0.042
Cast steel, $k = 10$.......	0.51	0.72	0.88	1.01	1.13	1.24
$= 14{,}220$...	0.013	0.019	0.023	0.027	0.031	0.033

NOTE.—The ratio of length to diameter of journal for cast and wrought-iron journals should, according to *Redtenbacher*,[*] be taken independent of the velocity and equal to

$$\frac{l}{d} = 1.21 + \frac{0.87}{d\text{ cm}} \left[= 1.21 + \frac{0.348}{d\text{ ins.}} \right],$$

which would give for a journal of 50 millimetres [2 ins.] diameter a ratio

$$\frac{l}{d} = 1.21 + \frac{0.87}{5} = 1.38.$$

According to *Wiebe*[†] the ratio λ should be assumed as independent of the material employed and equal to $\lambda = \frac{1}{3}\sqrt[3]{n}$, where n represents the number of revolutions per minute. This would give for 90 revolutions $\lambda = 1.5$ and for 400 revolutions $\lambda = 2.5$.

Reuleaux[‡] makes the ratio of length to diameter dependent on the material and on the number of revolutions n, and gives $\lambda = \beta\sqrt{n}$, when $\beta = 0.12$ for wrought-iron journals in gun or Babbitt metal bearings, and $\beta = 0.15$ for cast-steel journals in gun metal or Babbitt bearings. This makes, for $n = 150$, λ respectively equal to 1.5 and 1.8 and for 400 revolutions $\lambda = 2.4$ and 3. According to the same writer $\lambda = \frac{4}{3}$ for cast-iron journals and $\lambda = 1.75$ for wrought-iron journals in cast-iron bearings.

Examples. 1. When the cast-iron end-journal of the shaft of a water-wheel is subjected to a load of 3,000 kilograms [6,614 lbs.] and λ assumed equal to 1.33, we get for the diameter,

$$d = 2.26\sqrt{3{,}000\,\frac{1.33}{3}} = 82.5 \text{ mm } [3.3 \text{ ins.}]$$

and for the length of the journal, $l = 1.33 \times 82.5 = 110$ mm [4.4 ins.].

2. An end-journal (not subjected to torsion) of wrought-iron shafting making

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[*] *Redtenbacher*, Resultate für den Maschinenbau, 66.

[†] *Wiebe*, Die Lehre von den einfachen Maschinentheilen, § 105.

[‡] Reuleaux, Der Constructeur, III. Edition, § 79.

120 revolutions per minute is subjected to a load of 1,200 kilograms [2,650 lbs.]. What should be its dimensions?

Taking the ratio $\lambda = 1.5$, the above table gives

$$d = 1.13 \sqrt{1,200} = 39.2 \text{ mm } [1.56 \text{ ins.}],$$

and therefore the length $l = 58.8$ mm [2.34 ins.].

§ 4. **Pivots.**—By pivot is meant the end journal which sustains the pressure acting in the direction of the axis of an upright shaft. This pressure is composed of the weight of the shaft, the weight of the machine parts (wheels, etc.) attached to it, and the vertical components of forces acting obliquely to the axis. Pivots are generally the lowest parts of upright shafting, though there are cases in which a hollow shaft is provided at its upper end with a pivot which rests in a bearing or foot-step that is supported by a fixed spindle placed within the hollow shaft. Examples of this may be found in turbines and in centrifugal machines suspended pendulum fashion. In cranes with fixed post the foot-step attached to the movable frame of the crane turns about the fixed pivot on the upper part of the crane post.

Pivots are subjected to compression by the axial pressures which they must sustain. A calculation of their dimensions taking into account only their resistance to crushing, would in most cases give so small a cross section as to cause rapid wear of the journal and a still more rapid wear of the bearing. Particularly with high speeds would wearing and heating of the pivot be considerable, requiring that the diameter of the pivot be determined with reference to the number of revolutions of its shaft. For journals subjected to but a slight amount of motion, as the pivots of turntables, cranes, etc., it may suffice to determine the diameter d of the pivot, with reference only to resistance to compression. If for this case p represents the permissible pressure per unit of area and P the total pressure on pivot, we can get the diameter of the pivot from the equation

$$\frac{\pi}{4} d^2 p = P \qquad d = \sqrt{\frac{4P}{\pi p}} = 1.13 \sqrt{\frac{P}{p}}.$$

We can here, according to *Tredgold*, substitute for steel pivots $p = 5$ kilograms [7,110 lbs.], and for gun metal or brass $p = 1$ kilogram [1,420 lbs.]. For the pivots of more rapidly running shafts, Reuleaux gives the formula:

$$d = 0.17\sqrt{Pn}, \qquad [d = 0.0045\sqrt{Pn}],$$

where n represents the number of revolutions per minute. This corresponds to a permissible pressure p per unit of area, which may be obtained from

$$1.13\sqrt{\frac{P}{p}} = 0.17\sqrt{Pn}$$

as follows:

$$p = \left(\frac{1.13}{0.17}\right)^2 \cdot \frac{1}{n} = \frac{44.2}{n}\ \text{kg}\ \left[p = \frac{62850}{n}\ \text{lbs.}\right].$$

This gives the following table:

$n = 150$	300	450	600	1000	REVOLUTIONS.
Permissible pressure p per mm² = 0.30...	0.15	0.10	0.075	0.044	Kilograms.
per sq. in. = 420....	210	140	105	63	Pounds.

Pivots are generally made cylindrical with the circular edge of base taken off as in Fig. 39, or rounded as in Fig. 40.

FIG. 39.

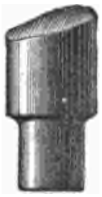
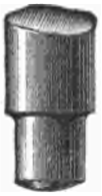


FIG. 40.



It is only when the shaft is subjected to considerable shock or lateral oscillation that it is customary to turn the pivot into a spherical shape as shown in Fig. 41, for in such a case a cylindrical pivot would give rise to great lateral pressures. The conical form of pivot, Fig. 42, is only to be found in certain machines, for example, on the spindles or arbors of light lathes, in which it is of special importance that the axis of the spindle shall remain invariable in position. The normal lateral pressure of such a conical journal on its bearing is, it is true, greater than the pressure acting in the direction of the axis because of the wedge action which occurs, the friction is consequently greater than that of a cylindrical journal with flat end surfaces; but since it is not necessary that the pivot should enter its bearing far, the path of pivot friction and consequently the work of friction may be greatly reduced. This is the principal reason why conical pivots are so often used for the spindles of spinning machines, for the friction of these numerous and rapidly rotating spindles (some mills having more than a hundred thousand running at 3,000 to 4,000 revolutions per minute) consumes a considerable quantity of work.

FIG. 41.

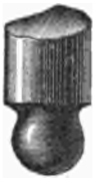
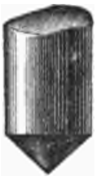


FIG. 42.



The attempts made to substitute for the conical pivot another having for profile the so-called anti-friction curve * (see I., p. 249) have not led to the universal adoption of the latter, though in individual cases, for example in rotary pumps, it has been applied.

Examples. 1. The movable jib of a crane capable of sustaining a load of 10,000 kilograms [22,050 lbs.] has together with the winding-gear and crane-post a weight of 2,500 kilograms [5,512 lbs.]; what should be the diameter of the cast-steel pivot?

If we assume the greatest permissible load on the pivot to be 5 kilograms per

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* On this point, and on the calculation of the moments of friction, see *Reye*, Civil-Ingénieur, 1860, Nos. 3 and 4, or Zeitschrift deutsch. Ing., Vol. V. p. 200.

square millimeter [7,110 lbs. per sq. in.] we shall obtain for the necessary area of pivot

$$\frac{12,500}{5} = 2,500 \text{ mm}^2 \text{ [3.875 sq. ins.]},$$

which corresponds to a diameter of pivot equal to 56.4 mm [2.25 ins.].

2. What should be the diameter of the pivot of a turbine whose wheel, shaft and toothed gear together weigh 600 kilograms [1,320 lbs.], and make 160 revolutions per minute? how much work is consumed by friction of pivot? Assuming the permissible pressure of pivot at

$$p = \frac{44.2}{160} = 0.276 \text{ kg [0.393 lbs.]},$$

we obtain for area of pivot

$$\frac{600}{0.276}\ 2,174 \text{ mm}^2 \text{[3.37 sq. ins.]},$$

and for its diameter

$$d = 52.6 \text{ mm [2.08 ins.]}.$$

Assuming 0.1 for coefficient of friction, the work consumed by friction will amount to

$$0.1 \times 600 \times \frac{2}{3} + 3.14 \times 0.052 \times \frac{160}{60} = 17.5 \text{ meter-kilograms per second}$$

[126.6 ft. lbs. per second].

§ 5. **Collar Journals.**—It is evident that the diameter d of the pivot, and consequently the moment of friction of the pivot $= \frac{1}{3} d \,.\, \varphi P$ (see I., § 188), will be greater, the smaller we assume the pressure per unit of area of pivot. The moment of friction increases not only with the work of friction but also with what is assumed as proportional to this work, namely, the *cubic wear;* in other words the cubic content of the material worn off the pivot and its bearing increases directly with the diameter of the pivot. In order that the *linear* wear δ may be reduced to a minimum, we can make d larger than d_0, the diameter of pivot obtained from considerations of strength only. This is justifiable because the work of friction and consequently the cubic wear increase *directly* as the *diameter*, while the linear wear varies *inversely* as the *square of the diameter* when the cubic wear is constant, as may be seen from the following discussion. If we assume instead of the diameter d_0 necessary for strength, a diameter of pivot

$$d = \nu\, d_0,$$

the work of friction per revolution will be

$$A = \frac{2}{3} d \pi \,.\, \varphi P = \frac{2}{3}\, \nu\, d_0 \pi \,.\, \varphi P = \nu A_0;$$

where

$$A_0 = \frac{2}{3} d_0\, \pi \,.\, \varphi P$$

represents the work of friction of a pivot of diameter d_0. If we designate the linear wear of the two pivots d_0 and d respectively by δ_0 and δ, the cubical wear V_0 and V may be obtained from:

$$V_0 = d_0^2\, \frac{\pi}{4}\, \delta_0,$$

and

$$V = d^2\, \frac{\pi}{4}\, \delta.$$

Since we may assume that

$$V_0 : V = A_0 : A,$$

we obtain by substitution of the foregoing values:

$$d_0^2\, \delta_0 : d^2\, \delta = A_0 : \nu A_0;$$

or putting $d = \nu\, d_0$:

$$\delta_0 : \nu^2 \delta = 1 : \nu,$$

i.e.,

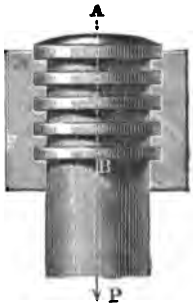
$$\delta = \frac{\delta_0}{\nu};$$

which justifies the above-made assertion that in spite of the increase of the moment of friction, the linear wear diminishes inversely as the diameter of the pivot.

If, therefore, the diameter of the journal be increased, a correspondingly more reliable construction will be obtained by reducing the hurtful play caused by the wear, but it will be at the expense of economy, for the larger journal necessarily involves a greater loss of mechanical work, and a greater expenditure of the lubricant and of the material of the bearing. Now in machinery, shafts frequently occur which are subject to great pressures and are run at high speeds; it is all-important that these shafts be as free as possible from vibration; in other words, the pressure p on their journals must be assumed as small as possible. Such shafts occur, for example, in turbines and in screw propellers as well as in the upright shafting of mills, factories, and other technical works. If we were to apply to these cases pivots of ordinary design, we might often obtain diameters considerably greater than the greatest diameter of shaft. This difficulty can in a certain degree be overcome by employing the collar journal AB, Fig. 43. Here the supporting surface is not placed on the end of the shaft, but is distributed over numerous annular surfaces formed by collars fixed to the shaft and resting on corresponding projections of the bearing; the latter must of course be composed of two parts in order that it may enclose the shaft. If d represents the mean diameter of these rings, b their width radially, z their number, and p the pressure per unit of area, we shall have evidently

$$z \pi d b p = P.$$

FIG. 43.



From this it is very evident that for a given pressure P, and a given diameter of shaft, the specific pressure p may be diminished at pleasure without at the same time increasing the lever arm of the friction. For this purpose it will only be necessary to choose an appropriate number, z, of rings, the lever arm of friction in all cases being very nearly equal to

$\frac{d}{2}$.[*] From this follows that the collar journal will be more economical than the ordinary pivot in all those cases in which the lever arm of friction of the latter equals $\frac{1}{3}d$ and exceeds the radius of the shaft at the journal by more than half the width of the annular ring.

Besides the property just determined, by virtue of which the linear wear can be diminished at pleasure by a suitable increase of bearing surface, without entailing an increased loss of work, the collar journal possesses an advantage often of the greatest importance in many designs, namely, it is not necessary to place the journal at the end of the shaft, any other part of the shaft being equally well adapted to receive it. For this reason the collar journal has been almost universally adopted for screw propellers, and is also often applied to upright shafts in which the lower end must be free to receive wheels, or in which it is desirable to have the bearing above water.

As regards the proportions of collar journals, *Reuleaux*, making d the mean diameter of the rings and n the number of revolutions per minute, gives the formula:

$$d = 0.04 \sqrt[3]{\frac{P^2 n^2}{z^2}}, \left[d = 0.307 \sqrt[3]{\frac{P \quad n^2}{z^2}}\right].$$

This assumes that

$$b = 1.2 \sqrt{d}, \quad [b = 0.24 \sqrt{d}].$$

This formula corresponds to the supposition that the highest permissible pressure $p = \frac{33}{n}$, [$p = 46{,}940$ lbs.], for we can obtain from it

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[*] Strictly speaking the lever arm of friction should be put equal to $\frac{2}{3} \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$ (see I., § 188) where r_1 and r_2 respectively represent the inner and outer radius; if, however, the width b is small relatively to the diameter, we can substitute for this lever arm $\frac{r_1 + r_2}{2}$.

$$P = \frac{z}{n}\sqrt{\left(\frac{d}{0.04}\right)^3} = 125\,\frac{z}{n}\,d\sqrt{d}, \quad \left[P = 34{,}450\,\frac{z}{n}\,d\sqrt{d}\right].$$

Since

$$P = z\pi d b p = z \times 3.14\,d \times 1.2\,\sqrt{d} \times p = 3.77\,z\,p\,d\,\sqrt{d},$$
$$[P = 0.754\,z\,p\,d\,\sqrt{d}],$$

we get by equating:

$$p = \frac{125}{3.77 \times n} = \frac{33}{n}, \quad \left[p = \frac{46{,}940}{n}\right].$$

This formula gives us the following little table:

Revolutions per minute = n =	150	300	450	600	1000
p kg per mm^2 of bearing......... =	0.22	0.11	0.073	0.055	0.033
p lbs. per sq. in. " .... =	313	156	104	78	47

Example.—The shaft of a screw propeller makes 300 revolutions per minute, and exerts a pressure of 6,000 kilograms [13,230 lbs.] on the vessel; what should be the dimensions of its collar journal? If, according to the foregoing, we assume that the permissible pressure should not exceed 0.1 kilogram per square millimeter we shall find 60,000 square millimeters [93 sq. ins.] of bearing surface will be necessary in order that the wear may be as small as is practicable for a simple flat pivot; this would require a diameter $d = 277$ millimeters [11 ins.]. Now since the shaft itself has only a diameter of 140 millimeters [5.6 ins.], it will here be advantageous to employ the collar journal. If we make the width $b = 15$ millimeters [0.6 ins.], the mean diameter will become

$$140 + 2 \times \tfrac{15}{2} = 155 \text{ millimeters } [6.2 \text{ ins.}],$$

and the bearing surface of each collar,

$$3.14 \times 155 \times 15 = 7{,}300 \text{ square millimeters } [11.3 \text{ sq. ins.}].$$

Therefore $\frac{60000}{7300} = 8$ collars will be needed. The lever-arm of friction of the collar journal will amount to

$$\frac{155}{2} = 77.5 \text{ millimetres [3.1 ins.]};$$

the work of overcoming friction per revolution will be

$$A = \varphi \times 6{,}000 \times 3.14 \times 0.155 = \varphi \times 2{,}920 \text{ meter-kilograms } [\varphi \times 21{,}120 \text{ ft. lbs.}];$$

where φ represents the coefficient of friction. Assuming the lubrication to be good, we may put $\varphi = 0.054$, and the work lost by friction of journal per second will be:

$$\frac{300}{60} \times 0.054 \times 2920 = 788 \text{ meter-kilograms} = 10.5 \text{ horse powers.}$$

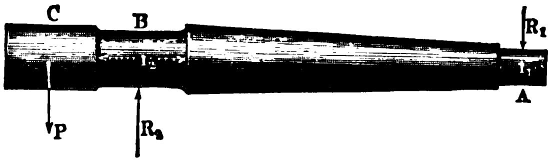
If we had chosen a flat pivot with a diameter equal to 277 millimeters, the work of friction would have amounted to

$$\frac{300}{60} \times 0.054 \times 6000 \times 3.14 \times \frac{2}{3} \times 277 = 939 \text{ meter-kilograms} = 12.5 \text{ horse powers.}$$

The friction of the collar journal therefore would absorb about two horse powers less of work than the flat pivot.

§ **6. Diameters of Axles.**—Rules have thus far only been given for determining the diameter of end journals, *i.e.* of such journals as A in Fig. 44, existing at the end of shafts, and consequently subjected to bending stress only, the bending force being the reaction R_1 of the support acting at the middle of the journal. A neck journal B so situated that its axle is subjected to bending stress on both sides of it, for example, at C by the force P and at A by the reaction R_1 of the support, must have its dimensions determined from the bending moments $P \,.\, BC = R_1 \,.\, AB$ of the external forces. This will generally cause the diameter d_2 to be greater than for an end journal subjected to the same load R_2 and having, according to the preceding paragraphs, the diameter d_0 and the length $l_0 = \lambda d_0$. Now it is not nec-

FIG. 44.



essary that the ratio $\frac{l_2}{d_2}$ should have the same value as $\lambda = \frac{l_0}{d_0}$ of the end journal, for the real pressure R_2 on the bearing is less than it would be for an end journal of diameter d_2. In this case we can take the length l_2 of the neck journal equal to l_0 of the end-journal, and shall then get as regards wear, as high a degree of security as with an end journal subjected to the load R_2. For in these cases the pressures on the bearings will be equal, the work consumed by friction at every revolution, and consequently also the cubic wear, will be proportional to the distances traversed, or to the diameters. The cubic wear may be regarded as a cylindrical shell whose diameter is d, and thickness δ, then if

$$l_0 d_2 \pi \times \delta_2 : l_0 d_0 \pi \times \delta_0 = d_2 : d_0,$$

we must have

$$\delta_2 = \delta_0 ;$$

that is, the linear wear (measured radially) is equal in the two cases. As will be seen hereafter, when bearings are discussed, practical considerations call for the same ratio λ in the case of neck journals as has been above established for end journals. This evidently does not require that the diameter of the neck journal be greater than the d_2 above determined, but it does require that l_2 shall be greater than l_0, the result being that the linear wear is diminished. The diameter at any point of a shaft subjected to bending stresses is to be determined in the same manner as the diameter of a neck journal, namely, from the moment M of the external forces. The cross section of the shaft is obtained from the well-known fundamental formula for resistance to flexure,

$$M = k \frac{W}{e},$$

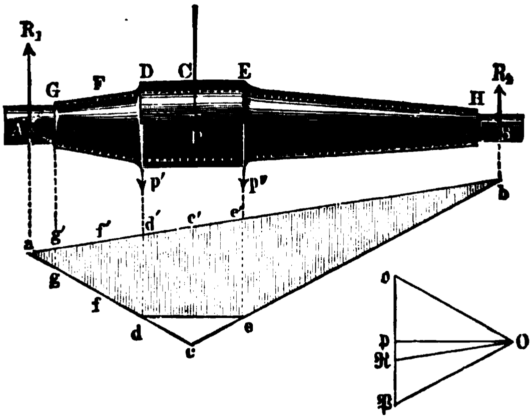
which, when the cross section is circular is easily changed into

$$M = \frac{\pi}{32} d^3 k.$$

If the diameters are determined in this manner for a sufficient number of points, the shaft will have the form of uniform strength, for which, however, an approximate form is generally substituted. Cylindrical forms in particular are used not only for the journals, but for the hubs or portions of the shaft at which such machine parts as wheels, levers, etc., are fastened. In order that these bending moments may be easily determined, use should be made of the methods of Graphical Statics (see Appendix to fifth edition of Vol. I.). We will illustrate by applying these methods to a few most frequently occurring cases.

§ 7. **Examples.**—Let *A* and *B*, Fig. 45, be the centres of the journals of a shaft which carries at *D E* the hub of a working beam. It is required to find the dimensions of this shaft from the load *P*, whose point of action we will assume to be at *C*, midway between *D* and *E*.

FIG. 45.



If in constructing the force polygon, we lay off the load *P* on a vertical o 𝔓 to whatever scale may have been chosen for the forces, and choose the pole *O* at a distance from o 𝔓 equal to the base assumed for the moments, we can easily construct the corresponding equilibrium polygon *a b c* (see § 43, Appendix, Vol. I.), by drawing the lines *a c* and *b c*, in equilibrium with the force P at c, parallel to 𝔬 O and O 𝔓. If the intersection of these lines with the verticals through the centres of the journals be connected by a straight line, this latter will represent the closing line of the equilibrium polygon. Parallel to this closing line $a\,b$ draw the pole ray Oℜ, then will ℜ𝔬 be the reaction R_1 at A, and 𝔓ℜ the reaction R_2 at B.

If we assume that the hub of the working beam transmits the pressure to the shaft at the points D and E instead of at the centre C, we must regard the force P as resolved into two components p' and p'' acting at D and E. Each of these components is equal to $\frac{P}{2}$ when, as in this case, their resultant P passes through the centre C; in any case their values can be ascertained by prolonging the verticals through D and E till they intersect the equilibrium polygon in $d'\,d$ and $e'\,e$, then drawing the new side $d\,e$ and parallel to it in the force polygon the ray O𝔭. Then will 𝔬𝔭 be the component p' and 𝔭𝔓 the component p'', which together can replace the force P. This construction follows immediately from the well-known principle that the resultant (P) of two or more components ($p'\,p''$) passes through the intersection of those two of the sides of the equilibrium polygon between which the components lie. For the four forces R_1, p', p'', R_2 in equilibrium, we have found the corresponding force polygon ℜ 𝔬 𝔭 𝔓 ℜ and the equilibrium polygon $a\,d\,e\,b\,a$.

By choosing the pole O at a distance from the line of forces 𝔬 𝔓 equal to the base assumed for the moments, that portion of the ordinates at any point of the shaft intercepted by the equlibrium polygon represents, as is well known, the resultant moment acting at that point.

For any cross section F of the shaft, therefore, the bending moment $f\,f' = M$ is first found, and then the diameter of the shaft for this cross section, by means of the formula

$$d = \sqrt[3]{\frac{32}{\pi}\frac{M}{k}}.$$

If we thus determine the diameter for a sufficient number of points between A and C and also between B and C, we can draw the curved profile for each of the two portions of the shaft, and thus find its form of uniform strength. Since the diameters, and consequently the ordinates, of each portion are proportional to the cube root of the moments, and therefore (as may be easily shown by means of the equilibrium polygon) to the cube root of the distance from A or B, it follows that the curved profile of each portion of the shaft must be a cubic parabola of the form $y = c\sqrt[3]{x}$; y denoting the ordinate or half diameter of the shaft, x the horizontal distance from the centre of the corresponding journal, and c a constant factor. If we first determine the diameter d_1 of the journal from the reaction R_1, and for the ratio $\frac{l_1}{d_1} = \lambda$, we can find the diameter at any point F by means of the formula

$$d = d_1 \sqrt[3]{\frac{ff'}{gg'}}.$$

In practice the shafts and axles are generally given approximate forms, for the determination of which only a few diameters are needed, for example, those at G, H and C. The journals are of course made cylindrical, and where they join the middle portion of the shaft or axle, there is a shoulder of about $\frac{1}{8}$ of the diameter of journal* so that the diameter at the shoulders is equal to $\frac{5}{4}\,d_1$ and $\frac{5}{4}\,d_2$ respectively. In like manner, by means of the above-given formula

$$d = d_1 \sqrt[3]{\frac{cc'}{gg'}},$$

the hub DE is given a diameter $\frac{5}{4}$ as large as the one obtained from the ordinate cc' for the middle point C. For the portions between the journals and hubs, frustums of cones are chosen which have at C a common base whose diameter

$$d = d_1 \sqrt[3]{\frac{cc'}{gg'}},$$

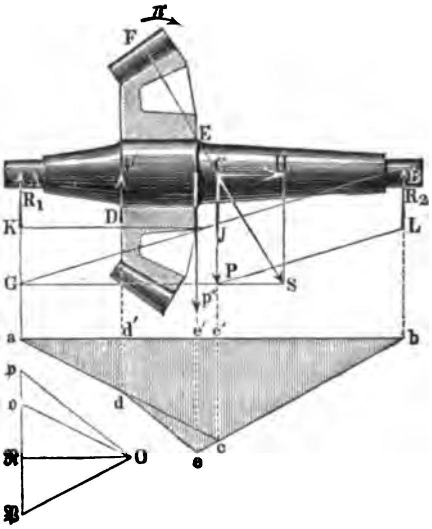
---

* See Redtenbacher's Resultate für der Maschinenbau, p. 77.

and in the immediate neighborhood of the journals have cross sections whose diameters are respectively $\frac{5}{4}\,d_1$ and $\frac{5}{4}\,d_2$.

Very often the direction of the forces acting on the axles or shafts is oblique to the axis, as in the case of bevel wheels. The determination of the bending moments for such a case, represented by Fig. 46, differs but slightly from the preceding one.

FIG. 46.



If $C\,S$ represents in magnitude and direction the pressure acting on a bevel wheel at F, this pressure may be resolved by the parallelogram of forces into the vertical pressure P and the horizontal thrust H. The latter, which is in equilibrium with the lateral component of the reaction at the bearing, tends to produce compression of the axle, and can be neglected in determining the dimensions of the axle, since in nearly every case its influence on the stress to which the material is subjected is insignificant in comparison with the bending force P. If we make the same construction as in the preceding case, by making $\mathfrak{o}\,\mathfrak{P} = C\,P$, we obtain as before the equilibrium polygon $a\,c\,b$. It may here be remarked that it is easy to so choose the pole O that the closing line $a\,b$ of the equilibrium polygon will be horizontal or parallel to the axis $a\,b$. We have only to assume the pole O in such a position that the perpendicular $O\,\mathfrak{R}$ let fall from it upon $\mathfrak{o}\,\mathfrak{P}$ shall divide the line $\mathfrak{o}\,\mathfrak{P}$ at $\mathfrak{R}$, in such a manner that $\mathfrak{R}\,\mathfrak{o}$ is equal to the reaction R_1 at A and $\mathfrak{P}\,\mathfrak{R}$ equal to the reaction at B. By a simple construction one can easily find both of the pressures at the supports, and their equal and opposite reactions R_1 and R_2. For this purpose it is only necessary to draw a horizontal line through P and transfer the length CP to AG and connect G with B. The connecting line GB then divides the force CP at J in such a ratio that

$$CJ : JP = CB : CA,$$

from which follows that CJ represents the pressure at the support A and JP that at B. If, therefore, we draw

$$JK \parallel CA \text{ and } PL \parallel JB,$$

we shall obtain from

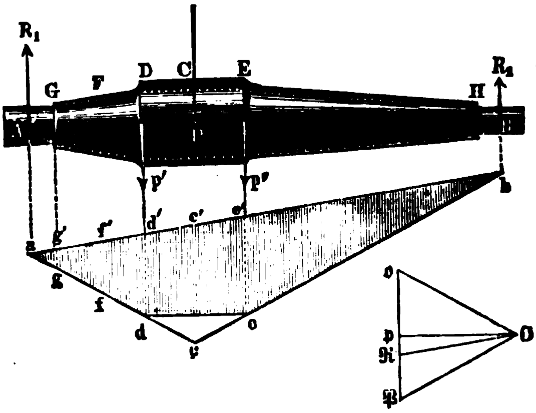
$$KA = R_1 \text{ and } LB = R_2,$$

the two reactions at the supports, which must be made equal to $\mathfrak{R}\mathfrak{o}$ and $\mathfrak{P}\mathfrak{R}$ respectively, in order to obtain a horizontal closing line ab in the equilibrium polygon (see Appendix, § 45, Vol. I.).

The resulting equilibrium polygon is abc. The load P must be again resolved into the two components p' and p'', which act at the two cross sections D and E, for the oblique force S tends to turn the wheel in the direction of the arrow π, causing the nave to be pressed against the axle at D and E with forces equal to p' and p''. The determination of these two components, which evidently must act in opposite directions, since their resultant P does not lie between their points of application, is accomplished exactly as in the previous example.

If here as there we draw from the points of application D and E of the components, the lines Dd and Ee to the corresponding sides ac and bc containing the resultant force P, the connecting line de will then give the new side of the polygon. Furthermore the ray $O\mathfrak{p}$ drawn parallel to it, cuts off on the line of forces the two components $\mathfrak{o}\mathfrak{p} = p'$ and $\mathfrak{p}\mathfrak{P} = p''$, the algebraic sum of which is again equal to $\mathfrak{o}\mathfrak{P} = P$. The resulting equilibrium polygon $adeba$ shows us that the maximum bending stress on the axle at E is now greater than it would be if the force P acted directly at C, while in the example given in Fig. 47, where the force P acts at the middle of the nave, a diminution of the bending moment is effected, this moment diminishing as the length of the nave increases. This increase, moreover of the bending moment, is not due to the obliqueness of the pressure, but is owing to the direction of the force FS intersecting the axis at a point C outside of the nave. Every time that this last mentioned condition occurs, the one-sided action of the load causes an increase of the bending moment, as compared with a direct load. This is often the case, for example, in those water-wheels in which the transmission of power is effected by means of a toothed ring bolted to the rim of the wheel and extending laterally beyond its nave. The dimensions of the axles, both in this and in the following cases, are determined in the manner already described by means of the fundamental formula:

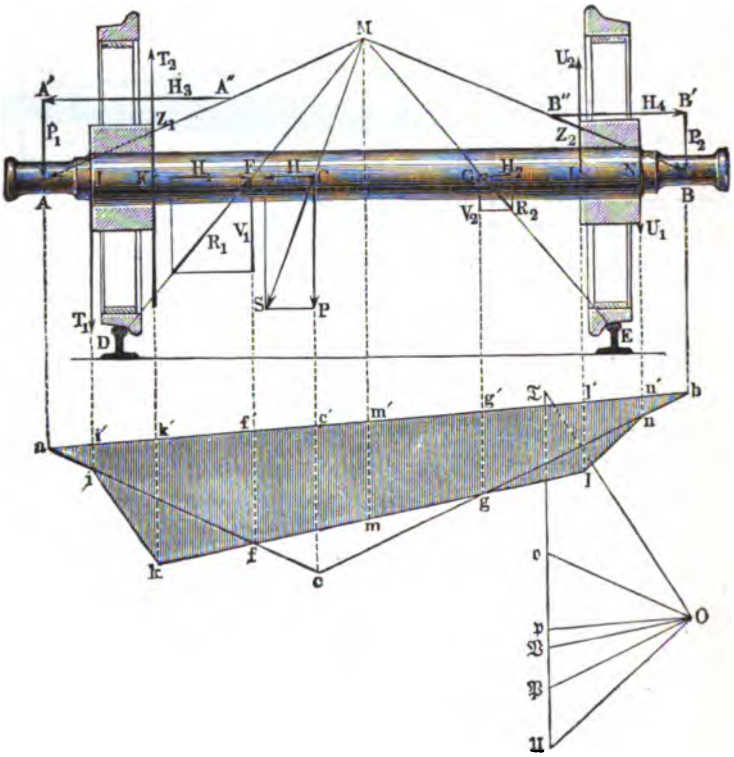
Fig. 47.



$$M = k\,\frac{W}{e}\,.$$

§ 8. An interesting example illustrating the effects of an oblique load, is that of a car axle, Fig. 48. If P be that part of the weight of the car which falls upon the axle $A\,B$ and acts at the centre of gravity M, we shall have to assume in addition the horizontal force H, whose point of application is also at the centre of gravity M; for under certain circumstances, for example, in passing around curves and over irregular portions of the roadway, a horizontal force will be developed which, according to Scheffler, may on an average be assumed

FIG. 48.



equal to about 0.4 P. The resultant MS of these two forces is oblique to the axis, cutting it at C. Now at this point of intersection C, resolve the resultant CS into the two components CP and CH as above mentioned. The vertical force P then tends to bend the axle, and the bending moment for every cross section can be easily determined by the equilibrium polygon. First of all it is evident that the force CS must be in equilibrium with the two reactions R_1 and R_2 of the rails against the wheels. For this purpose it is necessary that three forces S, R_1 and R_2 intersect at the same point. One of these forces S must always pass through the centre of gravity M, however much its direction may change with the changing speed of the car. If, therefore, we draw the lines DM and EM, they will give us the directions of the two reactions R_1 and R_2, and their intersections with the axis at F and G will give the points at which these reactions may be regarded as taking hold of the axles. The vertical components V_1 and V_2 of these forces must be in equilibrium with P, and the horizontal components H_1 and H_2 with H. In order to determine the magnitudes of these individual forces, we must again make $\mathfrak{o}\,\mathfrak{P}$ equal to the vertical force CP (we will for the present ignore the horizontal forces H, H_1 and H_2), choose the pole O at a distance from $\mathfrak{o}\,\mathfrak{P}$ equal to the base of the moments, and construct the equilibrium polygon acb. The closing line ab parallel to the ray $O\mathfrak{p}$ now again divides the force P or $\mathfrak{o}\,\mathfrak{P}$ at $\mathfrak{p}$ into two components $\mathfrak{o}\,\mathfrak{p} = P_1$ and $\mathfrak{p}\,\mathfrak{P} = P_2$, these components representing the portions of the total load P, which is distributed upon the journals A and B. In order to determine the vertical components V_1 and V_2 of the reactions at the rails, it is only necessary to draw perpendiculars from F and G until they intersect the equilibrium polygon at $f'f$ and $g'g$, and then parallel to the side fg thus obtained draw a ray $O\mathfrak{B}$. V_1 will then be given by $\mathfrak{B}\,\mathfrak{o}$ and V_2 by $\mathfrak{P}\,\mathfrak{B}$, and we shall also get, after making $FV_1 = \mathfrak{o}\,\mathfrak{B}$ and $GV_2 = \mathfrak{B}\,\mathfrak{P}$, the reactions R_1 and R_2, and their horizontal components H_1 and H_2 at H_1F and H_2G, the algebraic sum of which is of course equal and opposed to H.

We can now substitute for the vertical forces V_1 and V_2, the pressures exerted by the naves against the axle at J, K, L and N, the horizontal forces being neglected. These magnitudes may be found, if we draw through each of these points vertical lines which intersect the equilibrium polygon, and then join the intersection i with k and l with n. We shall then get by means of the ray $O\mathfrak{T}$ drawn parallel to ik, $\mathfrak{T}\,\mathfrak{o}$ equal to the pressure T_1 acting downward on the nave at J, and $\mathfrak{B}\,\mathfrak{T}$ equal to the pressure T_2 acting upward on the nave at K. In like manner the ray $O\mathfrak{U}$ drawn parallel to ln gives $\mathfrak{U}\mathfrak{V}$ equal to the pressure U_2 acting vertically upward on the axle at L, and gives $\mathfrak{P}\mathfrak{U}$ equal to the downward-acting pressure at N.

If, therefore, we substitute for the load P the journal pressures P_1 and P_2, and for the reactions at the rails the nave pressures T and U, we shall obtain the force polygon $\mathfrak{o}\mathfrak{P}\mathfrak{U}\mathfrak{V}\mathfrak{T}\mathfrak{o}$ and the corresponding equilibrium polygon $abnlkia$, the ordinates of which again give the moments of those external forces which produce bending. The maximum strain occurs at K, corresponding to the ordinate kk', and since the horizontal force H can act to the right as well as the left, it will be necessary to determine each half of the axle by means of half the equilibrium polygon $mkiam'$. This explains why it is allowable to make car axles smaller at the middle. If the horizontal force H were not present, every point of the axle between the wheels would be subjected to the same breaking moment, and would therefore be of cylindrical form. (See Vol. I., § 243.)

The influence of the horizontal forces H is generally so small in comparison with that of the bending forces, that they need not to be taken into consideration. In order, however, to learn the action of these forces, we will imagine the whole force S acting at M to be resolved into two components Z_1 and Z_2, which pass through the centre of the journals A and B. Since the vertical components $P_1 = A'A$ and $P_2 = B'B$ of these forces have been already determined, and the directions of the forces given by MA and MB respectively, the forces themselves may be found by projecting A' and B' horizontally on to AM and BM. We then get $A''A' = H_3$ and $B''B' = H_4$, which are the horizontal components of the forces Z acting on the journals. The axle, on the whole, is therefore subjected to four horizontal forces, namely the two last-mentioned ones H_3 and H_4 having their points of application at the journals A and B, and the two H_1 and H_2 arising from the reaction of the rails, which may be supposed to be acting at the naves, since these latter connect the wheels and the axle.

The relation $H_3 - H_4 = H = H_1 - H_2$, holds for these forces, and the axle is subjected by them to tensile strains, as follows:

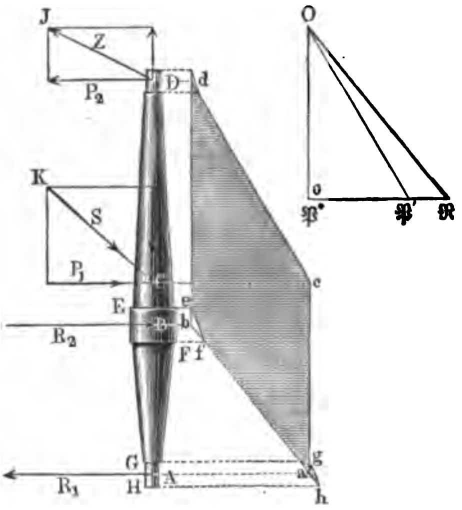
The portion AJ by H_3;

The portion BN by H_4;

The portion KL by $H_3 - H_1 = H_4 - H_2$.

§ **9.** The manner in which the bending moments on a crane-post AD, Fig. 49, is to be determined from the equilibrium polygon, ought now to be easily seen. Here the two external forces, namely the thrust $KC = S$ and the pull $DJ = Z$, act on the axle supported by the journals A and B.

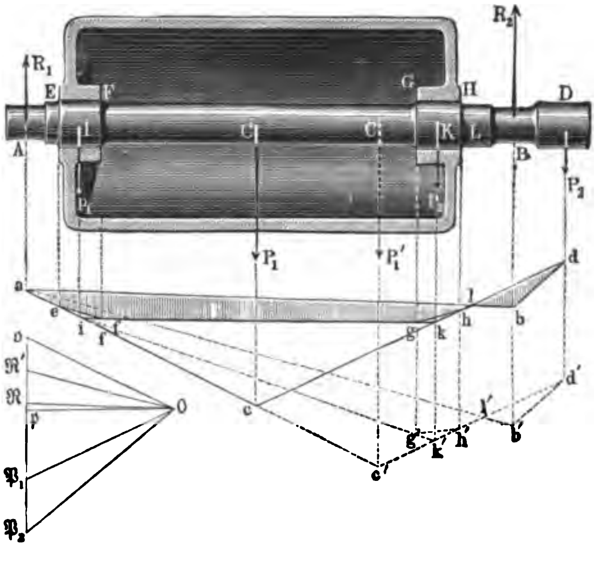
FIG. 49.



If we again resolve these two forces into their horizontal and vertical components, only the former, P_1 and P_2, will tend to break the post by bending, and are, therefore, the only ones to be considered. It is moreover evident (see Vol. I., Appendix, § 38, Example) that these two forces must be equal, and form a couple which can only be held in equilibrium by means of another couple formed by the reactions R_1 and R_2 of the journals $A\ B$. If we construct for P_1 and P_2 the force polygon $\mathfrak{o}\ \mathfrak{P}'\ \mathfrak{P}''$, and choose the pole O in the now wellk-nown manner, we shall get the equilibrium polygon $a\ c\ d\ b$, where a and b are the intersections of the outer sides with the directions of the reactions R_1 and R_2. Consequently $b\ a$ is the closing line of the polygon, and the ray $O\mathfrak{R}$ parallel to it, gives the two equal and opposite reactions $R_2 = \mathfrak{o}\mathfrak{R}$ and $R_1 = \mathfrak{R}\mathfrak{o}$. We therefore get corresponding to the force polygon $\mathfrak{o}\mathfrak{P}'\mathfrak{P}''\mathfrak{R}\mathfrak{o}$ the equilibrium polygon *acdba*, which changes into *hgcdefh* as soon as the reactions at the journals are replaced by their components acting at *H*, *G* *E*, and *F*.

As another example, we will investigate the strain on an axle for the drum of a hoisting apparatus. This axle, supported by the journals *A* and *B*, Fig. 50, carries between these journals the naves *EF* and *GH* of the drum; the power being communicated to a wheel which is attached to the free end of the axle at *D*. If we now assume that the load P_1

FIG. 50.



acts at the middle of the drum, and the driving force P_2 acts downward, and then construct the force polygon $\mathfrak{o}\mathfrak{P}_1\mathfrak{P}_2\mathfrak{o}$ with the pole O, we shall get the equilibrium polygon *acdba*.

If we draw the ray $O\mathfrak{R}$ parallel to the closing line *ba*, $\mathfrak{R}\mathfrak{o}$ will equal the reaction R_1 at *A*, and $\mathfrak{P}_2\mathfrak{R}$ will equal R_2 at *B*. The vertical lines passing through the middle of the naves *I* and K give in the connecting line ik a new side of the polygon which, by drawing the ray $O\mathfrak{p}$ parallel to it, can be used to determine the forces $p_1 = \mathfrak{o}\mathfrak{p}$, and $p_2 = \mathfrak{o}\mathfrak{P}_1$, with which the naves of the drum are pressed against the axle.

A further decomposition of these nave pressures into forces acting at the cross sections E, F, G and H, gives, in the manner so often explained, the points e, f, g and h, of the equilibrium polygon $acfghdba$. This shows us immediately that the bending moment of the axle at the point L, lying over l, is zero, and, consequently, the elastic line of the axle at this place will have a point of inflexion.

Since by the winding and unwinding of the rope the point of application of the load P is gradually shifted from one end of the drum to the other, it will be necessary to construct the equilibrium polygon for a few other positions of this point of application, in order that we may determine the most unfavorable strains to which the axle is subjected. The figure contains a second equilibrium polygon $aef'g'h'd'b'a$, drawn for the case when the load is at C', from which we see that a change in the individual moments also causes the point l, at which the moment equals zero, to change its place. It is, moreover, evident that such a point of inflexion, L, is not present when the force P_2 acts in a direction *opposite* to the load P_1. The dimensions of the axle of the drum are, however, principally determined with reference to the torsional stresses, which will be considered hereafter.

§ 10. The Construction of Axles.—Axles are best made of wrought iron or steel, and are then generally forged into one piece with their journals. Cast-iron axles are rarely used, and indeed only when the diameter of the axle is so great that it is too difficult or too costly to make it by forging. For this reason the axles of heavy water-wheels are sometimes made of cast iron. In order that the material may be economized as much as possible, it is usual to rib the axles between the journals in such a manner that their cross-sections have a cross or star-shaped form, as in Fig. 51 and Fig. 52. It is then only necessary to take care that the axles have a suitable form at those points where the naves of wheels, levers, etc., are to be fastened to the axles. A solid and convenient fastening is best obtained by making the axle cylindrical at these points. The dimensions are here determined

FIG. 51.

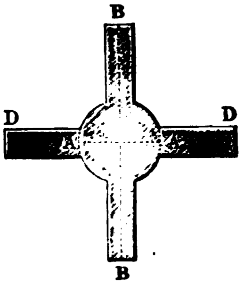
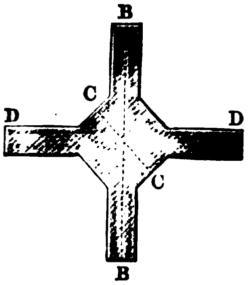


FIG. 52.



in the manner above given by finding the bending moment, *M*, either by calculation or by graphical statics, and then getting the dimensions of the cross sections from the fundamental formula $M = k\frac{W}{e}$. Here $\frac{W}{e}$ is to be determined for the particular cross section, in the manner given in Vol. I., § 225; and it must not be forgotten that during the turning of the axle the neutral axis assumes all possible positions within its cross section. Therefore those two values of $\frac{W}{e}$ will be determined which correspond to the positions in which the neutral axis either coincides with the principal ribs, as in (*A A* or *B B*) Fig. 51, or is inclined to them at an angle of forty-five degrees, and the *smaller* of these values will be used in the formula. Generally that value of $\frac{W}{e}$ will be the smaller which corresponds to the position of the neutral axis in one of the principal ribs.

If we designate the diameter of the inner circular portion of the cross section, Fig. 51, or the side of the square Fig. 52 by *d*, the height *B B* of the ribs by $D = \mu d$, and their breadth by $b = \nu d$, we shall have for the moment of inertia of the cross section, Fig. 51, taken with reference to an axis through the principal rib *A A*:

$$W = \frac{\pi d^4}{64} + \frac{b}{12}(D^3 - d^3) + \frac{D-d}{12} b^3$$

$$= \left(\frac{\pi}{64} + \frac{\nu(\mu^3 - 1) + \nu^3(\mu - 1)}{12}\right) d^4,$$

therefore

$$\frac{W}{e} = \frac{W}{\frac{1}{2}\mu d} = \left(\frac{\pi}{32} + \frac{\nu(\mu^3 - 1) + \nu^3(\mu - 1)}{6}\right)\frac{d^3}{\mu}.$$

For a neutral axis *C C*, which bisects the angle of the ribs, we get approximately, when x represents the distance of any element from the centre of the cross section:

$$W_1 = \frac{\pi d^4}{64} + 2\int_{\frac{d}{2}}^{\frac{D}{2}} 2 b\, d x \left(\frac{x}{\sqrt{2}}\right)^2 = \frac{\pi}{64} d^4 + \frac{b}{12}(D^3 - d^3)$$

$$= \left(\frac{\pi}{64} + \nu\frac{\mu^3 - 1}{12}\right) d^4;$$

and since

$$e_1 = \frac{1}{2}\frac{D}{\sqrt{2}},$$

we get:

$$\frac{W_1}{e_1}\left(= \frac{\pi}{32} + \nu\,\frac{\mu^3 - 1}{6}\right)\frac{\sqrt{2}}{\mu} d^3.$$

If we assume $\mu = 3$, and $\nu = \frac{1}{3}$, we obtain

$$\frac{W}{e} = \left(\frac{3.14}{32} + \frac{\frac{1}{3}(27 - 1) + \frac{1}{27}(3 - 1)}{6}\right)\frac{d^3}{3} = 0.518\, d^3,$$

and

$$\frac{W_1}{e_1} = \left(\frac{3.14}{32} + \frac{1}{3}\,\frac{27 - 1}{6}\right)\frac{1.414}{3} d^3 = 0.726\, d^3.$$

We must therefore assume the smaller value $\frac{W}{e}$ in the calculation.

In like manner, keeping the same symbols we find for the cross section with square core, Fig. 52:

$$\frac{W}{e} = \left(1 + \nu(\mu^3 - \sqrt{8}) + (\mu - \sqrt{2})\,\nu^3\right)\frac{d^3}{6\,\mu},$$

and

$$\frac{W_1}{e_1} = \left(1 + \nu(\mu^3 - \sqrt{8})\right)\frac{\sqrt{2}}{6\,\mu}d^3,$$

from which follow, when $\mu = 3$, and $\nu = \frac{1}{3}$, the values:

$$\frac{W}{e} = 0.506\,d^3,$$

and

$$\frac{W_1}{e_1} = 0.711\,d^3.$$

Since during the revolution of the axle the individual fibres are alternately stretched and compressed, we know from experience that there is a greater tendency to fracture than in stationary beams, in which all the fibres are either always extended or compressed. It will, therefore, be well not to choose the permissible stress, k, of the material of axles very high; particularly in cast iron should this stress k not be assumed greater than 3 kilograms per square millimeter [4,300 lbs. per sq. in.].

EXAMPLE.—What diameter must be given to the axle of an overshot water-wheel, which is not subjected to torsion, when weight of the wheel plus the weight of the contained water equals 8,000 kilograms [17,600 lbs.] and the centre of each of the journals is 0.25 of a meter [0.82 ft.] distant from the plane of the wheel arms?

Supposing a symmetrical distribution of the load, the pressure on each journal will be 4,000 kilograms [8,800 lbs.] and we shall get the diameter d of the journal from,

$$4{,}000\,\frac{l}{2} = \frac{\pi}{32}\,d^3k;$$

from which, when

$$\lambda = \frac{l}{d},$$

we get

$$d = 2.26 \sqrt{4{,}000 \frac{\lambda}{k}}.$$

If we substitute here

$$\lambda = \frac{4}{3},$$

and $k = 3$ kilograms [4,400 lbs.], we shall get

$$d = 2.26 \sqrt{\frac{4{,}000 \frac{4}{3}}{3}}, = 95.3 \text{ millimeters [3.8 ins.]},$$

for which we can assume in round numbers $d = 100$ mm [4.0 ins.], and the length of the journal $l = 133$ mm [5⅓ ins.].

The diameter of that portion of the axle between the systems of arms is to be made everywhere the same, for the breaking moment of every section is given by 4,000 × 0.25 meter-kilograms [8,800 × 0.82 ft. lbs.], and consequently its diameter must be obtained from

$$4{,}000 \times 250 = \frac{\pi}{32} d_1^3 k,$$

or

$$d_1 = \sqrt[3]{\frac{32}{\pi} \frac{4{,}000 \times 250}{3}} = 152 \text{ mm [6.08 ins.]}.$$

If we wished to have the axle hollow and the ratio of the external diameter D to the internal one d, as 4 : 3, we could get the external diameter D from :

$$4{,}000 \times 250 = \frac{\pi}{32} \frac{D^4 - (\frac{3}{4} D)^4}{D} k = 0.098 \times 0.684 D^3 k,$$

or

$$D = \sqrt[3]{\frac{4{,}000 \times 250}{3 \times 0.098 \times 0.684}} = \frac{100}{0.586} = 191.5 \text{ mm [7.66 ins.]}:$$

consequently the inner diameter,

$$d = \frac{3}{4} \times 191.5 = 143.6 \text{ mm [5.75 ins.]}.$$

If the axle is to have a cross-shaped section with cylindrical core, as in

Fig. 51, by assuming the diameter of the ribs $D = 3d$, and $b = \frac{1}{3}d$, we can get the diameter of this core from the equation :

$$4{,}000 \times 250 = \frac{W}{e}k = 0.518\, d^3 \times 3,$$

or

$$d = \sqrt[3]{\frac{4{,}000 \times 250}{0.518 \times 3}} = 86.36 \text{ mm } [3.45 \text{ ins.}].$$

Consequently the external diameter of the rib is

$$D = 3d = 259 \text{ mm } [10.36 \text{ ins.}],$$

and its breadth

$$b = \frac{1}{3}d = 28.8 \text{ mm } [1.15 \text{ ins.}].$$

§ 11. **Journal Fastening.**—Sometimes cast-iron axles are made hollow, a pipe form being given to them. To make the journals likewise hollow is not generally advisable, because the diameter of the journals and the consequent loss of work by friction is considerably increased. It is much better in such a case to make the journals of the strongest material, wrought iron or steel, and fix them firmly in the hollow shaft. Such a construction is often chosen for the pivots of wrought-iron upright shafts; for the insertion of separate journals into the shaft allows not only the use of hardened steel for the journals, but also allows the latter to be changed when they are worn.

The fastening of such a journal into the axle is most often and easily accomplished by giving to the journal a slightly conical prolongation *AB*, Fig. 53 and Fig. 54, which is accurately fitted to a hole bored in the axle *CD*. In the case of a vertical axle, Fig. 53, the weight of the latter, and of the machine parts on it, is sufficient to press the journal firmly against its seat, as if it were a wedge, and thus hold it fast by means of friction. But when the axle is in a horizontal position, a special key *E*, Fig. 54, is generally applied, the taper of which enables the journal to be firmly pressed against its socket. The key-way, *E*, Fig. 53, is only used to free or loosen the journal by means of a wedge. In using this kind of fastening, which, moreover, is often applied to other machine parts, care must be taken to avoid splitting the axle. The thickness sufficient for this purpose can be ascertained from the following calculation.

Let the radius of the conical bearing at the middle of its length be represented by r, and let l be the length of the conical bearing, measured along an element of the cone, and let α be the angle at the vertex of the cone. The pressure P acting on the journal in the direction of the axis, and due to the weight of the axle in Fig. 53, or to the key E in Fig. 54, presses the conical journal against the surface of its socket with a normal pressure per square millimeter (per square inch), represented by p. The resultant of all these normal pressures acting on the elements of the bearing, is evidently a force acting in the direction of the axis, of a magnitude

FIG. 53.

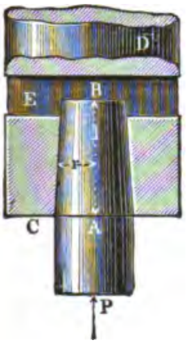
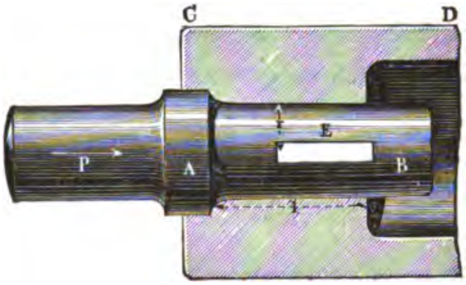


FIG. 54.



$$2 r \pi . l p . \sin \frac{\alpha}{2} = A.$$

Furthermore, let φ be the coefficient of friction (of rest), then will the resistance at every element of the surface be φp, and this, acting in the direction of a conical element, will resist the further penetration of the journal. The resultant of all these elementary forces of friction is likewise a force acting in the direction of the axle and equal to:

$$\varphi\, 2 r \pi \,.\, l p \,.\, \cos\frac{\alpha}{2} = B.$$

The condition of equilibrium in which no further penetration of the journal, due to the load *P*, can occur, will therefore be

$$A + B = P = 2 r \pi \,.\, l p \left(\sin\frac{\alpha}{2} + \varphi \cos\frac{\alpha}{2} \right),$$

from which we get the normal pressure:

$$p = \frac{P}{2 r \pi l \left(\sin\frac{\alpha}{2} + \varphi \cos\frac{\alpha}{2} \right)} = \frac{P}{f \left(\sin\frac{\alpha}{2} + \varphi \cos\frac{\alpha}{2} \right)},$$

where f represents the bearing surface $2 r \pi l$ of the journal.

We see from this formula, which agrees with the one developed for the ordinary wedge (See Vol. I., § 183), that if friction were not present the pressure p would be increased indefinitely by the diminution of $\frac{\alpha}{2}$, for by assuming that $\varphi = 0$, we should get $\alpha = 0$, and $p = \infty$. But when friction is taken into account, the greatest value which p can ever attain obtains when α is taken very small, that is, when we assume $\sin\frac{\alpha}{2} = 0$, and $\cos\frac{\alpha}{2} = 1$. We then get:

$$\max. p = \frac{P}{\varphi f}.$$

It is evident from this that the specific pressure p, tending to split the shaft, will be the greater the smaller the coefficient of friction φ. Closely connected with this is the fact that machinists lubricate with oil the conical steel drifts which they use for bursting castings, the bursting being effected by driving the drifts into holes drilled for the purpose.

At all events, the above formula shows us that in ordinary cases of journal fastening, special provision against the splitting of the (iron) shaft by conical pivot, as for example in providing special rings, will be unnecessary, and is only to be recommended in cases where the structure of the material (wood) or shocks would seem to make this arrangement desirable.

In those cases in which the axles are subjected to great shocks, they should not be ribbed or hollow, but solid, for shocks can only be resisted by a suitable *mass*, and not by a *large moment of inertia*, of the cross section. Conformably with this statement we find, for example, the vertical cast-iron shafts of millstones are solid cylinders.

EXAMPLE.—The cast-steel pivot of a mill spindle is fitted into the latter with a conical projection, 50 millimeters [2 ins.] long, and with a maximum diameter of 30 millimeters [1.2 ins.], and a minimum diameter of 25 millimeters [1 in.]. Supposing the weight of the millstone to be 1,500 kilograms [3,3co lbs.], what is the pressure on the socket of the mill spindle?

The angle α, at the vertex of the conically shaped pivot may be calculated from the relation

$$tan \frac{\alpha}{2} = \frac{\frac{1}{2}(30 - 25)}{50} = 0.05, \quad \text{or } \frac{\alpha}{2} = 3^\circ.$$

The bearing surface between pivot and socket amounts to

$$f = 3.14 \frac{30 + 25}{2} 50 = 4,320 \text{ mm}^2 \text{ [6.9 sq. ins.]}.$$

Assuming the coefficient of friction between the bearing surfaces to be 0.10, generally it will be much larger, because of the dryness of the bearing surfaces, we shall get

$$p = \frac{P}{f\left(sin \frac{\alpha}{2} + \varphi \, cos \frac{\alpha}{2}\right)} = \frac{1,500}{4,320\,(0.052 + 0.1 \times 0.999)}$$

$$= 2.29 \text{ kg [5 lbs.]}.$$

The pressure on the vertical projection of the journal socket amounts to:

$$R = \frac{30 + 25}{2} \times 50 \times 2.29 = 3{,}148 \text{ kilograms } [6{,}900 \text{ lbs.}].$$

At the lower end of the shaft of 80 millimeters diameter [3.2 ins.], there are two cross sections about 45 millimeters [1.8 ins.] high, and of an average breadth,

$$\frac{80 - 27.5}{2} = 26.25 \text{ mm } [1.05 \text{ ins.}],$$

which resist the splitting tendency. Together these cross sections have an area equal to

$$45 \times 2 \times 26.25 = 2{,}363 \text{ mm}^2 \ [3.78 \text{ sq. ins.}].$$

Consequently the material of the shaft is only subjected to a stress of $\frac{3{,}148}{2{,}363} =$ 1.33 kilograms [1,890 lbs. per sq. in.] by the wedge-like pressure of the pivot, provided the coefficient of friction is not more than 0.10.

§ 12.—Wooden Axles are now comparatively seldom constructed. They are, to be sure, still used for axles of water wheels in wooded regions, where the cost of the material is small and transportation of little account. Their application is also justified in those cases where they are required to absorb heavy shocks, for example, in the axles of trip hammers. In Vol. I., § 375, the superiority of wood for resisting living forces, on account of its large resilience, has been noted, and we find consequently in such hammers, oak axles of considerable length (6 to 8 meters or 19.7 to 26.2 ft.), the cross sections of which are much larger than the rules for resistance to flexure require, because shocks must be principally absorbed or resisted by masses. In such cases it has been found that wood could not be replaced by any more suitable material. Wood is also to be recommended for shafts of considerable length, which are not subjected to great stress, for example, in the posts of foundry cranes, where, from the height of the building, iron would be too costly. The use of wood for such purposes presupposes, however, a low price and cheap freights, for costly freights generally neutralize the advantage of low price. That the perishable nature of wood is a special disadvantage in substantial and durable constructions, is evident.

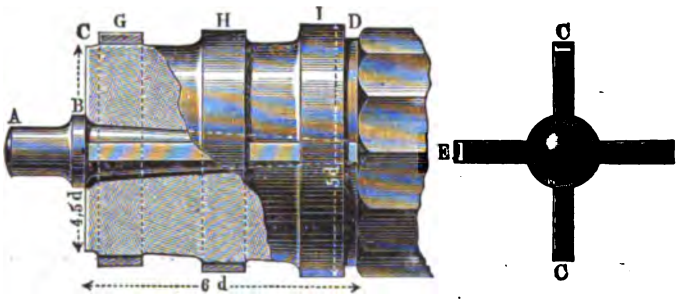
Wooden axles are almost always made cylindrical or prismatic, and consequently of the same cross section throughout their whole length; the cross section being either a circle or a polygon of eight or more sides (rectangular axles seldom occurring except in crane posts). We can, therefore, assume a circular cross section in calculating the strength, and then have

$$M = k\frac{W}{e} = k\frac{\pi d^3}{32},$$

where *d* represents the diameter of the inscribed circle. The permissible stress *k* per square millimeter [per square inch] we can assume, according to circumstances and the intended durability of the construction, say between 0.75 and 1 kilogram [1,070 and 1,420], for sound oak. *Reuleaux* prescribes that the diameter be taken 1.55 times as large as for a cast-iron shaft working under the same conditions. This corresponds to a stress *k* which is $1.55^3 = 3.724$ times smaller than the value $k = 3$ kilograms [4,270 lbs.], usually assumed for cast iron, *i.e.* for oak $k = 0.8$ kilograms [1,140 lbs.]. When the axle is provided with sockets, say, for fastening the arms of water wheels, it is of course necessary to take account of these diminutions of the cross sections, by choosing a correspondingly greater diameter.

The journals of wooden shafts are always made of iron, both for the sake of durability, and to diminish friction. Only for small forces are hooked and pointed wrought-iron journals used (see Vol. I., § 194). The cross-shaped journal, Fig. 55, is most often used because it allows of a very firm connection with the wooden shaft. The journal is of cast iron because it would be difficult to make of wrought iron. The real journal *A*, or the pin *A B*, is united by means of a collar *B* to the two wings *C C* and *E E*, which are at right angles to each other, and fit into two suitable slots of the shaft *CD*. By means of three wrought-iron rings, *G*, *H*, and *I*, the four tines formed on the shaft by these slots are pressed well against the wings, by being shrunk on to the shaft while in a red-hot condition. The contraction of the rings in cooling, presses the wood with great force against the cross-shaped end of the journal.

Fig. 55.



For the dimensions of this journal *Wiebe*[*] gives the following proportions, *d* representing the diameter of the journal:

Length of journal, $l = \frac{4}{3} d$;
Diameter of collar, $d_1 = \frac{1}{3} d$;
Length of wings, $6 d$;
Greatest diameter of wings, $5 d$;
Smallest " " " $4.5 d$;
Thickness of wings, $\frac{1}{8} d + 6.5$ mm [$\frac{1}{8} d + 0.26$ in.].

Many other constructions have been devised for the journals of wooden axles, but have not come into use because they are inferior both in simplicity and security to the cross-shaped journal (see reference to Wiebe just given).

§ 13. **The Flexure of Axles.**—In the foregoing the dimensions of axles have been determined with reference to their strength, that is in such a way that the greatest stress occurring in the fibres should not exceed at any place a certain maximum permissible stress k.

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[*] Die Lehre von der einfachen Maschinentheilen, I., § 107.

The deflection occurring in axles subjected to such a stress could be neglected in the foregoing discussion, because the deflections could be made sufficiently small by assuming the length and permissible stress as small as possible. Sometimes, say in order that a more certain transmission of power may be effected, it is necessary to limit the deflection, for example, to so proportion the axle that the greatest deflection s shall not exceed a certain aliquot part of the length l. The dimensions thus obtained are to be chosen, if they are larger than those obtained from the formulas for strength given above.

By using the expressions developed in Vol. I., § 235, etc., for the deflection of loaded beams, the required calculation can easily be made, and we will here make it only for the oft-recurring case in which the deflection at the middle of an axle supported by two journals is to be determined. Let l designate the whole length of the axle between the bearing centres, and P the load in the middle, then will the deflection s be given by:

$$s = \frac{P l^3}{48\, W E} \text{ (Vol. I. § 241),}$$

in which E represents the modulus of elasticity. If now it be required that the deflection s shall never exceed the value of αl, where α is a small fraction, we shall get the dimensions of the required cross section from:

$$W = \frac{P l^2}{48\, \alpha E}.$$

The formula for strength, however, gives:

$$M = \frac{P l}{4} = k \frac{W}{e},$$

or

$$W = \frac{Pl \, . \, e}{4k}.$$

Putting these two expressions for W equal to each other, we get:

$$\frac{l}{48 \alpha E} = \frac{e}{4k},$$

or

$$\frac{l}{2e} = \frac{6 \alpha E}{k}.$$

The expression gives that limiting value of the ratio between the length l and the greatest dimension $2e$ of the cross section which corresponds to the case in which both formulas give the same value for W. When $\frac{l}{2e}$ is larger, the calculation has to be made for deflection; when $\frac{l}{2e}$ is smaller, the calculation must be made for strength.

If we assume $\alpha = 0.001$, we shall obtain this limiting value:

For wrought iron, when $E = 19{,}700$; $k = 6$ [$E = 28{,}000{,}000$, and k 8,500],

$$\frac{l}{2e} = 19.7 = \text{about } 20.$$

For cast iron, when $E = 10{,}000$ and $k = 3$ [$E = 14{,}220{,}000$, and $k = 4{,}250$],

$$\frac{l}{2e} = 20.$$

For wood, when $E = 1{,}100$ and $k = 0.8$ [$E = 1{,}560{,}000$, and $k = 1{,}140$],

$$\frac{l}{2e} = 8.25.$$

Consequently, under the assumption made, iron axles must be calculated with reference to deflection when their length exceeds 20 times their greatest depth, and wooden axles when their length exceeds 8.25 times the depth. For every other case of permissible stress, and for all other permissible values of α, the calculation can easily be made in a similar manner.

§ 14. **Shafts and Shafting.**—A shaft is an axle which is used for the purpose of transmitting the rotary motion of a body fixed to it to another also fastened to it, or to transmit the motion of rotation from one point to another.

When used in shops, for the transmission of motion from the motor to the individual machine, it is known as shafting; it is consequently a very important element in machinery.

A shaft is supported by bearings, which permit its rotation. When the length is great, as is so often the case, it is generally not sufficient to support the shaft only at the ends, as in axles, but it must have intermediate supports. The shaft therefore receives, according to its length, a greater or less number of journals, which are all neck journals, with perhaps the exception of the end journals; that is, they are such as transmit the moment of the force, and are subjected by the latter to a torsional strain. Each one of the end journals, moreover, can also be a neck journal, when the pulley or wheel receiving or transmitting the power lies outside of the journal. If the pulley or wheel is attached to that side of the end journal toward which the shaft extends, the journal is properly an end journal, which is not subjected to torsion (except in very slight degree, due to the moment of friction on the journal), and is therefore to be treated as a journal for an axle.

Shafting is not only subjected to torsion due to the transmitted moment of the force, but is also subjected to a bending stress due to its own weight and the weight of parts fastened to it, as wheels, etc. The pressures of toothed wheels, the tension of belts, etc., likewise tend to bend the shafting. This last stress is generally small, however, for shop shafting, in comparison with that due to torsion, and can be generally neglected, because usually there is opportunity to put bending weights and forces *close to the bearings*, by which means the bending moment is diminished. The determination of the number of bearings and their distance apart, will likewise be governed by the distribution of the bending forces. It nevertheless happens that sometimes the influence of the bending forces is not to be neglected in comparison with those which cause turning; for instance, this is often the case with shafts of water wheels and steam engines, though it more rarely occurs in shafting proper. When the bending force is not neglected, the shaft must be treated according to the rules for compound stress, as will be hereafter shown in an example.

The method of determining the dimensions of a body subjected to a twisting moment has already been shown in Vol. I., § 269.

If we let t represent the greatest shearing stress of the material, and W the measure of the twisting moment of the cross section (its polar moment of inertia), and e the distance of the outermost fibre from the axis of rotation, we shall have (Vol. I., § 264):

$$Pa = t\frac{W}{e},$$

where P represents the twisting force, a its lever-arm, and, consequently, Pa the twisting moment. Let us now represent, as heretofore, by k the greatest permissible tensile stress of the material of the shaft; an equation obtained from the theory of shearing will then give us the following value for the permissible shearing stress t:

$$\frac{t}{k} = \frac{m}{m+1} = \frac{4}{5},$$

or,

$$t = \frac{4}{5}k.$$

Assuming the same values for k as in axles, viz.:

	WROUGHT IRON.	CAST IRON.	WOOD.
$k=$	6 kg [8,500 lbs.]	3 kg [4,250 lbs.]	0.8 kg [1,140 lbs.]

we get the same security against torsion by taking $t=\frac{4}{5}k$, which gives the following table:

	WROUGHT IRON.	CAST IRON.	WOOD.
$t=$	4.8 kg [6,820 lbs.]	2.4 kg [3,410 lbs.]	0.64 kg [910 lbs.]

Introducing these values, and remembering that the twisting moment, or polar moment of inertia, of a circular cross section with diameter d is given by

$$W=\frac{\pi d^4}{32},$$

and

$$\frac{W}{e}=\frac{\pi d^3}{16},$$

we shall have for round shafting of wrought iron:

$$d=\sqrt[3]{\frac{16}{4.8\,\pi}Pa}=1.02\sqrt[3]{Pa}\quad[d=0.0907\sqrt[3]{Pa}];$$

of cast iron:

$$d=\sqrt[3]{\frac{16}{2.4\,\pi}Pa}=1.28\sqrt[3]{Pa}\quad[d=0.114\sqrt[3]{Pa}],$$

and of wood:

$$d=\sqrt[3]{\frac{16}{0.64\,\pi}Pa}=2.0\sqrt[3]{Pa}\quad[d=0.178\sqrt[3]{Pa}].$$

Cross sections other than circular seldom occur in shafting; when they do occur the calculation may be readily made by substituting in the above formulas the corresponding values of $\frac{W}{e}$; for the particular case in which the cross section is a square having b for its side, we get:

$$W = \frac{b^4}{6}; \quad e = \frac{b}{2}\sqrt{2}; \quad \frac{W}{e} = \frac{b^3}{3\sqrt{2}} = 0.236\, b^3.$$

When the cross section is annular, with diameters d_1 and d_2 we have:

$$\frac{W}{e} = \frac{\pi (d_1^4 - d_2^4)}{16 d_1}.$$

It is often the case in shafting that the twisting moment Pa is not directly given, but we have given instead the work in horse-powers and the speed in revolutions per minute. A horse-power being equal to 75 meter-kilograms, or 75,000 millimeter-kilograms [6,600 inch-lbs.] per second, we shall have for the relation between the work L per second and the pressure P acting with the lever-arm a, the following expression:

$$L = P \times a\pi \frac{n}{60} = 75{,}000\, N \text{mm-kg} \quad [L = 6{,}600\, N \text{in.-lbs.}],$$

N representing the horse-power transmitted, and n the number of revolutions per minute; we now get

$$Pa = \frac{75{,}000 \times 60}{2\pi} \frac{N}{n} = 716{,}200 \frac{N}{n} \quad \left[Pa = 63{,}000 \frac{N}{n}\right].$$

Substituting this value of Pa in the above formulas for d, we get, for wrought iron:

$$d = 1.02\, \sqrt[3]{716{,}200}\, \sqrt[3]{\frac{N}{n}} = 91.3 \sqrt[3]{\frac{N}{n}}, \quad \left[d = 3.65 \sqrt[3]{\frac{N}{n}}\right];$$

for cast iron:

$$d = 1.28\sqrt[3]{716{,}200}\sqrt[3]{\frac{N}{n}} = 115\sqrt[3]{\frac{N}{n}} \quad \left[d = 4.6\sqrt[3]{\frac{N}{n}}\right];$$

and for wood,

$$d = 2.0\sqrt[3]{716{,}200}\sqrt[3]{\frac{N}{n}} = 179\sqrt[3]{\frac{N}{n}} \quad \left[d = 7.16\sqrt[3]{\frac{N}{n}}\right].$$

In calculating the dimensions of a shaft subjected to torsion we can use either the set of formulas which have d expressed in terms of P and a, or the set which express d in terms of N and n. Both sets will give the same result provided P is a constant force acting at the extremity of a constant lever-arm, as is, for example, the case in a turbine, where P is the pressure of the water against the buckets of a wheel with radius a. Here the work performed by the *constant* pressure P at the end of the invariable radius a is given by the equation

$$75{,}000\,N = P\,2\,a\,\pi\frac{n}{60}, \quad \left[6{,}600\,N = P\,2\,a\,\pi\frac{n}{60}\right].$$

The two sets of formulas will no longer give the same result, however, when either the turning force P or the lever-arm a, or both simultaneously, have varying values, as is the case in a steam-engine crank. Here we have not only the varying pressure of steam during expansion, but an effective lever-arm which varies from zero at the dead point up to the full radius of the crank. Were we to calculate the crank-shaft from the formulas containing $\frac{n}{N}$, we should get too small a diameter for the shaft, since the expression $\frac{N}{n}$ corresponds only to the mean value of the twisting moment. In all such cases as the present one we must use in the calculation the maximum twisting moment, which can be found for each special case by an examination into the relations between the crank and the action of the steam. Similar remarks apply, of course, not only to steam engines, but to crank trains in general, such, for example, as are used to work pumps.

§ **15. Angle of Twist.**—The formulas found in the previous paragraph for the diameters of shafts take no account of the angle of torsion, having been developed only subject to the condition that the greatest stress occurring in any fibre should not exceed a certain permissible value, t. In shafts which are, however, both small and long, the angle of torsion can assume a value inconsistent with an efficient and sure transmission. In such cases, therefore, it is not sufficient to determine the diameter of the shaft according to the above formulas for strength, but the diameter must be determined with reference to the greatest permissible value of the angle of torsion. According to *Reuleaux* the value of this angle expressed in degrees, for shafts not longer than 3 meters [9.84 ft.], is to be assumed equal to $\alpha^\circ = \frac{L}{4}\left[\alpha^\circ = \frac{L}{13.1}\right]$, where L represents the length of the shaft in meters [feet], consequently

$$\alpha^\circ = \frac{l}{4{,}000} = 0.00025\, l^* \ [\alpha^\circ = 0.0063 l],$$

when l is expressed in millimeters [inches]. In Vol. I., § 263, there was obtained the formula:

$$Pa = \frac{\alpha W C}{l},$$

where α is the angle of torsion of the shaft expressed in circular measure, and therefore equal to $\frac{\alpha^\circ \pi}{180}$, and C is the modulus of elasticity. Introducing this angle expressed in degrees, the above formula will reduce to;

---

* According to Redtenbacher, the angle of torsion may amount to $\frac{l}{397}$, when l is given in centimeters, that is $\alpha^\circ = 0.000253\, l$ mm.

$$Pa = \frac{\alpha^\circ \pi W C}{180 l},$$

and when the cross section is circular, we get, after substituting

$$W = \frac{\pi d^4}{32},$$

the following:

$$Pa = \frac{\alpha^\circ \pi^2 d^4 C}{180 \times 32 l} = 0.00171 \frac{\alpha^\circ d^4 C}{l}.$$

The modulus of elasticity C for shearing is (Vol. I., § 260, Fifth German Edition), equal to $\frac{2}{5}E$, when E is the modulus of elasticity for tension, and we can therefore put for

Wrought iron,	$C = 8{,}000$	$[C = 11{,}376{,}000]$,
Cast "	$C = 4{,}000$	$[C = 5{,}688{,}000]$,
Wood,	$C = 400$	$[C = 568{,}800]$.

Introducing these values, and assuming

$$\alpha^\circ = \frac{l}{4{,}000} = 0.00025\, l \quad [\alpha^\circ = 0.0063\, l],$$

we shall get for wrought iron,

$$d = \sqrt[4]{\frac{l P a}{0.00171 \times 0.00025\, l \times 8{,}000}} = 4.13 \sqrt[4]{Pa}$$

$$[d = 0.3 \sqrt[4]{Pa}];$$

for cast iron,

$$d = \sqrt[4]{\frac{l P a}{0.00171 \times 0.00025\, l \times 4{,}000}} = 4.92 \sqrt[4]{Pa}$$

$$[d = 0.357 \sqrt[4]{Pa}];$$

and for wood,

$$d = \sqrt[4]{\frac{l P a}{0.00171 \times 0.00025\, l \times 400}} = 8.75 \sqrt[4]{Pa}$$

$$[d = 0.63 \sqrt[4]{Pa}].$$

If we wish to determine the dimensions from the horse-power N, and the revolutions n, it will only be necessary to introduce,

$$Pa = 716{,}200 \frac{N}{n} \quad \left[Pa = 63{,}000 \frac{N}{n}\right],$$

and we shall get, for wrought iron,

$$d = 120 \sqrt[4]{\frac{N}{n}} \quad \left[d = 4.80 \sqrt[4]{\frac{N}{n}}\right];$$

for cast iron,

$$d = 143 \sqrt[4]{\frac{N}{n}} \quad \left[d = 5.72 \sqrt[4]{\frac{N}{n}}\right];$$

and for wood,

$$d = 255 \sqrt[4]{\frac{N}{n}} \quad \left[d = 10.20 \sqrt[4]{\frac{N}{n}}\right].$$

In order to ascertain when the formula for strength will give the same value for d as that for stiffness, it is only necessary to put the two expressions for d equal to each other and deduce the corresponding value for Pa, or $\frac{N}{n}$. When this is done, we get for wrought iron,

$$d_0 = 91.3 \sqrt[3]{\frac{N}{n}} = 120 \sqrt[4]{\frac{N}{n}};$$

from which we have,

$$\frac{N}{n} = \left(\frac{120}{91.3}\right)^{12} = 26.6,$$

and this value substituted in either of the two formulas will give $d = 273$ mm [10.9 ins.]. The very same value for d is also obtained for cast-iron shafts when we make

$$d_0 = 115\sqrt[3]{\frac{N}{n}} = 143\sqrt[4]{\frac{N}{n}}.$$

From this follows that for all iron shafts having a diameter less than $d_0 = 273$ mm [10.9 ins.], the formulas for stiffness must be used, since these give the greater values, and for diameters greater than d_0 the formulas for strength must be used.

For wooden shafts the corresponding limiting value for d_0 is obtained from;

$$d = 179\sqrt[3]{\frac{N}{n}} = 255\sqrt[4]{\frac{N}{n}},$$

which gives:

$$\frac{N}{n} = \left(\frac{255}{179}\right)^{12} = 73.5,$$

and this introduced into both formulas will give the same value $d = 749$ mm [29.75 ins.]. Since larger diameters probably never occur, it will be well to always calculate wooden shafts from the formulas for stiffness.

In very short shafts, as in winches used in hoisting apparatus and worked by hand, the angle of torsion α is quite small, and may be neglected, because the shaft is short; in such cases the formulas for strength must always be used.

EXAMPLE I. What should be the diameter of the shaft of a drum for a crane whose chain exerts a pull of 4,000 kilograms [8,800 lbs.] at the circumference of the drum, the radius being equal to 0.20 meter [8 ins.]?

Since, on account of the shortness of the drum, the angle of torsion need not be considered, we can obtain the diameter from

$$d = 1.02 \sqrt[3]{4,000 \times 200} = 94.7 \text{ mm } [3.79 \text{ ins.}].$$

2. What diameter must be given to the wrought-iron shaft of a turbine transmitting 30 horse-powers at 45 revolutions per minute?

We must here take, according to the formula for stiffness,

$$d = 120\sqrt[4]{\frac{30}{45}} = 108.5 \text{ mm } [4.34 \text{ ins.}].$$

3. What should be the diameter of a vertical wooden shaft turned by four horses at the average rate of two revolutions per minute?

If we neglect the bending stresses, we shall get, by using the formula for stiffness against twisting, for the required diameter,

$$d = 255\sqrt[4]{\frac{4}{2}} = 304.8 \text{ mm } [12.2 \text{ ins.}].$$

§ **16. Twisting and Bending of Shafts.**—When a shaft is subjected to torsion, and is at the same time subjected to bending forces which are not so small that they may be neglected, the diameter of the shaft must be determined according to the rules for compound stress, which were developed in Vol. I., § 277. This case principally occurs in the shafts of water-wheels and steam-engines. Let M_1 represent the bending moment, and M_2 the twisting moment, for any cross section whose moment of inertia is W, then the required dimensions of this cross section will be obtained from the relation (see Vol. I., § 284, Fifth Ger. Ed.);

$$\tfrac{3}{8} M_1 + \tfrac{5}{8} \sqrt{M_1^2 + M_2^2} = k \frac{W}{e},$$

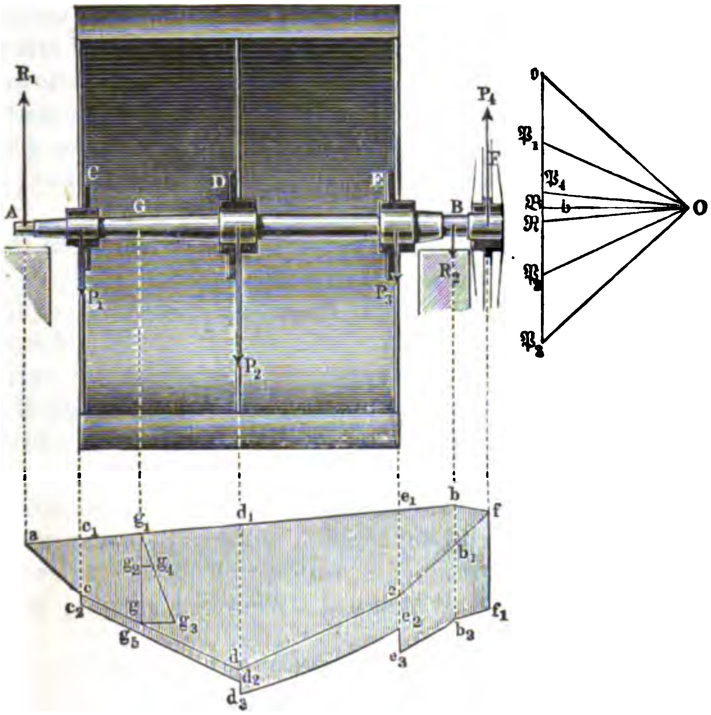
where k represents the greatest permissible bending stress. In order to determine the bending and twisting moments, and from these the above expression

$$\tfrac{3}{8} M_1 + \tfrac{5}{8} \sqrt{M_1^2 + M_2^2}$$

for the various cross sections, we can with advantage again use the methods of graphical statics, as will be illustrated in the following example.

Let *A B*, Fig. 56, be the axle of an overshot water-wheel, and let *C*, *D*, and *E* be the naves to which the three systems of arms are fastened, and let a toothed wheel be attached at

FIG. 56.



F, which transmits the power to the shafting. The weight of the water-wheel and of the water contained in its buckets is transmitted through the systems of arms at *C*, *D* and *E*, to the axle, and we may represent the forces acting at these points by P_1, P_2 and P_3. If the central system of arms at *D* is situated symmetrically with reference to those at *C* and *E*, we can assume that half the total weight is acting at *D*, and a quarter at each of the points C and E. The power of the water being transmitted through the wheel F, whose radius is r, there will be a pressure of P_4 on the teeth of the wheel F, which will likewise tend to bend the axle and will in addition exert a twisting moment $M_2 = P_4 r$ on the axle. We must here remark that it is not the whole shaft between F and A which is subject to twisting, but only the part between F and C, and even this part is not at all points equally subjected to torsion. We must assume that the power of the water is transmitted to the axle through the three systems of arms C, D and E, and that one half is transmitted at D, and one quarter at each of the points C and E. Consequently the part of the axle between E and F will be subjected to the whole twisting moment $M_2 = P_4 r$, that between D and E to $\frac{3}{4} M_2$, and that between C and D to $\frac{1}{4} M_2$.

Now, in order to determine the moments, and from them the dimensions, we will construct from the forces P_1, P_2, P_3 and P_4, the force polygon $\mathfrak{o}\,\mathfrak{P}_1\,\mathfrak{P}_2\,\mathfrak{P}_3\,\mathfrak{P}_4$, and then draw the equilibrium polygon $a\,c\,d\,e\,f\,b\,a$ for a pole, O, assumed at a distance equal to the base of the moments. The ray drawn parallel to the closing line $b\,a$, gives at the point of intersection $\mathfrak{R}$ the two reactions $R_2 = \mathfrak{P}_4\mathfrak{R}$ at B, and $R_1 = \mathfrak{R}\,\mathfrak{o}$ at A. In the present case in which the force P_4 acting vertically upward (the train of gearing being situated on the same side as that portion of the wheel which is filled with water, the reaction R_2 is downward, *i.e.* the journal B is pressed up against the cap of the pedestal during the running of the wheel.

The ordinates $c\,c_1$, $d\,d_1$, etc., of the equilibrium polygon $a\,c\,d\,e\,f\,b\,a$ are proportional to the bending moments M_1 for the corresponding cross sections, if we regard the pole distance $\mathfrak{B}\,O = b$, as unity. If we know the whole twisting moment $M_1 = P_4 r$, we can reduce it to the same unit b by putting

$$P_4 r = P_t b, \qquad \text{and} \qquad P_t = P_4 \frac{r}{b},$$

which can be determined by a simple construction. (If we had made the base of moments, $\mathfrak{B}O = b$ equal to the radius r of the wheel F, we should have $P_t = P_4$.)

Now if we know P_t, *i. e.*, the force which, acting at the end of a lever-arm $\mathfrak{B}O = b$, corresponds to a twisting moment $M_2 = P_4 r$, we can easily find the moment of the combined stress, by constructing the expression

$$\frac{3}{8} M_1 + \frac{5}{8} \sqrt{M_1^2 + M_2^2}.$$

In order to make this construction for any cross section, for example for the one at G, divide the corresponding ordinate gg_1 of the equilibrium polygon at g_2, so that $g_1g_2 = \frac{3}{8} gg_1 = \frac{3}{8} M_1$, then draw gg_3 perpendicular to gg_1, and equal to $\frac{1}{4} P_t = \frac{1}{4} M_2$, and draw g_3g_1 and divide it at g_4 so that

$$g_4g_3 = \frac{5}{8} g_1g_3 = \frac{5}{8} \sqrt{M_1^2 + \left(\frac{1}{4} M_2\right)^2},$$

which can be easily done by drawing g_2g_4 parallel to g_3. Now lay off g_4g_3 from g_2 along g_2g_5, we shall then get in g_1g_5 the ideal bending moment

$$\frac{3}{8} M_1 + \frac{5}{8} \sqrt{M_1^2 + \left(\frac{1}{4} M_2\right)^2},$$

always under the supposition that the pole distance $\mathfrak{B}O = b$ is the base of moments. If we make this construction for a sufficient number of points, we shall get a combined equilibrium polygon $a\,c\,c_2\,d_2\,d_3\,e_2\,e_3\,b_2\,f_1\,f\,b\,a$, of which the upper part, $a\,c\,d\,e\,f\,b$ corresponds to the bending forces P_1 P_2 P_3 and P_4, while the lower part $c\,c_2\,d_2\,d_3\,e_2\,e_3\,b_2\,f_1\,f\,e\,d\,c$ shows the influence of the twisting moment M_2. The sudden increase d_2d_3 and e_2e_3 at the points d and e of this latter portion of the equilibrium polygon, depends, of course, on the peculiar transfer of power to the shaft at C, D and E, already explained above, and it is therefore necessary to so conduct the construction $g g_1 g_3$, that the horizontal perpendicular $g g_3$ shall equal $\frac{1}{4} P_t$ between the points c and d, equal to $\frac{3}{4} P_t$ between d and e, and equal to P_t between e and f. The determination of the required dimensions now follows in the manner so often described, by means of the formula:

$$k \frac{W}{e} = \frac{3}{8} M + \frac{5}{8} \sqrt{M_1^2 + M_2^2},$$

the right member being given by the ordinates of the combined equilibrium polygon.

§ 17. **Resistance of Shafts to Shocks.**—It is often necessary to discuss the resistant capacity of shafts to living forces. It is then of less importance that the greatest stress in the fibres or the angle of torsion shall not exceed certain values, than that the material of the shaft be capable of taking up the mechanical work due to the change of form within the limit of elasticity.

This straining of the shaft by living forces occurs whenever there is a sudden change of velocity, *i. e.*, whenever there is a shock. For example, when the axle of a trip hammer in rapid motion is struck by a tappet on the tail of the hammer, the axle will be subjected to a shock, which it can only resist by a sufficient mass. Let us designate by M the mass of the axle and that of the water wheel reduced to the point at which the tappet acts, and let M_1 be the mass of the hammer, helve and ring, reduced to the same point, and let c be the velocity of the tappet at the moment of impact; we then have, according to Vol. I., § 335, for the loss of mechanical work

$$L = \frac{M M_1}{M + M_1} \frac{c^2}{2}.$$

This work, which tends to destroy the impinging parts, must in every case be absorbed by the elastic action of these parts. Since the small mass in the helve of the hammer cannot take up a large amount of work, the shaft must take up much the greater part of the mechanical work, and is consequently subjected to a wrenching strain. We will therefore assume that the total loss of work due to the shock is to be absorbed by the shaft. The mechanical work necessary to twist a prismatic shaft whose length is l and whose polar moment of inertia is W, is expressed by (see Vol. I., § 262 and § 264)

$$L = \frac{S^2}{C} \frac{Wl}{2e^2},$$

where S represents the greatest working stress, and C the modulus of elasticity for shearing. If we now substitute for S the greatest permissible shearing stress t, we shall get the equation

$$L = \frac{MM_1}{M+M_1} \frac{c^2}{2} = \frac{t^2}{C} \frac{Wl}{2e^2},$$

which will serve to determine W and l, or the volume V of the shaft. For a circular cross section, in particular, we have

$$W = \frac{\pi d^4}{32}, \quad \text{and} \quad e = \frac{d}{2},$$

consequently for this case

$$L = \frac{MM_1}{M+M_1} \frac{c^2}{2} = \frac{t^2}{C} \frac{\pi d^4 l}{32 \times 2 \left(\frac{d}{2}\right)^2} = \frac{t^2}{C} \frac{\pi d^2 l}{16} = \frac{t^2}{4C} V,$$

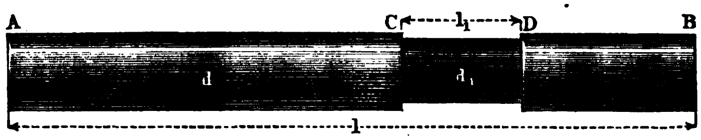
where V is the volume $\frac{\pi d^2}{2} l$ of the shaft.

In like manner, as in hammers, a shock occurs when a shaft at rest is suddenly connected with a shaft going at high speed, by means of a *rigid* coupling (see couplings), and the calculation is to be made in precisely the same manner. A shock occurs also when a rotating axle is suddenly checked by a great resistance, as is the case in stamping presses, where the resistance which the metal offers to the punch or die absorbs a portion of the living force of the stamping apparatus, while another portion twists the shaft, etc.

§ 18. **Influence of Contractions.**—From the above formulas we see that the torsional resistance of a shaft to a shock is proportional to the volume $V = \frac{\pi d^2}{4} l$, which is likewise the case for the tensile and flexile resistance to shock.

We must here remark that the formula $L = \frac{t^2}{4C} V$ is only true for the condition that the material be subjected to the same stress t in the outermost fibres of all the cross sections, which can only be the case when the shaft is cylindrical, *i. e.*, of constant diameter. Every deviation from the cylindrical form, namely, every diminution caused by cutting journals into the shaft, as is so often done, lessens in a marked degree the resistance which the shaft can oppose to the action of shocks. To illustrate this point let $A B$, Fig. 57, be a cylindrical shaft, whose diameter is d and length l. If this shaft were of the same diameter throughout its whole length, every fibre on the surface would be subjected to the same stress, say the greatest permissible one t, whenever the shaft is twisted. The mechanical work necessary to cause a twisting which will produce the stress t in every fibre, *i. e.*, the mechanical work which the shaft can absorb, is, according to what has preceded, equal to

FIG. 57.



$$L = \frac{t^2}{4C} \frac{\pi d^2}{4} l = \frac{t^2}{4C} V.$$

Now let us suppose that the shaft is turned down at CD, for the purpose, say, of forming a journal, the turned-down portion being of a length l_1, and having a diameter $d_1 = \nu d$; it will then be much less capable of resisting living forces. Let us again imagine the shaft to be twisted to such a degree that the outermost fibres in CD, are subjected to the permissible stress t, the stress in the stouter portion of the shaft will then not be t, but a smaller value t', which may be calculated from the general formula for torsion,

$$Pa = t \frac{\pi d_1^3}{16} = t' \frac{\pi d^3}{16},$$

or

$$t' = \frac{d_1^3}{d^3} t = \nu^3 t.$$

The mechanical work performed by the shaft during this twisting is composed of the work performed by the piece CD, with diameter d_1 and length l_1, when subjected to the stress t, and that work which is performed by the rest of the shaft $AC + DB$ with diameter d and length $l - l_1$, when its greatest stress is $t' = \nu^3 t$, which total work can be written

$$L_1 = \frac{t^2}{4C} \frac{\pi d_1^2 l_1}{4} + \frac{\nu^6 t^2}{4C} \frac{\pi d^2 (l - l_1)}{4}.$$

Since the length of the portion $CD = l_1$ is insignificant, as compared with the length l of the whole shaft, the first term of the equation may be neglected, and in the second term we may substitute l for $l - l_1$, and will then get

$$L_1 = \nu^6 \frac{t^2}{4C} \frac{\pi d^2 l}{4} = \nu^6 \frac{t^2}{4C} V.$$

The shaft can now only absorb a living force $L_1 = \nu^6 L$.

If we assume for example, $d_1 = 0.8\, d$, the shaft will have a capacity for resisting shock which is $0.8^6 = 0.266$ less than the original shaft before being partially turned down. It is an interesting fact, that the diminution of the shaft at *one* place CD, diminishes its resisting capacity even more than would be the case if the shaft were turned down *along its whole length* to the diameter $d_1 = \nu d$. In this case the stress t occurring in its fibres, would take place simultaneously in the outermost fibres of all the cross sections, exactly as in the original, and the diminution of its capacity for resisting shocks or living forces, would only be the result of its diminished volume. For this case, therefore, we should have for the living force absorbed:

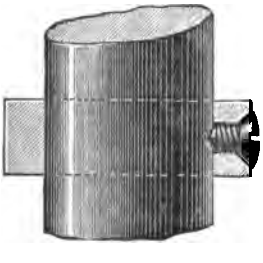
$$L_2 = \frac{t^2}{46} \frac{\pi (\nu d)^2 l}{4} = \nu^3 \frac{t^2}{4C} V;$$

or putting $\nu = 0.8$:

$$L_2 = 0.64\, L_1.$$

These results sufficiently explain the importance of the rule, that in all shafts subjected to the action of shocks, all changes of cross section, and particularly all sudden sharply formed cuts should be avoided, and that the shoulders on the journals should never be made by turning down the shaft, but by welding or screwing on collars. An example of collar fastened by set screw is shown in Fig. 58. The wrought-iron collar AB is cylindrical, and is accurately fitted to the shaft C, to which it is fastened at the proper place, by means of a steel set screw D, with hardened point.

FIG. 58.



§ 19. **Arrangement of Shafting.**—In arranging the shafts of a shop, we must always try to make the diameter of the shaft as small as considerations of durability will allow.

Larger shafts not only increase the weight of the shafting, bearings, couplings, etc., thus increasing the first cost, but also consume a greater amount of energy by the greater amount of friction due to the larger journals. This last circumstance is of the greater importance because the shafting is often of extreme length when an extensive shop is to be provided with power. In arranging lines of shafting, therefore, all the circumstances which influence the diameter of the shaft must receive due consideration. We will, in what follows, consider more closely the conditions which most influence these important dimensions.

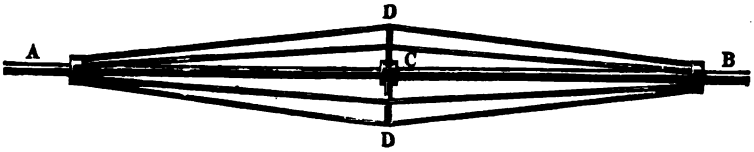
First of all, as regards the material, it is customary to make shafting of wrought iron; only in rare cases, say where great forces are to be transmitted, is cast-iron shafting used. This, however, is to be avoided if possible, for its larger journals cause great loss of work by friction, and the small resilience of cast iron does not offer sufficient security against shocks. We therefore usually find cast-iron shafting only in those establishments which were built at a time when the production of rolled iron of large dimensions was accompanied by great difficulties and high cost.

Since at the present time we are deviating more and more from the formerly popular system of centralization, in which the whole power for an extensive establishment was obtained from a central motor, and are setting up several motors as near as possible to the machinery utilizing the power, the causes have disappeared which necessitated large cast-iron shafting. For small sizes of shafting cast iron is *never* used, because of its liability to break; and, in like manner, wemust regard the wooden shafting still found in old mills as historical reminders of a past age.

From what has been established in previous paragraphs, we see that the size of the shafting must usually be determined with reference to its resistance to torsion, since the bending forces due to its own weight and that of the machine parts fastened to it have generally little influence. It is only necessary that the shafting be supported by bearings in a sufficient number of places, and we can assume that the distance apart of the bearings should be from 2 to 2½ meters [6.56 to 8.2 feet]. Besides the weight of the shaft and such machinery as wheels, pulleys, etc., other forces tend to bend the shafting, for example the pressures exerted at the circumference of toothed wheels, and the tension on belts and ropes. With reference to these bending forces, it will be well to put the wheels, pulleys, etc., transmitting the larger forces, whenever it is possible, close to the bearings; in very large pressures, such as occur in toothed wheels, the particular wheel is often put between two bearings specially made for it. At all events we should never put a coupling (see Couplings) which unites two shafts midway between two bearings at the ordinary distance apart, but put it as closely as possible to one of the bearings, or, when the forces are very large, put a bearing on each side of it.

Sometimes it is necessary for shafting to cross long spaces in which it is not convenient to place bearings, as, for example, in leading shafting over passageways, or through open workrooms, in which we cannot or will not put up pillars; we then use, in order that the shafting be not too large, *trussed shafts*, Fig. 59. In this case the shaft *A B* is provided at the middle of the unsupported portion with a star-shaped piece having from four to six arms or spokes, the ends of which are provided with rings, through which the tie rods *A D B* are drawn. This arrangement allows the shaft to be unsupported for a considerable distance, but power cannot be taken from it between the points *A* and *B*.

FIG. 59.

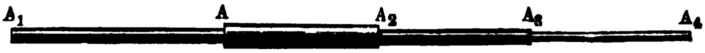


§ 20. **Distribution of Power by Shafting.**—According to the above formulas for strength and stiffness with respect to torsion, it is evident that, other things being equal, the shaft must be made the larger the greater the value $\frac{N}{n}$. Here N must be understood to represent the number of horse-powers which are to be transmitted by the portion of the shafting under consideration, but not the whole power received by the shafting. The only case in which N represents the whole power is when the shaft receives at one end N horse-powers, and transmits them all to the other end. In this case the shaft is subjected, by the force received, to a twisting throughout its whole length. But if we suppose the shaft to receive N horse-powers at some intermediate point, and of this amount N_1 horse-powers be transmitted to one end, and $N_2 = N - N_1$ horse-powers to the other end, then of course the first portion of shafting will be subjected to a stress measured by N_1, and the second by a stress measured by N_2 horse-powers. This gives rise to the practical rule, to arrange shafting, wherever possible, so that the motor will communicate its power to the shaft at some point situated near the middle of the latter, thus permitting the transmitted force to be distributed in both directions. This allows a smaller diameter of shaft, and consequently the weight, the loss of work through friction, wear of bearing and consumption of lubricants will be less than if the motor communicated its power at one end of the shafting.

The usual requirements of technical establishments are such that the power received from the shafting by the motor at a given point is transmitted to each side of the latter, and is given off not at the end of the shaft, but at intermediate points, according to the arrangement of the machinery to be driven. In this case different portions of the shaft will be subjected to different moments of torsion, and the diameters of the shafting can gradually diminish from the point at which the power is received, the diminution of size corresponding to the diminution of the moment of force occurring at any point. For example, if at the point A, Fig. 60, of a shaft, N horse-powers are to be distributed in such a manner that N_1 horse-powers will be received by A, and N_2, N_3 and

N_4 horse-powers successively received at A_2, A_3 and A_4, for the machinery situated at these points, it will be necessary to calculate the size of $A\,A_1$, for a twisting moment corresponding to N_1, the size of $A_3\,A_4$ for N_4, $A_2\,A_3$ for $N_3 + N_4$, and $A\,A_2$ for $N_2 + N_3 + N_4$ horse-powers. Each of these portions of shafting could, therefore, be made of a different diameter. In order, however, that there be not too great diversity of bearings, hub bores, etc., it is usual to make long portions of the shaft of the same diameter, and this is particularly desirable when the number of points $A_2\,A_3\,A_4$ . . . , at which power is taken off, is very great, and when the power taken off at each point is at the same time very small (as in the shafting of spinning rooms, etc.). These considerations give us another practical rule, namely, that in arranging a machine plant, those machines requiring the greatest amount of power should be placed as near as possible to the motor; that is, to the point A, Fig. 60, receiving the power, in order that the diameters and weight of the shafting and the friction be reduced as much as possible. Even if this rule cannot be always strictly followed, on account of the requirement that the position of a machine shall be determined with reference to convenient manufacturing, it is nevertheless desirable when possible to group the most powerful machines nearest the motor.

FIG. 60.



We consequently find the stones in grist mills placed in the immediate neighborhood of the motor, while the lighter machines, such as sieves and hoists, etc., are driven by lines of shafting of more or less length. In like manner in saw mills, the saws absorbing most power are to be driven as directly as possible by the motor, while the spaces farther from the motor must be used for setting up the lightest frame and circular saws. Similar remarks are applicable to spinning factories and other technical establishments.

§ **21. Speed of Shafting.**—The speed of shafting, or number of revolutions, n, has an important influence on the diameter of the shaft, as may be seen from the formula

$$d = c \sqrt[4]{\frac{N}{n}},$$

where the diameter d becomes smaller the greater the number of revolutions n. Although the speed of the shafting is partially determined by manufacturing considerations, there remains in most cases considerable choice as to the speed at which it may be run. In order to determine the influence which the speed has on the transmission with respect to size of shaft and loss of power by friction, we will assume that shafting transmitting N horse-powers at n revolutions, must have a diameter d, and a corresponding weight g. Now suppose that the arrangement be so changed that the shafting makes v times as many revolutions as before, and therefore $v\,n$ revolutions per minute; the formula

$$d = c \sqrt[4]{\frac{N}{n}},$$

will require a diameter

$$d_1 = \frac{d}{\sqrt[4]{v}}.$$

Corresponding to this diameter, the weight of the shaft must now amount to

$$G_1 = G \frac{d_1^2}{d^2} = \frac{G}{\sqrt{v}}.$$

If we now designate *the friction generated by the weight of the shaft itself* by $F = \varphi\, G$, where φ is to be understood as the coefficient of friction, the friction due to v times the original speed will be

$$F_1 = \varphi\, G_1 = \varphi \frac{G}{\sqrt{v}}.$$

Finally, the work L, due to this friction at n revolutions, will be

$$L = \pi d n \,.\, \varphi G,$$

and with v times the speed,

$$L_1 = \pi d_1 v n \,.\, \varphi G_1 = \pi d n \varphi G \sqrt[4]{v} = L \sqrt[4]{v}.$$

From this consideration it follows that the diameter and weight of shafting are diminished by an increased speed, but that the work of friction due to the shaft's own weight is increased in the ratio

$$1 : \sqrt[4]{v}.$$

It must be remarked, however, that the friction due to the shaft's own weight is but a small part of the total friction, the latter being principally due to the very considerable pressures arising in toothed gearing, and in the tensions of belts. Since these pressures for a given horse-power, N, are inversely proportional to the speed n, we see that when P represents the sum of these pressures, the friction from this source will be expressed by

$$L_p = \pi d n \varphi P,$$

while the work of friction due to v times the speed will be

$$L_{p1} = P d_1 v n \varphi P_1 = \pi d n \varphi P \frac{1}{\sqrt[4]{v}}.$$

We see from this that the work of friction and the loss of power and consumption of lubricants will generally be less the more the speed is increased. Since an increase of speed is accompanied by other advantages, namely lighter and less costly wheels, pulleys, bearings, and transmission pieces, by smaller pitches for the teeth, and by narrower belts, etc., it is not customary to run shafting at speeds below a certain limit.

It is usual to run shop shafting at 90 to 120 revolutions per minute, but still higher speeds are employed for rapidly running machinery; on the other hand, for slow running machinery, the speed is sometimes far below these values. Upright shafting in particular, when it extends through several stories, is usually run slowly, seldom more than 60 revolutions per minute. Perfectly general rules concerning this matter cannot, of course, be laid down, since the most suitable arrangement will always be dependent on special circumstances.

The determination of the diameters of shafting must be always obtained from the formula for stiffness (§ 15):

$$Pa = 0.00171 \frac{\alpha^\circ d_4 C}{l}.$$

Since shafting is generally very long, the angle of torsion α° would $[\alpha^\circ = 0.0063\, l]$ be always very large if we were to assume for it the value $\alpha^\circ = \frac{l}{4{,}000}$, above given for shafts. In order that the angle of torsion, even for long transmissions, may be kept within moderate limits, *Reuleaux* gives the rule that the angle α° for shafting should be:

$$\alpha^\circ = \sqrt{\frac{L^{\text{m}}}{8}} = \sqrt{\frac{l^{\text{mm}}}{8{,}000}} \quad \left[\alpha^\circ = \sqrt{\frac{l^{\text{ins.}}}{315}}\right].$$

This assumption should give the same value for α° (namely, $\alpha^\circ = \frac{1}{2}^\circ$) that was given by the former rule,

$$\alpha^\circ = \frac{l}{4{,}000}, \quad \left[\alpha^\circ = \frac{l^{\text{ins.}}}{160}\right],$$

when $l = 2{,}000$ mm $[\,l = 78.74$ ins.$]$. We must understand the symbol l to mean the whole length of the shaft when the force is received at one end of the shaft and given out at the other. But it must be understood as the half length of the shaft when the delivery of power is uniform along the shaft.

In general, l is the distance of the centre delivery from the point of admission of the power, *i. e.*, that centre which is obtained by considering weights as acting at the points of delivery, which weights are proportional to the moments of the forces at these points. This point, S, Fig. 60, would be given by the equation:

$$A\,S = l = \frac{N_2 \,.\, A\,A_2 + N_3 \,.\, A\,A_3 + N_4 \,.\, A\,A_4}{N_2 + N_3 + N_4}.$$

This length being understood as l, and substituting

$$\alpha^\circ = \sqrt{\frac{l}{8{,}000}}, \quad \left[\alpha^\circ = \sqrt{\frac{l}{315}}\right],$$

we get for wrought-iron shafting the formula:

$$P\,a = 0.00171\,\frac{d^4\,C}{l}\sqrt{\frac{l}{8{,}000}}, \quad \left[P\,a = 0.00171\,\frac{d^4\,C}{l}\sqrt{\frac{l}{315}}\right]$$

from which

$$d = 1.60\,\sqrt[4]{P\,a}\,\sqrt[8]{l}, \quad [d = 0.174\,\sqrt[4]{P\,a}\,\sqrt[8]{l}],$$

or

$$d = 1.60\sqrt[4]{716{,}200\,\frac{N}{n}}\,\sqrt[8]{l} = 46.6\,\sqrt[4]{\frac{N}{n}}\,\sqrt[8]{l},$$

$$[d = 2.75\,\sqrt[4]{\frac{N}{n}}\,\sqrt[8]{l}\text{ ins.}].$$

§ **22. Couplings.**—A line of shafting is usually of considerable length, and must, therefore, be composed of several pieces suitably united, for the difficulties of construction, convenience of transportation, and of setting up, forbid its being made in a single piece. It is customary, therefore, to take into account these circumstances of construction and transportation, by forming the shafting of several pieces whose length is usually not more than 5 or 6 meters [16 or 19 ft.]. The ends of the different pieces are united by means of couplings; these force the whole shaft to rotate whenever any one part turns. The various constructions devised for this purpose may be divided into *fixed* and *movable* couplings; the former are such as retain their position, always uniting the shafts, while the latter permit any amount of motion or disengagement whenever a shaft is to be detached from the driving shaft. There is a third class of couplings which, in a certain sense, lies between the other two, in that it transmits motion only so long as the force to be transmitted remains within prescribed limits, the action of the coupling ceasing whenever these limits are exceeded. These couplings are called friction couplings, because the turning of the driven shaft is effected by means of friction; they can be constructed either as fixed or movable couplings.

The fixed couplings may be divided into *rigid* and *flexible*, of which the former unite the two ends of the shaft into one rigid piece, so as to exclude all relative motion between them, while the latter do permit of some slight motion or sliding of the shafts. Rigid couplings can only be applied when the axes of the shafts to be united, lie and remain exactly in the same straight line, for it is evident that the slightest deviation in the direction of the shafts would cause great pressures whenever the shafts were turned, if they were rigidly connected, and that these pressures would not exist if the union of the two permitted a certain amount of motion. If we cannot be sure that the axes of the two shafts will retain their position unchanged, as is the case for example, in rolling mills, where the motion of one shaft relatively to the other is unavoidable, then flexible couplings are applied, as well as in the case of long shafts where the expansion due to changes of temperature must be allowed for.

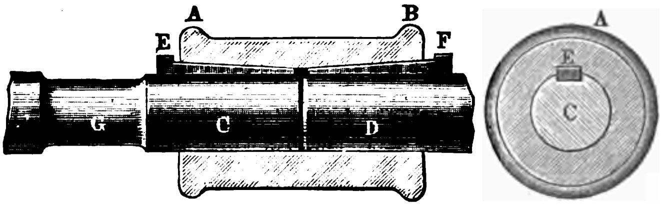
The coupling of two wooden shafts by means of a cast-iron piece, *C*, provided at each of its ends with crossed ribs or leaves, Fig. 61, is now only of historic interest, since wooden shafting is no longer constructed.

FIG. 61.

To unite wrought-iron shafts rigidly by allowing one end to lap over the other, and then uniting them by a bolt, is not at all advisable, because the cross section of the shafts is thereby reduced by one half. If we seek to overcome this disadvantage by a corresponding enlargement of the ends of the shafts, we not only increase considerably the cost of the shafts, which are usually turned out of round bars, but also render it impossible to push the hubs of wheels or pulleys from one position on a shaft to another. For the same reasons those couplings must be rejected, in which one end of the shaft has welded to it a wrought-iron sleeve, into the square aperture of which the end of the other shaft is fitted; it is only for very small forces, or for cast-iron shafts in which the construction of the sleeve is easily accomplished, that this is allowable, although even in this last case it were better to use some one of the couplings about to be described.

§ 23. **Box or Muff Coupling.**—Most of the couplings occurring in practice are so constructed that the ends of the shafts can be united by means of an intermediate piece composed of one or more parts. A simple and consequently often applied coupling is shown in Fig. 62, a cast-iron sleeve or muff *A B* is passed over the two ends of the shafts *C* and *D*, and united with them by the wedges *E* and *F*. These wedges or keys are of rectangular cross section, and move or lie in grooves which are cut partly in the shafts and partly in the muff, thus preventing all turning of the latter with reference to the shafts. When the shaft *C* is turned, the key *E* transmits the motion to the muff *A B*, and this transmits the

FIG. 62.



rotation by means of the key F, to the shaft D. A union of the two shafts in the direction of their length does not exist in this case, and is not necessary, since any tendency of the shafts to move in the direction of their length is so slight that the collars or shoulders of the bearing at G suffice to overcome it. The notching of the two ends of the shaft within the muff, which is sometimes proposed, is unnecessary, and is not advisable because of the diminished strength involved in such a construction. That it is desirable to place such a coupling as near as possible to the bearing G, has been already mentioned.

The sleeve of the coupling is subjected to torsion by the force which twists the shafts, and it is therefore permissible to determine its dimensions by placing the value $t\dfrac{W}{e}$ for the muff, equal to that for the shaft. If d represents the diameter of the shaft determined by the formulas for torsion, D the external diameter of the muff, t the greatest stress in the wrought-iron shaft, and t_1 the permissible stress in the cast iron, we can put,

$$\frac{\pi}{16}\cdot\frac{D^4-d^4}{D}t_1=\frac{\pi}{16}d^3t;$$

or, since we may assume

$$t_1=2.4,\quad t=4.8\quad [t_1=3{,}400 \text{ and } t=6{,}800],$$

that is $t_1=\frac{1}{2}t$, we have:

$$\frac{D^4-d^4}{D}=2d^3,$$

or

$$D^3-\frac{d}{D}d^3-2d^3=0.$$

If we put here $\dfrac{D}{d}=v$, we obtain the equation:

$$v^3-\frac{1}{v}-2=0,$$

from which we get $v = 1.39$. But the muff is also subjected to a shearing or tearing stress due to the driving in of the key; but the stress thus arising cannot be determined by calculation, and as the muff is also weakened by the keyway or groove, it is customary to give the muff a greater thickness than would follow from the above calculation. We can make the thickness δ of the muff equal to $\frac{1}{3}d + 5$ to 10 millimeters [or $\frac{1}{3}d + .2$ to .4 in.], consequently $D = 1.67d + 10$ to 20 millimeters [$D = 1.67d + .4$ to .8 in.]. It is, moreover, usual to strengthen the muff at its edges, by the projecting collars A and B, in order to prevent the breaking of the edges during the driving of the key. The keys are subjected to shearing, and the calculation for this stress can be made so that the resistance of the keys to shearing shall be just as great as that to twisting. Representing the diameter of the shaft by d, obtained from the formula for strength,

$$d = 1.02\sqrt[3]{Pa}, \quad [d = .092\sqrt[3]{Pa}],$$

we get

$$Pa = \left(\frac{d}{1.02}\right)^3 = 0.94\,d^3, \quad [Pa = 1{,}290\,d^3],$$

putting $a = \frac{d}{2}$, we get for the force tending to shear the key and acting at the circumference of the shaft:

$$P = 1.88\,d^2, \quad [P = 2{,}585\,d^2].$$

We have, therefore, when the length of the key $l = d$, the breadth $b = v\,d$, and when k represents the permissible stress (4.8 for wrought iron [6,830 lbs.]),

$$P = 1.88\,d^2 = l\,b\,k = d\,.\,v\,d \times 4.8, \quad [P = 6{,}830\,v\,d^2],$$

from which follows:

$$v = \frac{1.88}{4.8} = 0.392;$$

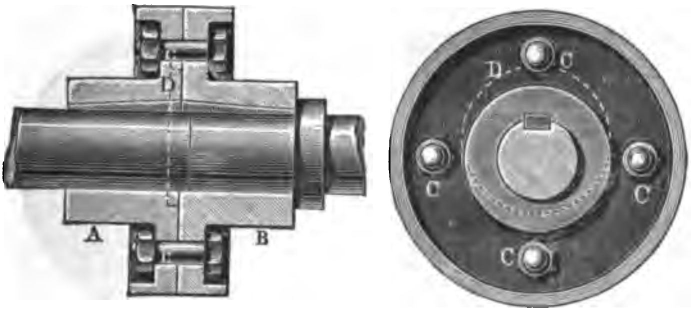
hence the key must have a breadth $b = .39\,d$. This case, calculated on the supposition that the diameter of the shaft was determined with reference to strength, its maximum permissible stress being taken at $t = 4.8$ kilograms [6,830 lbs.], is the most unfavorable case, and is not applicable to shafting determined from the formulas for stiffness, $d = 4.13\sqrt[4]{Pa}$ [$d = 0.3\sqrt[4]{Pa}$], because in this latter case the maximum stress in the shaft is much smaller than t, consequently the force P acting at the circumference, which tends to shear the key, is also much less. It is customary, therefore, to determine the dimensions of keys in shafts used for the transmission of power from empirical rules. The following are given by *Redtenbacher:*

$$\delta = \tfrac{1}{3}d + 5 \text{ mm} \quad [\delta = \tfrac{1}{3}d + .2 \text{ in.}],$$
$$l = 2d + 30 \text{ mm} \quad [l = 2d + 1.2 \text{ ins.}],$$
$$b = 0.9\,\delta = 0.3\,d + 4.5 \text{ mm} \quad [b = 0.3\,d + 0.18 \text{ in.}].$$

Thickness of key $= \tfrac{1}{2}b = 0.15\,d + 2$ mm [$= 0.15\,d + 0.08$ ins.].

§ **24. Flange Couplings.**—The objection to the box or muff coupling just described, is that the shaft cannot be put up or taken down without shifting the muff along the shaft.

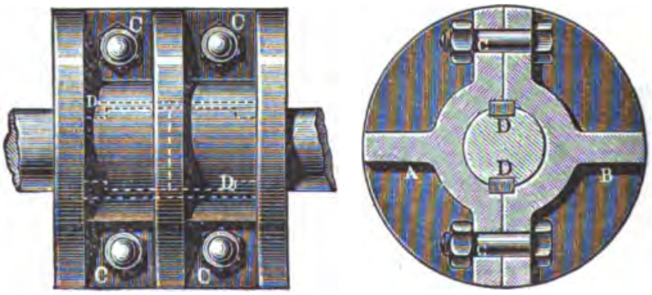
Fig. 63.



To obviate this defect, the couplings are often made in two parts as shown in the following examples. Fig. 63 represents a flange coupling, it consists of two disk-shaped pieces, *A* and *B*, which are keyed to the ends of the two shafts, and which touch each other along their faces. The transfer of motion from one shaft to the other is effected by bolts, *C C*, passing through both disks, the number of bolts varying from 4 to 8, according to the size of the coupling. The heads and nuts of these bolts are placed in circular grooves formed by the rims and hubs of the coupling, the object being to avoid accidents that might arise from the catching of belts or other objects.

Since the bolts in this coupling have to transmit the moment of the force, it will be well whenever the moments are large, to avoid putting the bolts, too near the axis, for by making the lever-arm *a* as large as practicable, the force *P* transmitted through the bolts and tending to shear them, will be diminished as much as possible. In order to relieve the bolts of this very unfavorable stress, which tends to make the bolt holes oval, and diminish the rigidity of the combination, the coupling has been greatly improved and made applicable to very large shafts, by turning an eccentric depression *D* into the face of the part *A*, and providing the other part with a corresponding projection exactly fitting the depression. The transmission of the moment is now effected by the projection, and the bolts have only to prevent an accidental separation of the two disks.

Fig. 64.

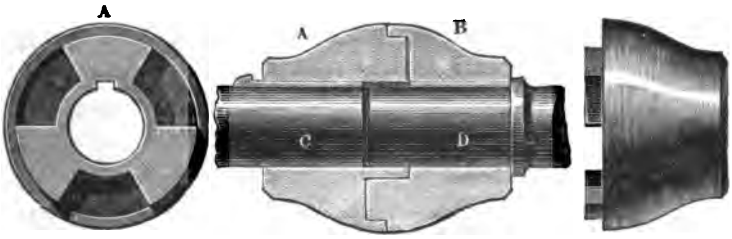


The coupling shown in Fig. 64 is, in many respects, an excellent one. In this case the union of the shafts is effected by means of a cast-iron muff, which is divided into two parts, *A* and *B*, which are fastened to the shaft by means of the bolt *C*. The keys *D* are placed in grooves and transmit the moment, their hook-shaped heads resisting any tendency to separation of the ends of the shafts. The ease with which this coupling can be attached to or removed from the shafting, and the possibility of very accurate construction, constitute the special advantages of this coupling, and render it suitable for the largest shafting.

§ 25. **Flexible Couplings.**—The couplings hitherto mentioned all belong to the so-called rigid couplings, and can therefore be applied only when it is reasonably certain that the axes of the shafts will always fall into the same straight line. If for any reason this is not the case, a flexible coupling is preferable, because it avoids the pressures and cramping action which would accompany the rigid connection.

A favorite form of flexible coupling is shown in Fig. 65, and is known as Sharp's Claw Coupling.* In this case also the muff consists of two parts, *A* and *B*, whose faces are provided respectively, as may be seen from the figure, with sector-shaped projections and depressions which mutually clutch each other. As a small amount of play has been left between the teeth, the slight flexibility desired is attained. If *D* is the driving-shaft, it must be supported by a bearing placed close to the coupling, and the end of *D* must be

FIG. 65.



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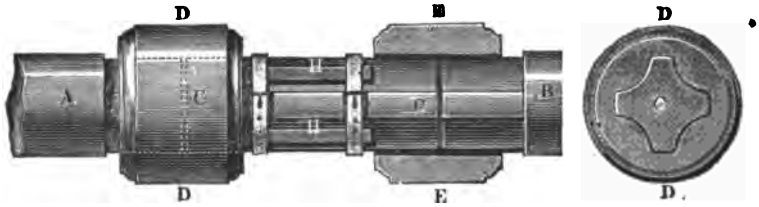
* For table of dimensions of this coupling, see *Wiebe: Die Lehre von den einfachen Maschinentheilen*, p. 303.

allowed to extend from 10 to 15 mm [0.4 to 0.5 in.] into the socket of the other half *A* of the coupling, so that it may offer some support to the free end of the shaft *C*, which it would not otherwise have, on account of the play left between the sector-shaped teeth.

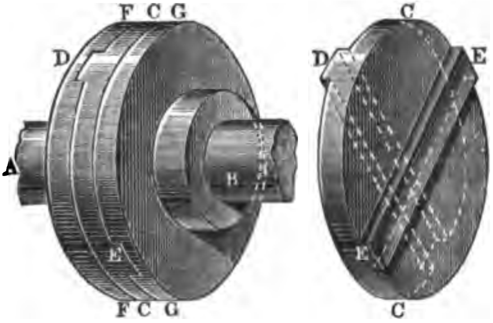
In rolling mills the cast-iron journals of two adjacent rolls, *A* and *B*, Fig. 66, are united by an intermediate piece *C*, and two muffs *D* and *E*. The cross section of the interior of the muffs is star-shaped, and exactly fits the exterior of the prolongations on each journal, but is made a trifle larger than the exterior of the intermediate piece, so as to allow a slight rise and fall of the latter, as is necessary in rolling-mill work. The muffs are prevented from shifting by strips of wood placed between the muffs and upon the intermediate piece, and fastened to the latter. Here we may incidentally remark that as the intermediate piece is weaker than the roll, it greatly diminishes the danger of breakage of the latter, for any excessive resistance would cause the intermediate piece to break rather than the roll. In like manner the general principle that all parts of a machine should be of uniform strength, is frequently departed from, by making small and easily replaced parts, like journals, relatively weaker than those difficult to replace, such as working beams, connecting rods, etc., whose breakage would occasion considerable damage and delay.

FIG. 66.



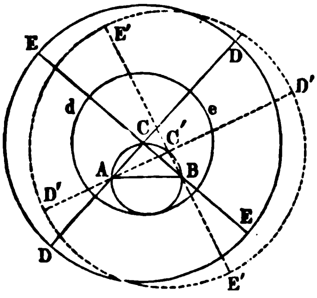
Oldham's Coupling, shown in Fig. 67, possesses great flexibility. In this device two flat circular disks, *F F* and *G G*, are keyed respectively to the ends of the two shafts, *A* and *B*, which are to be united. The two disks are connected by means of an intermediate disk *C C*, which connection is effected by placing on one face of the disk *C C*, and along its diameter, a prismatic bar *D D*, and upon the other face of the disk a similar bar *E E*, which is normal to *D D*. Since these bars fit into corresponding grooves on the disks *F* and *G* fastened to the shafts, we see that when the shaft *A* turns it will carry with it the bar *D D* of the disk *C*, that the latter then communicates this motion to the bar *E E*, and that this transmits the motion to the end disk *G* of the other shaft *B*, and thus causes the latter to turn.

Fig. 67.



This coupling possesses the peculiarity that the transmission of motion from one shaft to another can be accomplished even when the two shafts do not fall into the same straight line, provided that they are parallel. In this case, moreover, the motion transmitted will be uniform, as may be easily seen from Fig. 68. In this figure, *A* and *B* are the intersections of two parallel straight lines, representing the axes, with a plane passing through the middle of the coupling, and *D D* and *E E* are the projections of the centre lines of the above-mentioned bars upon this plane. We now see that when the shafts turn, the intermediate disk *C* will have such a motion that the two straight lines *D D* and *E E* drawn upon it, perpendicular to each other, will pass through the fixed points *A* and *B* respectively, a kind of motion which has been more fully discussed in the introduc-

Fig. 68.



tion (§ 11). From this discussion we see that the two centroids representing this motion are two circles, one of which belongs to the connecting line *A B*, and is drawn on *A B* as a diameter, while the other is a circle *d e* belonging to the cross bars *D D* and *E E*, having a diameter equal to twice *A B* and a centre *C* at the intersection of *D D* and *E E*. In the present case the two axes *A* and *B*, and their connecting line *A B* must be regarded as fixed unchangeably, and the motion of the intermediate disk will therefore correspond to the rolling of the larger circle *C d e* upon the smaller circle *A C B*. During this rolling, the point of intersection *C* of the two bars remains constantly upon the smaller centroid or circle described on *A B* as a diameter. Now let us suppose the large circle *C d e* to roll a little way on the smaller one *A B C*, it will correspond to a slight turning of the coupling, the point of intersection *C* shifting its position to *C'*, and the straight lines *C D* and *C E* changing to *C' D'* and *C' E'*. It now follows from the equality of the two angles *C A C'* and *C B C'*, that the two axes *A* and *B* must have turned through exactly the same angle, for they were compelled to rotate with *D D* and *E E*, while the latter were turning into the positions *D' D'* and *E' E'* respectively. When the two axes are in the same straight line, *A* will coincide with *B*, the centroids will contract to a mere point (the centre of the shaft), and the motion of the cross-bar disk will not differ from the turning motion of the two shafts.

In the case where the two parallel shafts do not fall into exactly the same straight line, their connection may be effected as shown in Fig. 69, a crank being placed on each end *A* and *B* of the shafts, the two journals of the cranks being joined by a link-bar, *C D*, and the whole combination called a Knee Coupling. This arrangement will not have the property of transmitting a uniform rotation from one shaft to the other. On the contrary, if the driving shaft *A* turns uniformly, the driven shaft *B* will alternately move faster and slower than *A*. The mechanism just described belongs to the crank train, and there it will be more fully discussed.

If in the coupling of Fig. 69 we give to the crank pins and their bearings spherical forms, the transmission of motion will still be possible though the shafts are not exactly parallel but form a slight angle with each other. The same holds with the disk coupling shown in Fig. 70, in which the coupling bolts *A* are keyed to the disk which is fastened to the shaft *C*, their pear-shaped heads, *B*, fitting loosely, and capable of slightly turning in the disk fastened to the other shaft, *D*. When there are greater deviations in the directions of the shafts, the devices just described, Figs. 69 and 70, cannot be applied, and use must be made of the following means:

FIG. 70.

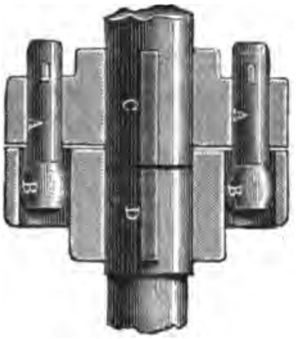
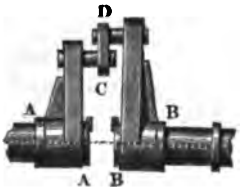


FIG. 69.



§ **26. The Universal Joint.**—This device is also known as *Hooke's* Coupling or *Hooke's* Gimbal Joint, and is shown in Fig. 71, where *C* represents the intersection of the two shafts *D* and *E*. The two axes are provided at their ends with forks *A A* and *B B* of equal size, whose bearings, *A*, *A*, *B*, *B*, fit the four journals of the cross *C*, the arms of which are at right angles. When the driving shaft *D* turns, the centre lines of the bearings *A A* of the fork attached to *D* must remain in a plane perpendicular to the shaft *D*, the middle points of the bearings describing in this plane a circle, *A F B G*, Fig. 72. At the same time, by means of the coupling cross *C*, the driven shaft *E* must likewise turn, and the centres of the bearings *B B* of the fork attached to *E*, must also remain in a

FIG. 71.

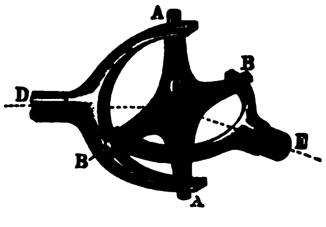
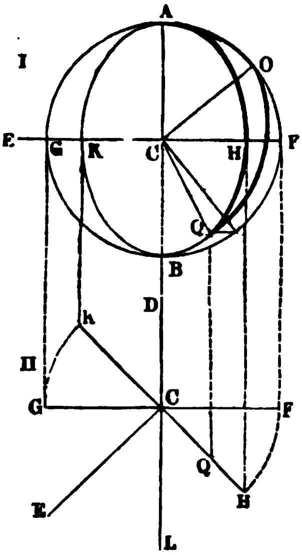


FIG. 72.



plane KH perpendicular to the shaft E and passing through C, the points describing in this plane another circle, $HBKA$. When the fork of the driving shaft is in the position BA, the fork of the driven shaft will be at KH. If the driving shaft has turned through 90°, that is if its fork has reached the position GF, the fork of the driven shaft will be at AB, etc., for the angles ACH, FCB, BCK, and GCA, are right angles equal to those formed by the arms of the cross. As may be easily seen, the two circles are great circles of a sphere whose centre is at the intersection, C, of the two axes, and whose diameter is the width $AA = BB$ of either fork. The angle KCG, formed by these planes, is equal to the angle $ECL = c$, formed by the axes. The cross $ABAB$, Fig. 71, connecting the forks, has a motion of such character that the centre line of the journals AA is constrained to move in the plane $AFBG$, and the centre line BB normal to AA in the plane $HBKA$. Evidently this is the same kind of motion which was more fully discussed in the Introduction (§ 24), and we may therefore make direct use here of the formulas there developed. As before, let us designate by α the angle ACO, representing the amount of turning of CA in its plane of motion, in a given time, from its initial position CA; also let $\beta = HCQ$ be the angle of turning of the straight line CB of the cross in its plane, estimated from the initial position HC. Then we shall have, as before, from the spherical triangle $CAOQ$:

$$\cos OCQ = \cos ACO \cos ACQ$$

$$+ \sin ACO \sin ACQ \cos O\,(AC)\,Q$$

or

$$\cos 90^\circ = 0 = \cos \alpha \cos (90^\circ + \beta) + \sin \alpha \sin (90^\circ + \beta) \cos c,$$

which may be reduced to

$$\cos c = \frac{\tan \beta}{\tan \alpha}.$$

The ratio of the paths β and α of the centres of the journals of the cross are therefore not constant, but it is the ratio of the tangents of their angles of rotation which is constant and equal to the cosine of the angle included between the axes. Since $\cos c$ is always a proper fraction, we shall always have $\tan \beta < \tan \alpha$; that is in the first and third quadrants we shall have $\beta < \alpha$, and in the second and fourth quadrants $\beta > \alpha$.

Accordingly, the velocity of the two axes is not the same, and when the driving shaft D turns with uniform velocity, the shaft E moves with greater and less speed alternately. In § 24 of the Introduction, the ratio of the angular velocities has already been determined, and is as follows:

$$\frac{\omega_2}{\omega_1} = \frac{(1 + \tan^2 \alpha) \,.\, \cos c}{1 + \tan^2 \alpha \,.\, \cos_2 c} = \frac{\cos c}{1 - \sin^2 a \,.\, \sin^2 c},$$

where ω_1 represents the angular velocity of the driving shaft D, and ω_2 that of the driven shaft E.

This ratio attains its greatest value when

$$\sin \alpha = \pm 1,$$

or

$$\alpha = \frac{\pi}{2}, \text{ and } \frac{3\pi}{2},$$

that is, when the driving fork is in $G F$, and the driven one in $A B$, which obtains whenever the cross lies in the *plane*

A F B G perpendicular to the driving shaft D. For this position we have:

$$\text{max.}\ \frac{\omega_2}{\omega_1} = \frac{\cos c}{1 - \sin^2 c} = \frac{1}{\cos c}.$$

The smallest value of $\frac{\omega_2}{\omega_1}$ is obtained where $\sin \alpha = 0$, or for

$$\alpha = 0 \text{ and } \pi,$$

hence

$$\text{min.}\ \frac{\omega_2}{\omega_1} = \cos c,$$

which corresponds to the position of the driving fork in *A B*, and the driven one in *H K*, that is when the plane of the cross is *normal to the driven shaft E.*

Finally, the two velocities ω_1 and ω_2 will be equal, when

$$(1 + \tan^2 \alpha) \cos c = 1 + \tan^2 \alpha \cos^2 c,$$

from which we get

$$\tan^2 \alpha \cos c\,(1 - \cos c) = 1 - \cos c,$$

or

$$\tan \alpha = \sqrt{\frac{1}{\cos c}},$$

and

$$\tan \beta = \sqrt{\cos c}.$$

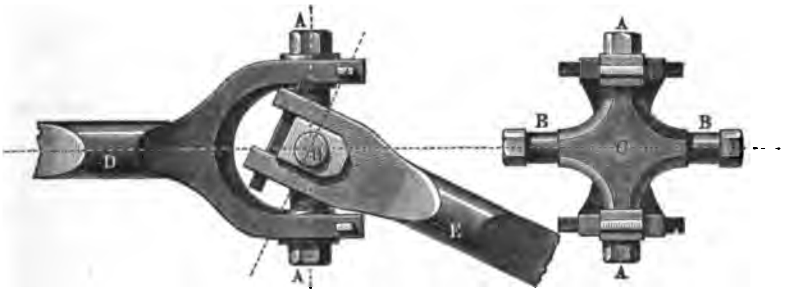
The velocity ratio consequently oscillates between the two extreme values $\frac{1}{\cos c}$ and $\cos c$; indeed if the velocity ω_1 of the driving shaft is uniform, the driven fork attains its maximum angular velocity $\frac{\omega_1}{\cos c}$ twice in every revolution, namely, when it is in the positions *A B*, and its minimum velocity

$\omega_1 \cos c$, also twice in every revolution, namely, whenever the driven fork passes the position *HK*. Between these limiting positions of the forks, the driven shaft is subjected alternately to acceleration and retardation.

The variation in the motion of the driven shaft increases with the angle of inclination c of the two shafts, and disappears when this angle is zero. Consequently the universal joint is mostly employed when the directions of the two shafts deviate but slightly ($c \overline{<} 30^\circ$), otherwise the inequalities of motion, and the resulting friction, shocks, and other disadvantages will become too great. The universal joint is frequently applied in agricultural machinery (for example, threshing machines) whose portable character does not permit of their being set up with the care demanded when rigid couplings are employed. Recently, *Hooke's* joint with properly increased dimensions, has been applied to war vessels for coupling the propeller shaft with the engine shaft, so that the screw may be inclined to the axis of the ship, and thus effect a rapid turning of the same.

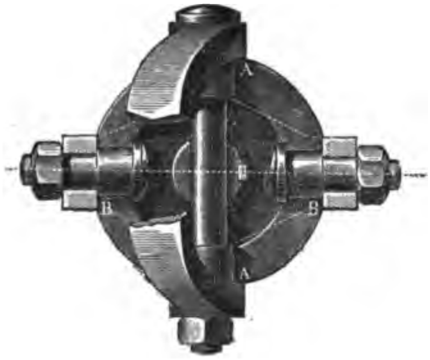
The details of the universal joint may be seen from the two views given in Fig. 73. The cross *C*, and the tines of the fork are of wrought iron, and the journals *A* and *B* are provided with devices for taking up the wear in the bearings.

FIG. 73.



Where the stresses are small, as in agricultural machinery, it is customary to reduce the first cost by making the forks and the cross of cast iron, but the journals of wrought iron, the forks being keyed to the shafts. To connect the forks, a wrought-iron ring, Fig. 74, is often used instead of a cross. This ring contains the sockets *A*, *A*, *B*, *B*, for holding the journals, one pair of which, *A A*, are turned on a bolt passing through the ring, while the other pair are made of two shorter bolts *B B*, but if desired, all the journals may be short bolts or pins. This ring is chosen in preference to the cross whenever it is necessary to have several systems of journals in one plane, as in the following constructions.

FIG. 74.



In *Taylor's* * universal joint, one of the shafts is provided at its end with a cylindrical sleeve with four axial grooves on the interior, while the other shaft has a spherical-shaped end with four corresponding teeth which gear into the grooves of the sleeve. The spherical-shaped end gives the desired flexibility to the combination, while the teeth effect the driving.

EXAMPLE.—In a universal joint whose axes make the angle $c = 30°$ with one another, we have, for the greatest velocity ratio:

$$\text{max. } \frac{\omega_2}{\omega_1} = \frac{1}{\cos 30°} = 1.155;$$

and for the smallest,

$$\text{min. } \frac{\omega_2}{\omega_1} = \cos 30° = 0.866;$$

and hence the ratio of these two values is;

$$\left(\frac{1}{\cos 30°}\right)^2 = \frac{4}{3}.$$

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* *Dingler's polyt. Journal*, Bd. 173, 1864.

Consequently, if the driving shaft turns with the uniform velocity ω_1, the velocity ω_2 of the driven shaft will vary between 1.155 ω_1 and 0.866 ω_1, and the measure of the variation will be 1.155 − 0.866 = 0.289.

The two shafts will have the same velocity of rotation when

$$\tan \alpha = \sqrt{\frac{1}{\cos 30^\circ}} = \sqrt{1.155} = 1.074 .$$

$$\tan \beta = \sqrt{\cos 30^\circ} = \sqrt{0.866} = 0.931.$$

Since these values give $\alpha = 47^\circ\ 3'$ and $\beta = 42^\circ\ 57'$, the greatest deviation will be $\alpha - \beta = 4^\circ\ 6'$, and will correspond respectively to the two cases when the driven shaft moves faster and slower than the driving shaft, each of these two cases occurring twice in every revolution.

§ **27. The Double Universal Joint.**—The inequality of motion of the driven shaft resulting, according to the above, from the single universal joint, gives rise to many disadvantages in the machines driven by it. For instance, the driving shaft has to overcome widely differing resistances, because in accelerating the driven shaft, not only must the work of the machinery be performed at greater speed, but work is also absorbed in accelerating the masses the living force of which in the following period of retardation acts to overcome the resistance due to the work, thus still further diminishing the load on the driving shaft, and increasing the difference in the amount of work given off in the two periods. As a consequence of this, there arises in the whole system an irregular jarring action producing much friction and wear. The result has been that of late numerous efforts have been made to construct universal joints free from this variation of angular velocity; particular mention may be made of various constructions called forth by a prize offered by the *Verein zur Beförderung des Gewerbfleisses in Preussen.*[*]

The simplest means of obtaining a uniform transmission of motion, when the universal joint is to be employed, consists in making use of a second universal joint to neutralize the inequalities of the first; that this is possible will appear in what follows.

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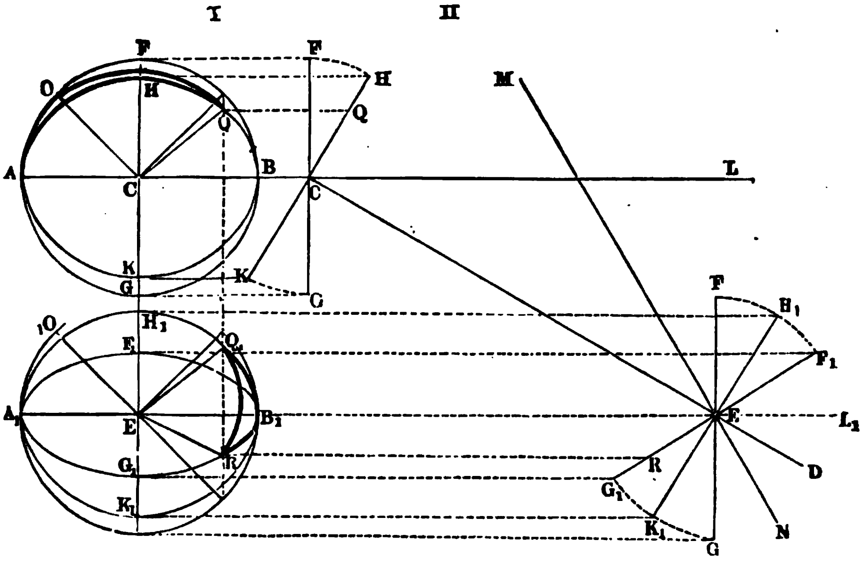
[*] See *Verh. d. V. z. Bef. d. Gew. in Preussen.* Vol. 44, 1865.

FIG. 75.



In Fig. 75, let C be the centre of the universal joint connecting the two shafts CA and CE, the driving shaft having the direction CA. All that was said in the previous article applies to the motion transmitted from CA to CE. The driving pins move in the plane $AFBG$, while the driven ones move in the plane $HQBKA$, and in such manner that the rotation of the driving shaft through the angle $ACO = \alpha$ will cause the driven shaft CE to rotate through an angle $HCQ = \beta$, the relation between the angles being given by

$$\tan\beta = \cos c \tan\alpha,$$

where *cos c* represents the angle $LCE = FCH$ included between the two shafts. Now let us suppose that at any point E of the driven shaft CE a third shaft EN is so connected with CE, that it makes with the latter an angle $DEN = c$

and equal to that contained by the first and second shafts AC and CE. We will also assume that the three shafts lie in the same plane (that of the paper). The two shafts at E are likewise connected by a universal joint, and in such a way that the fork at E belonging to CE is parallel to the fork at C on the same shaft. When the fork at C takes up the position CQ, the fork at E, whose pins move in the plane $H_1B_1K_1A_1$, will take up the position EQ_1 parallel to CQ, while the axis of the pins belonging to the fork of the shaft EN and moving in the plane $A_1F_1B_1G_1$ will take up the position ER at an angular distance from the assumed initial position EB_1, represented by the angle $B_1ER = \gamma$. In order to determine γ we must make use of the expression deduced from the spherical triangle EQ_1B_1R namely:

$$\cos Q_1ER = \cos Q_1EB_1 \cos B_1ER$$
$$+ \sin Q_1EB_1 \sin B_1ER \cos Q_1(EB_1)R,$$

which since

$$Q_1ER = \frac{\pi}{2}; \quad Q_1EB_1 = \frac{\pi}{2} - \beta,$$

and

$$Q_1(EB_1)R = \pi - c,$$

reduces to

$$0 = \sin\beta \cos\gamma - \cos\beta \sin\gamma \cos c,$$

whence

$$\cos c = \frac{\tan\beta}{\tan\gamma}.$$

But because we also have

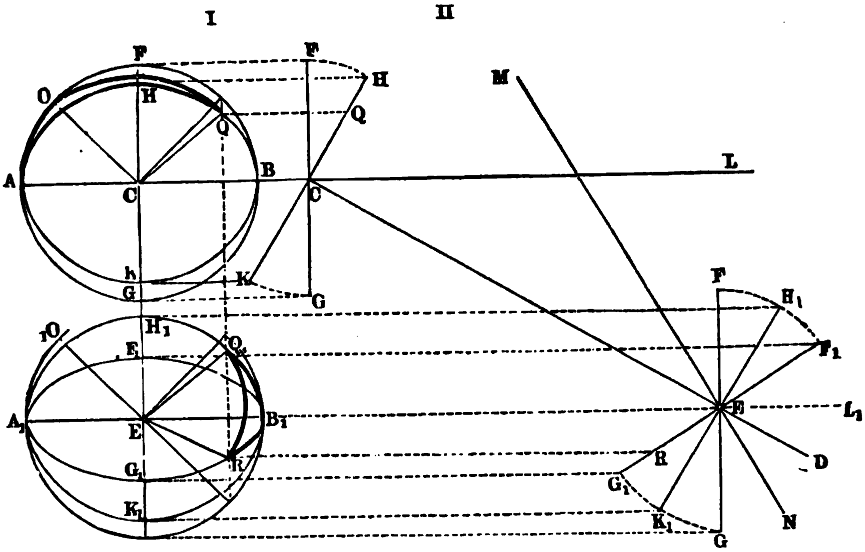
$$\cos c = \frac{\tan\beta}{\tan\alpha},$$

it must follow that $\tan\gamma = \tan\alpha$ or $\gamma = \alpha$. Since this relation holds for every value of A, we see that the angle γ

through which the third shaft has turned, must at every instant be equal to the angle of rotation α of the first shaft. Consequently, if the driving shaft $A\ C$ moves with uniform velocity, the third shaft will also so move, while the intermediate shaft $C\ E$ will retain the inequality of motion belonging to a simple universal joint having the angle of inclination c.

The assumptions underlying the foregoing calculation, show us that if the uniform rotation of the driving shaft is to be transformed into a uniform rotation of the driven shaft, it will be necessary not only to make the angles of inclination formed by these two axes with the intermediate axes, equal, but also to so place the two universal joints that their positions relatively to the shafts will be similar. By *similar* position we here understand, that when the fork of the intermediate axis coincides with the plane $A\ C\ E$ of the two axes

FIG. 76.



$A\ C$ and $C\ E$, which meet at C, the fork of the intermediate shaft at E will also coincide with the plane $C\ E\ N$ of the shafts intersecting at E, or that the two forks of the intermediate shaft at C and E must be *simultaneously* perpendicular to the planes of the intersecting shafts just mentioned. That is to say, the forks must always form equal angles with these planes.

This last condition, and also the one requiring that the angles of inclination at C and E should be equal, by no means prescribe, however, that all three shafts should lie in one and the same plane, as is assumed in Fig. 76. For, if we imagine two conical surfaces to be generated about the intermediate shaft CE, with their vertices at C and E, the generation being effected by revolving the lines of direction AC and EN about CE in such a way as to maintain the constant angle of inclination c, then any two elements of these conical surfaces may be taken for the direction of the shafts AC and EN. These two directions need not intersect nor be parallel, they will nevertheless transmit uniform motion between the two shafts which they represent, provided only that the universal joints have *similar* positions in the sense defined above.

§ 28. Universal Joints for Transmitting Uniform Motion.—The property of the double universal joint just determined also underlies other constructions whose object is to transmit motion without change of velocity.

Since the length of the intermediate shaft CE, Fig. 76, is wholly without influence on the transmission of motion, we can reduce this length as much as we please, and even shorten it so far that two forks at CE will fall into one and the same plane. The two fork-pins or bolts will therefore be reduced to a single one, for we know from what has preceded that the forks must be parallel.

For the sake of brevity we will call this resultant bolt, the *driving bolt.* It is evident that its motion must be confined to a plane which is parallel to the planes HK and H_1K_1, and hence must make equal angles $\frac{\pi}{2} - c$ with the driving and the driven shaft, that is, it must bisect the angle ACB

formed by the shafts *A C* and *B C*, Fig. 77 and Fig. 78. The shafts must now be provided with forks *similarly* placed, *i. e.*, the fork *a a* must be so fastened to the shaft *A*, and the fork *b b*, to the shaft *B*, that they will simultaneously lie in the plane of the shafts, as shown by Fig. 77.

Fig. 77.

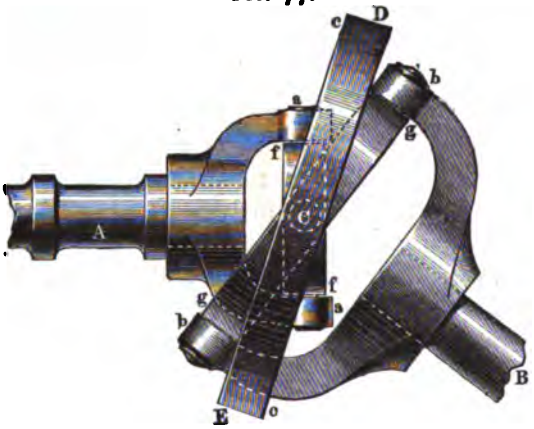
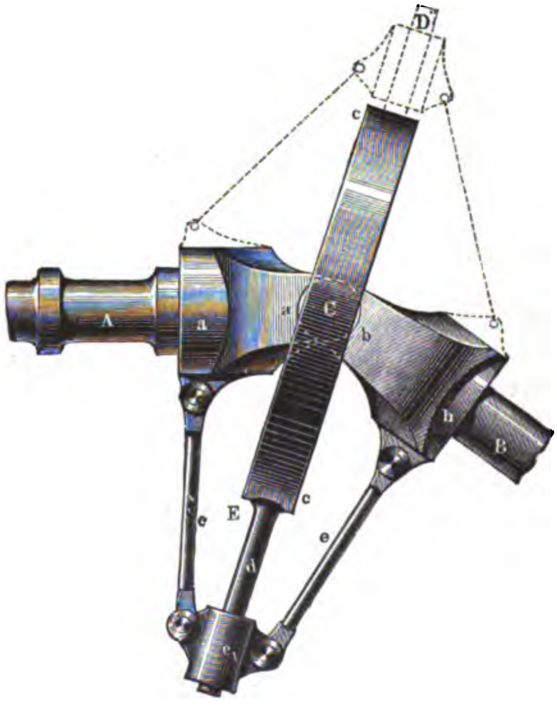


Fig. 78.



If we suppose the shafts to turn through 90°, these forks will take the position shown in Fig. 78, and their bolts or joint-pins will be perpendicular to the plane of the shafts, and will be projected at *C*. In this construction the bolts do not pass across the forks, but are merely short joint-pins fitting the eyes of the forks. In the construction shown in Fig. 79, however, one fork has a through bolt. Each of the forks *a a* and *b b* must be properly united by a cross with the driving bolt, which represents the contracted form of the forks of the intermediate shaft *C E* of Fig. 76. Since this driving bolt moves in the plane *D E* bisecting the angle of the axes, it is, of course, always perpendicular to the fork bolts *a a* and *b b*; consequently, in Fig. 80, it is perpendicular to the plane of the shafts at *C*, while in Fig. 81 it lies in this bisecting plane at *D E*. This driving bolt need not extend entirely through the fork, but may consist of two short joint-pins lying at *c c*, Fig. 81, connected by a wrought-iron ring *D E*. The two short joint-pins shown in projection at *c c* Fig. 80, must also be jointed to the crosses of the forks *a a* and *b b*. As in Fig. 79, these

FIG. 79.

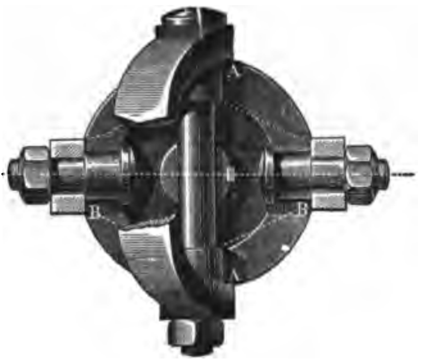
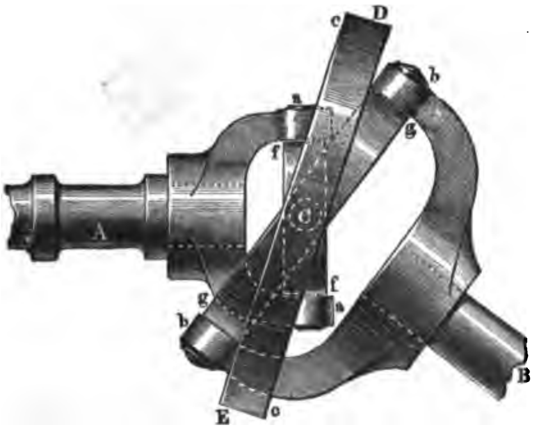
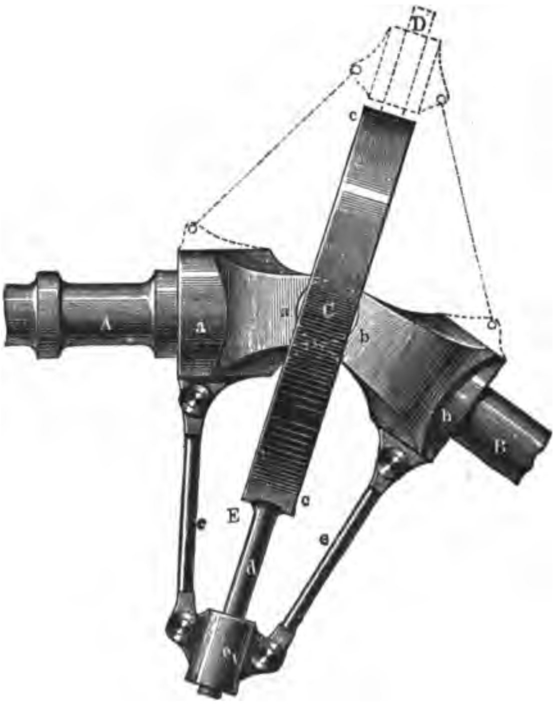


FIG. 80.



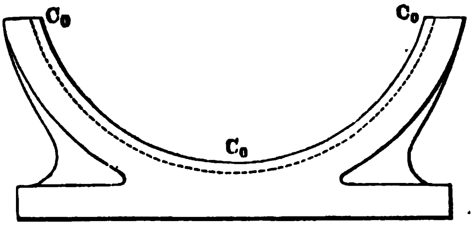
crosses are transformed into rings, which are shown at *ff* and *gg*, Fig. 80. A comparison of the construction thus obtained with Fig. 76, shows that all the parts of the former exist in the latter, *a a* and *b b* being the forks of the driving and the driven shafts, *ff* and *gg* the crosses in the form of rings, while the ring *D E* corresponds to the driving bolts of the two forks at *C* and *E* Fig. 76. For this driving bolt or ring *D E* permits the rings *ff* and *gg* to turn as freely as the crosses of the forks on the shafts *C E*, Fig. 76. Here, as there, transmission of motion without change of velocity can take place if we assume the ring or driving bolt to remain always in the plane which bisects the angle of the axes. In Fig. 76 the intermediate shaft and its forks fulfil this condition, for

FIG. 81.



the position of this axis is unchangeably fixed by the two points *C* and *E*, but in Fig. 80 these two points coincide at *C*, and the driving ring *DE* will not remain in the bisecting plane unless special means are provided for keeping it there. For this purpose various devices may be employed, one of which is due to *Reuleaux*,[*] and consists of a circular groove $C_0 C_0 C_0$, which guides the driving ring, *DE*.

Fig. 82.



In the construction given by *Bredt*,[†] the same object is attained by the arrangement shown in Fig. 81, which consists of two equal connecting rods *e e* jointed to the two forks *a a* and *b b* at equal distances from the axes of the shafts, and attached to a sleeve e_1 sliding on the guiding rod *d* fixed to the driving ring *DE*. It is evidently the symmetrical arrangement of $a e e_1 e b$, which of course takes part in the rotation, which prevents the driving ring from leaving the bisecting plane of the axes.

The universal joint devised by *Blees*[‡] has a somewhat different arrangement, but is also constructed in such a way that the two shafts have forks so united by a driving bolt that this driving bolt must remain in the plane bisecting the angle of the axes. An idea of the arrangement may be obtained from Fig. 83. The two shafts *A* and *B* have ring-shaped ends *D* and *E*, having the intersection *C* of the axes for their common centre. These rings, the outer of which cannot, of course, be completely closed, are united by a bolt *FCF*, whose axis bisects the angle *A C B*. It is evident from what has preceded, that a uniform transmission of motion will be obtained if we make sure that the driving bolt *FF* during the rotation of the shafts remains continually in the plane which bisects the angles *A C B* of the shafts. The bisecting plane is perpendicular to the plane of the paper at *F F*. To maintain *FF* in this bisecting plane, it is not sufficient to provide the ring-shaped ends *D* and *E* with round holes for the driving bolt, but we must give to them long slits, *d d* in *D*, and *e e* in *E*, in order that they may move relatively to the driving bolt. To keep the bolt in the bisecting plane, small bevel wheels, *f f*, whose vertices are at *C*, are fastened to the bolt *F F*, and one side of the slit in each ring is provided with teeth which gear into these bevel wheels. Since the point at which the bevel wheel is acted upon by the teeth of the inner ring *E* is diametrically opposite to the point at which the bevel wheel is acted upon by the exterior ring *D*, and since the motion of the rings relatively to each other corresponds to a rotation about a perpendicular at *C*, it follows that the bevel wheels of the driving bolt *F F* will roll upon the toothed portions of the slits, and in such a way that the axis of the driving bolt will continually remain in the bisecting plane which is perpendicular to the plane of the drawing at *F C F*. It is evident that instead of employing this toothed mechanism, we may make use of some other arrangements, for example, of that given in Fig.

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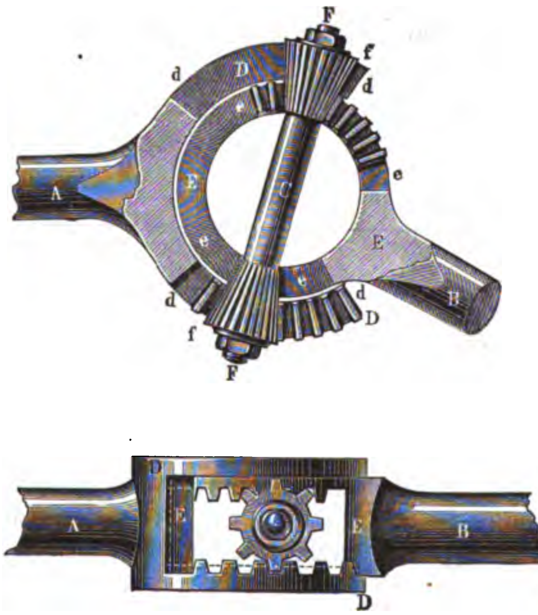
[*] *Verh.* d. *Ver. z. Bef.* d. *Gewerbfl.*, Vol. 44, 1865.

[†] *Civil Ingénieur*, Vol. 11, 1865.

[‡] *Verhandl. d. Ver. z. Bef. d. Gewerbfl.* Vol. 44, 1865.

FIG. 83.

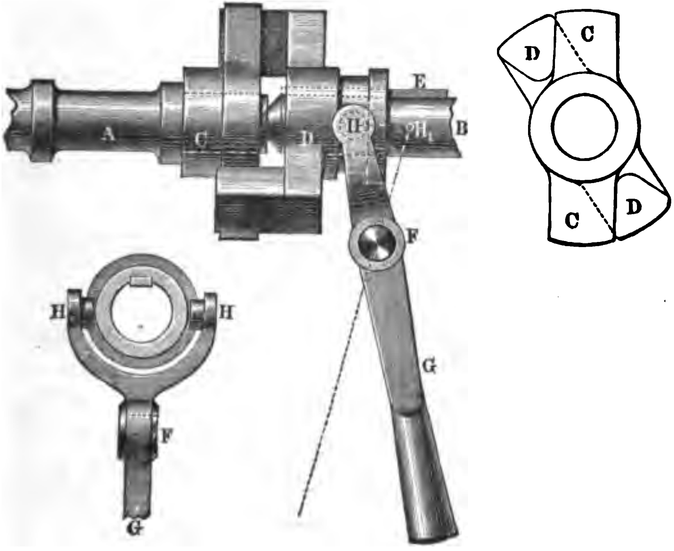


81 or Fig. 82. It follows, from what has been previously established, that when the driving bolt constantly makes *equal* angles with the two axes, the uniform motion of one shaft may be transmitted to another.

§ **29. Disengaging Gear.**—It often happens that one shaft must be coupled with another in such a way that they can be easily disconnected at any moment. The couplings used for this purpose are called *loose* couplings, or simply *clutches.* They generally consist of two parts, one of which is fixed to the driving shaft, while the other has such a motion on the driven shaft (usually a sliding one) as will start or stop the latter, according to the position of the second part relatively to the first. As the movable part has a groove which fits a feather on the driven shaft, the latter must necessarily partake of the rotation communicated to the movable or sliding part of the coupling.

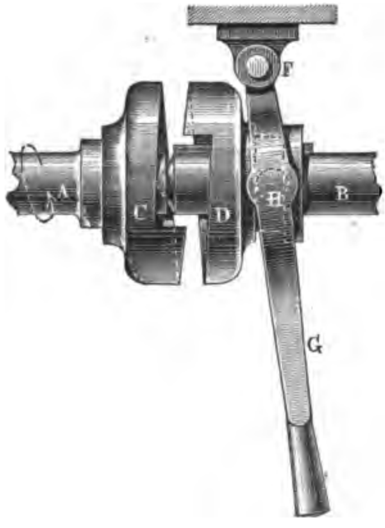
A simple disengaging gear is shown in Fig. 84. The driving shaft *A* is supported by a bearing placed as closely as

FIG. 84.



possible to the coupling, and has keyed to its end a cross-piece, *C*. The driven shaft *B* carries a fork *D*, which slides on the shaft, permitting the tines of the fork *D* to be acted upon or withdrawn from the arms of the cross-piece *C*, as may be desired. The feather *E* of the shaft fits into the groove of the nave *D*, and in all couplings of this class causes the shaft *B* to turn with the fork. The sliding of the fork along the shaft *B* for the purpose of throwing the latter in and out of gear, is effected by a lever *G*, which has a forked end embracing the nave *D*, the tines carrying two pins *H*, which fit into a groove turned on *D*, so that when the lever turns on *F* by moving from *H F* to $H_1 F$, the nave *D* is compelled to slide without preventing the rotation of *B*. Such forked levers are common to all disengaging gear. For large-sized couplings transmitting great forces, it is always well to support the driven shaft *B* by a second bearing placed close to the coupling. Sometimes when the forces are small the same object is attained by fitting the end of the driven shaft to a conical depression in the driving shaft, as shown in the figure.

FIG. 85.



The *clutches* shown in Fig. 85 and Fig. 86 are much used, the coupling parts consisting of disks *C* and *D*, whose faces carry one (Fig. 85) or more (Fig. 86) helical shaped teeth which fit each other, and act as drivers. The sliding part of the clutch is moved back and forth by means of a groove and a forked lever, as in Fig. 84, the arrangement in Fig. 85 differing a little, however, because of the differently placed fulcrum.

It is evident, from the oblique form of the teeth of these couplings, that motion can be transmitted from *A* to *B* only in one direction, as shown by the arrow, for an opposite rotation of A would throw the shafts out of gear. This is sometimes desirable, for example, when the shaft B drives machinery whose direction of motion must not be reversed. On the other hand, however, this property may become troublesome; for example, when the driven shaft B by virtue of its own living force or that of the masses fastened to it moves more quickly than the shaft A, and thus causes disengagement of the shafts. If a clutch is used in such a case, the teeth must be straight, similar to those in *Sharpe's* clutch, Fig. 87. The rigid coupling there used becomes a loose or movable one when the half A, instead of being keyed to its shaft, is provided with a groove and feather for sliding, and with a circular groove for a disengaging lever. Couplings with straight teeth cannot be thrown into gear as easily as those with oblique ones.

FIG. 86.

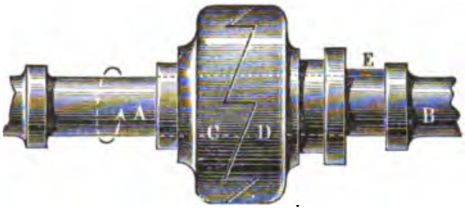
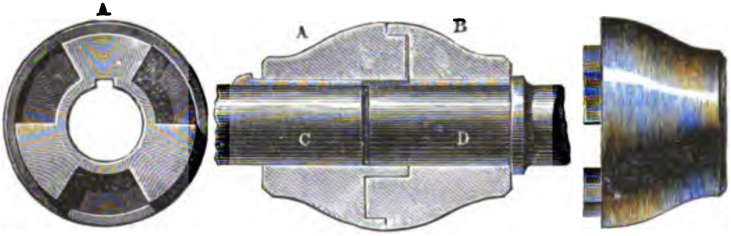


FIG. 87.



The couplings in Fig. 85 and Fig. 86 also differ in their action, in that more time is required to throw the single-toothed coupling into gear than the one that has several teeth. For since the driving does not begin until the tooth C touches the tooth D, it may take a whole revolution to throw a single-toothed clutch into gear, while it will at most require only $\frac{1}{n}$th of a revolution to throw into gear a clutch having n teeth. This is of little consequence in high speeds, but in slow speeds it is better to use clutches having several teeth, particularly where it is often necessary to throw the machine in and out of gear.

Clutches having a great number of teeth are also employed where it is necessary that the driven shaft should be started promptly, as in self-acting spinning machinery, where the faultless action of the individual parts largely depends on the promptitude with which they obey the reversing apparatus.

If the forces and speeds transmitted are small, these clutches may be thrown in and out of gear when the machine is running as well as when it is at rest. But if the force transmitted is considerable, it will be difficult to disengage the clutch because of the sliding friction between the teeth and along the feather. If the shafts move rapidly, care must be taken while engaging the clutch, otherwise breakages will be very apt to occur, particularly if the mass of the parts fastened to the driven shaft is large. For if we suppose the driven shaft *B* to be at rest while the driving shaft moves rapidly, then, at the moment of engagement, a shock will occur between the teeth of the coupling, attended by a loss of living force whose magnitude is

$$\frac{M_1 M_2}{M_1 + M_2} \frac{c^2}{2g},$$

(see Vol I., § 335), M_1 and M_2 representing the masses of the driving and driven shafts reduced to the teeth of the clutch, and c the velocity of the teeth at the moment of engagement. This loss of living force corresponds to the mechanical work

$$\frac{P_1 P_2}{P_1 + P_2} \frac{c^2}{2g},$$

and is expended in molecular work, and consequently not only the teeth of the clutch, but the shafts and other transmitting parts, wheels, etc., must absorb this living force. It follows from this, that suddenly starting a shaft, when the driving shaft moves quickly, is very apt to cause fractures, particularly if the mass to be set in motion is large. On this account the machinery of a flour mill driven by toothed gearing is never thrown into gear while the driving shaft is running, because with the great mass of the millstone (30 to 40 hundred weight), and high speed (about 120 revolutions per minute), the mill-spindle or some other part would surely break. Such machinery is therefore thrown into gear only when the driving wheel is at rest. On the other hand, when machines move slowly, they may be started while the driving shaft is running, as in vertical grinding mills, although the stones to be set in motion are much heavier. The very low speed (3 to 4 revolutions per minute) is the reason why this may be done without special danger, the violence of the shock not being sufficient to cause fracture. Such a procedure is, however, always dangerous when clutches or other other rigid couplings are used, for it is always attended by a shock.

§ 30. **Friction Couplings.**—It often happens that resistances arise in a machine which far exceed the average resistance existing under normal circumstances. For example, this occurs when a comparatively large stone gets between the rolls of an oil mill, or when the bucket of a dredging machine takes hold of a stone or stick which is larger than the breadth of the aperture through which the bucket must pass. If the motion is produced by moderate forces, confined within very narrow limits, as is the case when animal power is employed, then, as a rule, the motion will cease because of the sudden great resistance, and the obstacle can be removed. But when the driving force is not confined within narrow limits, as is the case when a little oil mill receives its motion from a large steam engine, or when the chain of the dredging machine is driven by an engine possessing heavy fly-wheels which contain a large amount of stored-up energy always available, then damage will result in both these cases. In the first case, the stone will be forcibly drawn between the rolls, and will either spoil their smooth surfaces by the dents which it makes, or will break some other part, while in the second case, there is danger of badly bending or tearing the chain.

To meet such accidents it is common to employ couplings which will only transmit motion so long as the pressure producing motion does not exceed the strength of the machine parts. In dredging machines* this coupling consists of wooden keys having a particular cross section and consequently a definite shearing strength, the keys being cut through when the resistance exceeds the upper permissible limit. Friction couplings are also often employed for this purpose, and are constructed in various ways. But whatever the arrangement, the essential features of a friction coupling are two parts fitting into and touching each other along a certain surface of revolution, the parts being kept in contact by a given pressure, Q. If one of these parts is set in motion, the other will move with it as long as the moment of resistance to be overcome is less than the moment of friction existing at the surface of contact when the driven shaft is at rest and the driving shaft in motion. With such friction couplings the hurtful effects described in previous paragraphs as attending the throwing into gear of rigid couplings at high speeds, are avoided. We consequently find them in frequent use, for example, in pile drivers driven by engines. Again, let us imagine that the driving shaft turns with any desired velocity, ω, and that the friction coupling is thrown into gear by pushing one part against the other with a pressure, Q. Let φ be the coefficient of friction at the surface of contact of the parts of the coupling, then the greatest pressure which can be transmitted from one part to the other will be $P = \varphi Q$, that is, it will be equal to the friction. Corresponding to this force, $P = \varphi Q$, there will still be a shock at the moment the coupling is thrown into gear, but no harm will be done, because P is under control, and it is evident that this shock will be wholly independent of the mass of the driven shaft. The latter will not at once assume the velocity of the driving shaft. A knowledge of the time needed to bring this about would be of no practical value, but could be ascertained by supposing the mass M of the driven shaft reduced to the point of application of the force of friction and then subjected to the influence of an accelerating force, $P = \varphi Q$, till the driven shaft had the same speed as the driving one. Until the driven shaft has attained this speed the two disks will slide on each other, the one belonging to the driving shaft moving faster than the other. It is also possible that this sliding may not cease, for instance, when the force $P = \varphi Q$, necessary to turn the driven shaft, is reached before the two shafts have attained the same velocity. Then this slower motion of the driven shaft will be accompanied by a sliding of the disks, the amount of sliding corresponding to the excess of the velocity of the driving shaft over that of the driven one. But such a condition is not a normal one, for the work of friction at the disks causes an unnecessary loss of energy. In such a case we may increase P by increasing Q or φ, or both at once, or we may so far diminish the work of the driven shaft (by giving the grinding or crushing mill less to do) that both shafts will turn with the same velocity, and consequently have no sliding of the disks.

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* See Dampfbaggermaschine in Hagen's *Handbuch der Wasserbaukunst.* Vol. IV., Part 3.

§ 31.—Friction couplings may be divided, according to their design, into two classes, the first class consisting of those in which the friction is generated by a pressure acting in the direction of the axis, and the second of those in which the pressure is exerted in a plane normal to the axis. An arrangement of the latter kind is shown in Fig. 88. Here a circular disk C is keyed to the driving shaft A, the disk having a groove in its circumference. Two semicircular wrought-iron straps $D\ D$ are set into this groove, and are firmly pressed against the disk C by means of the bolts $E\ E$, thus generating friction between them and the disk C. On the driven shaft *B* is fastened a fork-shaped claw *F*, whose arms press against the beveled heads of the bolts of the straps, and move with them. If the coupling is to be made so that it can be thrown out of gear, the nave of the fork must slide on a feather *H*, and be provided with the usual groove *G*. By means of the bolts *E* we may tighten the straps more or less, and thus regulate the magnitude of the force transmitted.

FIG. 88.

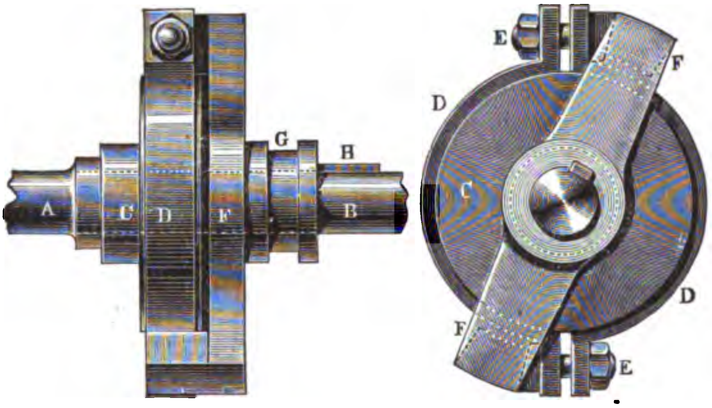
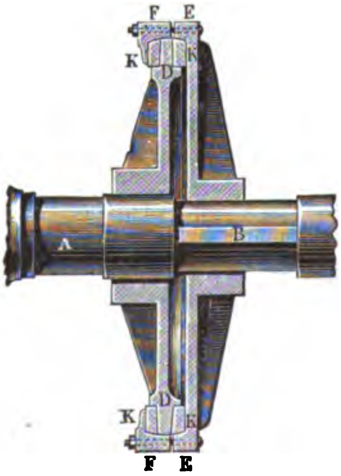


FIG. 89.



Those couplings in which the friction is generated by a pressure acting in the direction of the axis, are either disk or cone couplings, according as the pressed portions are flat or conical. A disk coupling is shown in Fig. 89. Here the disk *D* is keyed to the driving shaft *A*, the disk having projecting washers *K* of wood on both sides. By means of the ring *F* and screw-bolts, the sliding disk *E* of the shaft *B* is pressed against the wooden washers K. If Q represents the force with which the screws press the disk E and ring F against the wooden washers, then before sliding can take place, the friction φQ must be overcome along each of the two wooden washers, so that the whole resistance may be expressed by $P = 2 \varphi Q$. Since the breadth of the wooden washers is narrow in comparison with their radius, we can assume that the friction acts at the middle of the washers, but if these are of greater breadth, we must take for the lever arm of the friction

$$r = \frac{2}{3} \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2},$$

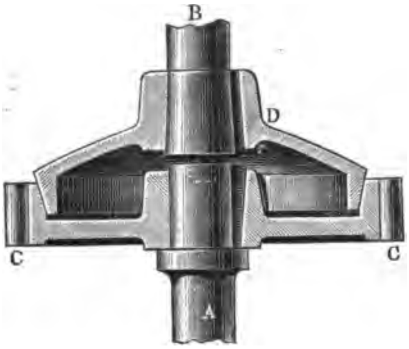
(see Vol. I. § 188), where r_1 and r_2 are the inner and outer radius respectively of the surface subjected to friction.

So long as the resistance to the driven shaft, reduced to the point of application of the friction, does not exceed the above value $2 \varphi Q$, the coupling will be effective.

As shown in the drawing, the coupling is not a disengaging one; to make it such, it would be necessary to remove the ring F, and press the disk E against the disk D by some arrangement for producing pressure, for instance the disengaging fork. But this is not to be recommended, for even if we ignore the fact that the pressure Q which must be exerted is twice as great as before, because the driving is effected by friction on one side only, the fact remains to be considered that the collar of the shaft A, the pins of the disengaging fork, and the side of the disengaging groove are subject to great friction, which causes considerable loss of work. It is principally this circumstance which renders the friction couplings with axial pressure unsuitable for disengaging gear, at least for horizontal shafts. It is otherwise with upright shafts, whose own weight and that of the parts fastened to them, suffice to generate the necessary friction.

In such cases it will be advantageous to employ cone couplings, Fig. 90 giving one form used in grinding mills for driving the stones.* Here A is a spindle driven by the spur wheel C, which gears into a toothed wheel on the main driving shaft. This spindle keeps the stone, resting on the short shaft B, in rotation as long as the moment due to the resistance of the work between the grinding surfaces is smaller than the friction between the cone D and the hollow cone turned on the inside of the rim of the wheel C. The weight of the stone will generate sufficient friction between D and C, and will moreover permit the grinding wheel to be thrown in and out of gear while the main shaft is running, without danger of shock or breakage.

FIG. 90.



The determination of friction follows here from the theory of the wedge, in a manner exactly similar to that employed in finding the normal pressure due to the fastening of conical pivots (§ 11). Let Q again represent the weight of the shaft B and the stone, α the angle at the vertex of the cone, and f the surface of contact of the cone, then we again have, as in § 11, for the normal pressure per unit of surface of contact:

$$p = \frac{Q}{f\left(\sin\frac{\alpha}{2} + \varphi\cos\frac{\alpha}{2}\right)},$$

and for friction,

$$F = \varphi p f = \frac{\varphi Q}{\sin\frac{\alpha}{2} + \varphi\cos\frac{\alpha}{2}}.$$

In some of the text-books the normal pressure between the conical surfaces is erroneously given as

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* See *Wiebe*: Die Mahlmühlen.

$$pf = \frac{Q}{\sin\frac{\alpha}{2}},$$

which gives approximately correct values for large values of the angle α, and very incorrect values for small values of the angle α. For, according to this formula, very small values of α would make p extraordinarily large and even equal to infinity when $\alpha = 0$, whereas § 11 shows that the maximum value of p at the limit ($\alpha = 0$) is:

$$\text{max.}\, p = \frac{Q}{\varphi f},$$

or,

$$\text{max.}\, pf = \frac{Q}{\varphi}.$$

The maximum of friction can, therefore, when the cones are very pointed, reach the amount

$$\text{max.}\, F = \varphi f \,\text{max.}\, p = Q,$$

while for plane disks we have

$$\text{max.}\, F = \varphi Q.$$

Friction couplings play a very important part in limiting the tension in the manufacture of thread, and this is true with regard to the manufacture of all such articles as are subjected to only a given slight tension—for example, paper and textile fabrics.

§ 32. **Couplings for Motors.**—It often happens in practice that two or more motors are employed simultaneously to run the works, both acting upon the same line of shafting. If these motors are connected with the shafting, and consequently with each other, by rigid couplings, fractures and other undesirable incidents will occur. For example, one motor may move more slowly than another, as in the case of a water-wheel with a diminished supply of water; now if the motors are joined by a rigid coupling, the slower one will be dragged along by the more quickly moving one, a steam engine, for example, and, consequently, the intermediate shafting must be of sufficient size and weight to stand such a stress. This form of coupling also causes considerable hurtful resistance in the steam engine, due to the dragging of the water-wheel, which greatly reduces the economy of power of the combination. Many measurements of power made for this purpose show that the sum of the work produced by two or more motors taken *separately*, is greater than the total power exerted when they are connected by a rigid coupling.* Whenever it is practicable, therefore, it is customary to avoid coupling several motors, giving to each motor some special work instead. This is in the direction of the now growing tendency to give up the formerly popular system of centralization of power, in which the most extensive works were driven from *one* place, and to use instead, whenever it is practicable, several independent motors. Nevertheless, cases may occur where it is desirable to couple several motors, and for such cases couplings are so constructed as to obviate the difficulties invariably attending a rigid connecting of the motors. These constructions are usually of a character that will permit each motor to act on the common shaft only so long as the power of the leading motor is sufficient to move the shaft at its normal speed. If for any reason one motor runs slower, it will disengage from the common shaft, which will then be entirely driven by the others. This latter condition will last until the disengaged motor, by reason of its diminished work, has accelerated its motion to the normal speed of the shaft, when it will again act upon the latter. On the other hand, if the leading motor, by reason of a greater supply of water, or a diminution of its usual work, acts with greater power and tends to accelerate the shaft, the other motors will disengage, and the whole work will be done by the first motor.

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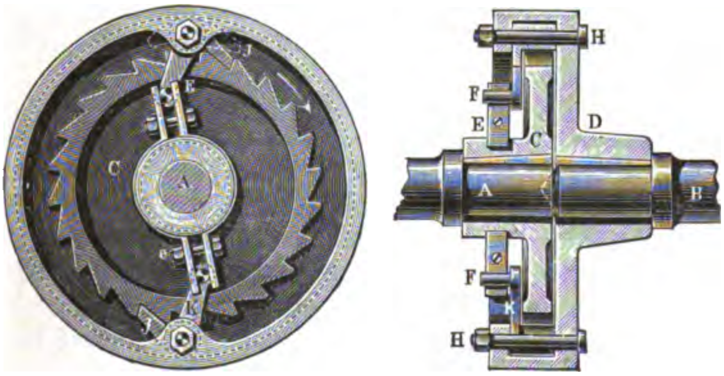
* See article by *J. Köchlin*, in Bulletin de la Société de Mulhouse, No. 108, 1850.

This will last until the condition of equilibrium is re-established, in which all the motors again take part in driving the shaft. It is evident that if the couplings act in the manner supposed, such an arrangement will conduce to greater economy in running; for example, if a water-wheel has a larger supply of water, the steam engine will be partially unloaded and the governor will act to economize steam. A dragging of the steam engine cannot now occur as when rigid couplings were used, nor is there the loss of power which would result if the motors were separate and a part of the abundant supply of water were wasted. There are two excellent couplings used for these purposes, one by *Pouyer-Quertier*, of Bordeaux, and the other by *G. Uhlhorn*, of Grevenbroich, which will now be described.

In Pouyer-Quertier's coupling, Fig. 91, the driven shaft *A* has a ratchet wheel *C* keyed to it, the nave of which contains a circular groove for holding the wrought-iron strap *E*. Since the two halves of the strap can be tightened more or less by the screws *e*, we may generate the desired friction in the groove of the nave. To the driving shaft *B* is fastened the nave *D* of a disk carrying two bolts *H* and jointed to these bolts two pawls *JJ*. These pawls *J*, which work with the teeth on the ratchet wheel *C*, are also provided with the

FIG. 91.



little lever-arms *K*, carrying pins *F* fitting in between the two halves of the strap. If we imagine the shaft *B* to move in the direction of the arrow, the pawls *J* will drive the ratchet wheel *C* and consequently the shaft *A*, the strap *E* moving with the nave of *C* as if the two parts were one. But if, as described above, the shaft *B* slows down, together with its disk *D* and pawls *J*, then by means of the strap *E* and the pins *F*, the more rapidly moving shaft *A* will lift the pawls out of the ratchet wheel, and will bring them into the positions shown by the dotted lines. A pin prevents further motion of the pawls, consequently the strap kept back by the pins *F* will move with the slower velocity of the shaft *B*, while the nave *C* slides in the strap with a velocity corresponding to the excess of motion of *A* over *B*. The motor *B* no longer acts on the shaft *A*, and therefore its velocity increases. As a result the work of friction between *C* and *E* will constantly grow smaller, and will cease at the instant in which the velocity of *B* again becomes equal to that of *A*. The slightest further increase in the velocity of *B* and *D* will cause the bolts *H* to move faster than the pins *F*, which are held back by the strap, consequently the pawls *J* will gear into the ratchet wheel, and the shaft *B* will again take part in driving the shaft *A*.

Uhlhorn's coupling* is shown in outline in Fig. 92. Here the driving shaft *A* has a keyed disk *B* carrying two jointed pawls *C*. These pawls enter into suitable slits in the ring *D* fastened to the driven shaft and drive this ring and the shaft in the direction of the arrow. If the shaft *A* and the disk *B* slow down, the more quickly moving ring *D*, by means of the inclined surfaces of its slits, will cause the pawls to fall into the position shown by the dotted lines, where they will remain till the shaft *A* again moves with a somewhat greater velocity than the ring *D*, when they will once more rise and enter the slits. The springs *E* inside of the ring *D* help the pawls to rise and enter the slits. The covering strips *G* serve to guide the pawls *C* while they are turning.

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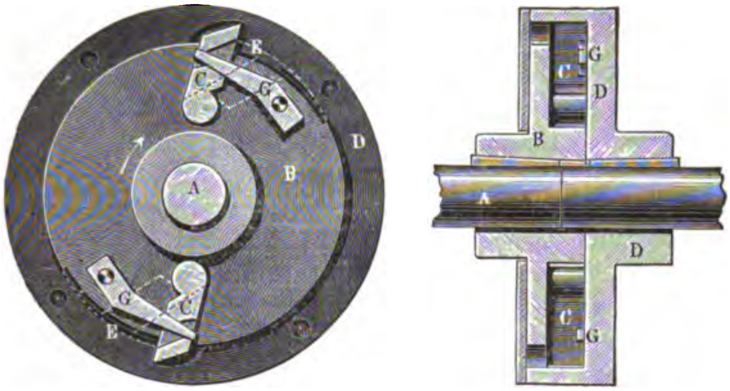
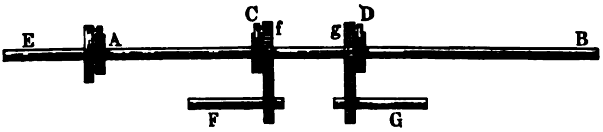
* *Dingler's* Polyt. Journal, Vol. 144, p. 461, 1857, and Vol. 179, p. 419, 1866.

FIG. 92.



Each of the motors is to be united with the shafting by some such coupling as has been described; it is not, however, necessary that the motors should act at the end of the shafting, for they may act at any point whatever. For example, if *A B*, Fig. 93, is a line of shafting which is to be driven by three motors whose shafts are *E*, *F*, and *G*, the couplings *A*, *C*, and *D* may be so placed that *e*, *f*, and *g* will be their driving parts (the disk *D* with the pawls, Fig. 91, or the disk *B*, Fig. 92). The part *e* may be directly fastened to the shaft of the motor *E*, while the driving parts *f* and *g* are so placed as to turn loosely on the shafting *A B*, and provided with teeth which gear into the toothed wheels on the shafts *F* and *G* of the other motors.

FIG. 93.



Other couplings for motors have been designed, which act by friction, for example, a toggle-jointed brake effecting the union of the shafts when they are running normally, the brake becoming ineffective when the motor in question lags.

(See *Reimann's* Coupling, in Dingl. polyt. Journal, Vol. 188, p. 452, 1868.)

§ 33. **Bearings.**—Each journal is enclosed by a machine part which serves to support the journal, or, as is the case with axles for railway carriages, to support the load borne by the journal. This part is called the *bearing*. In every case the internal form of the bearing is geometrically the same as the external form of its journal, that is, it must be the whole or a part of a surface of revolution. Cases in which the bearing only partially encloses the journal may be found in the journals of heavy water-wheels, which only need to be supported from below, and also in the axle-boxes of railway rolling stock, in which only the upper half of the journal is enclosed by the bearings. We will now consider only bearings for fixed axes, leaving the others for future treatment under the head of carriages.

The external form of a bearing depends principally on the ease and firmness with which it may be fastened to foundations or other supports, and also upon the ease with which it may be constructed. Within these limits manifold forms are possible and current. The dimensions of the bearing naturally depend on the length and diameter of the journal to be supported, the proportions of the individual parts of the bearing being determined from the forces acting upon them, according to the principles governing the strength of materials. We will not give the formulas for determining the dimensions of the bearings, since they belong specially to machine design, and will only give a few typical and fundamental forms of the many possible and current constructions.

In order that the shaft, while in motion, may be well supported by the bearing, it is necessary for the bearing to fit the journal as accurately as possible, thus avoiding all play between them, for when there is play the journal runs very unsteadily, and at high speeds causes shocks and vibrations and consequent loss of power and rapid wear. Since the most accurate constructions will wear with use, we must make the bearing in two or more parts, which can be moved so as to diminish, as much as possible, the dead space or play caused by the wear. A bearing is sometimes simply a hole bored through a piece, but this is done only in comparatively unimportant machines, when the forces are small and the turning speed is low. In more substantial constructions, whole (*i. e.* undivided) bearings are rarely used except with joint pins which are subject to but slight oscillating motions, also in regulating-gear and stop-gear which are seldom set in motion and then only for very brief periods.

The material of the bearing must be chosen with special reference to diminishing the friction at the circumference of the journal as much as possible, thus reducing not only the loss of work, but the wear of the material of the bearing. We must also make the interior of the bearing of a soft material which will wear the journal but slightly, for a rapid wear would in time cause a fracture of the shaft and, besides, it would generally be more difficult to renew the journal than to renew the interior of the bearing. In order that the worn part may be readily renewed, the bearings are provided with easily exchanged *boxes* or *brasses* which surround the journal, have a small coefficient of friction, attack the journal but slightly, and can be adjusted so as to take up the wear, thus fulfilling all the conditions enumerated above. These boxes or brasses are usually metallic alloys, the most common being *gun-metal* or *bronze*, composed of different proportions of copper, zinc, and tin. Brass, which consists of copper and zinc, is less suitable for the so-called *brasses* on account of its hardness and rigidity. Neither are cast-iron boxes often employed except in rough work, for the hard cast iron rapidly cuts away a wrought-iron journal, although the coefficient of friction between them is small. Besides bronze, certain easily fused alloys known as *Babbitt*, or *white metal*, are much used: they contain principally tin and antimony, and sometimes a little copper. One advantage of these alloys is that, on account of their fusibility, they may be poured directly into the space between the journal and the body of the bearing, and thus readily take their required form. On the other hand there is danger that the material will melt when the journal is running at high speed and the lubrication is imperfect.

Certain hard woods are also used for bearing surfaces, namely white beech-wood and lignum vitæ. The first of these makes a passable bearing surface, while the very hard, firm lignum vitæ is an excellent material for this purpose, and is preferable to the metallic bearings where the latter are inclined to heat. The pitchy character of this wood seems to be conducive to smooth running.

The greater the velocity of the journal the more necessary it is to keep the bearings lubricated. By this means the friction and the heating may be greatly diminished, and the wear of the material reduced to a minimum. Sometimes solid material, like lard, tallow, etc., is used for lubricants, but most often fluid substances like animal or mineral oils, are employed. In general the rule will hold that the fluidity of a lubricant should increase with the velocity of the journal and the diminution of the pressure on the lubricated surfaces. Consequently solid lubricants are to be recommended only in cases where the motion is slow and the pressure considerable; at all events it must be regarded as a defect of solid lubricants that they must be melted before they can get between the surfaces in contact. This presupposes sufficient heat to melt the lubricant, although one of the principal objects of lubrication is to prevent heating. There are many different devices for effecting the lubrication continuously and automatically. Such devices should not waste the costly lubricant, but should renew it as rapidly as it is rendered useless by the oxidation of its pitchy constituents, and by becoming thick with the worn-off dust of the metal. A good and regular supply of oil is necessary, especially for high speeds, if the bearings and their journals are to be kept in good condition, for the wear which results when heating takes place is extraordinarily large, and very rapidly leads to the total destruction of the one part or the other. Sometimes the journals of very heavy axles, such as are used in rolling mills, are kept sprinkled with water, the object being not so much to lubricate them as to keep them cool.

An important requirement of all bearings is that they shall be easily set up and adjusted to their correct position. That this is a necessary condition is evident from the following considerations. Let *A*, *B*, and *C* be three bearings on a line of shafting, and let us suppose that they have been incorrectly set up so that they are not in line. It is evident that the axis of the middle bearing at *B* is subjected to a lateral deflection at *A* and *C*, and consequently the shaft will react against the bearing *B* with a force corresponding to the amount of deflection, and will cause considerable friction there. During every revolution, all of the individual fibres of the shaft are alternately stretched and compressed, for the bending always takes place in the same direction. If the line of shafting is supported at many points, the same observation holds with reference to any three successive bearings, and we see that the shaft will be subject to periodical bendings in all directions. As a result, particularly when the shaft is run at high speed, hurtful vibrations will occur, which will cause the bearings to heat and the shafts to squeak. This produces a great loss of power, for it is well known in practice that considerable power is required to run a line of shafting that has been imperfectly set up, even when the load is thrown off, and that, on the other hand, when a long line of shafting is easily set in motion, it is a sure sign *that all the bearings are very nearly in the same straight line.*

In order to meet this last requirement as conveniently as possible, it is customary to arrange the foundation plates of the bearings so that they can be slightly shifted in one or more directions, and thus adjusted to their correct positions. In some good designs this matter of adjustment has been carried still further, the outside of the *boxes* being made to fit a spherical seat, which is concentric with the centre of the journal. This arrangement securely supports the box and permits the cylindrical bearing surface of the latter to adjust itself accurately to the direction of the journal. In this manner is removed all that lateral stress on the journal which would exist if the boxes or brasses were fixed to the body of the bearing and their axis did not coincide with the axis of the journal. These bearings are particularly desirable for long lines of high-speed shafting; they are, however, more costly than the ordinary constructions, and this is probably why they are not more commonly used. After these general remarks concerning bearings, we will proceed to describe their most common forms.

§ 34. **Bearings for Horizontal Shafts.**—The arrangement of bearings for the journals of horizontal water-wheels has been frequently illustrated in Vol. II. of this series. We also know from the preceding paragraph that such journals are not always completely enclosed by their bearings, the upper part being absent, as in Fig. 94. Here *A* is a cast-iron bearing which is provided with two ribs *a* that fit into corresponding grooves of the wooden block *B B*. The block *B B* is fastened to a larger wooden block *C C* by keys *b b*. By drawing or driving these keys, the bearing may be adjusted. Such uncovered bearings are now seldom used, and occur principally in wooden water-wheels, where there is no upward pressure on the journals and the weight of the wheel is sufficient to keep the journals in contact with their bearings. Evidently the bearing cannot be open above when the journal is subjected to an upward pressure, such as often obtains when toothed gearing is employed to transmit the water power to a line of shafting. In these cases closed bearings are employed, that is, such as have covers or caps, and this construction is in common use for all other shafts.

FIG. 94.

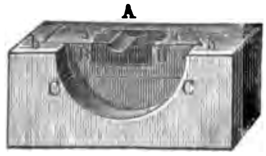


Fig. 95 is a top and side view of a closed bearing designed by *Redtenbacher*,* which is much used. The base *B* of the cast-iron casing of the bearing rests on the cast-iron bedplate *A*, which is bolted to the foundation. To fasten the bearing to the bed-plate, screw bolts are used which pass through elongated holes in the base *B*, and permit a slight lateral shifting of the bearing on the bed-plate. When the bearing has been correctly placed, it is secured to the bed-plate by wrought iron or wooden keys driven in between the base *B* and the shoulders *G* of the bed-plate. The casing of the bearing consists of the lower part *C* and the cap *D*, and is hollowed to fit the brasses *E E*. The cap *D* is likewise united to the lower part *C* by two or four bolts, two for small and four for large journals (the top view shows one arrangement of cap-bolts on the right, and the other on the left). The figure shows a small space left between the lower part *C* and the cap *D*, as well as between the two brasses *E E*. This allows the wear on the brasses, caused by the friction of the journals, to be taken up by screwing down the caps. Unless some provision is made for taking up the wear, the journal will soon run unsteadily. The cup *F* on the cap contains the oil for lubricating.

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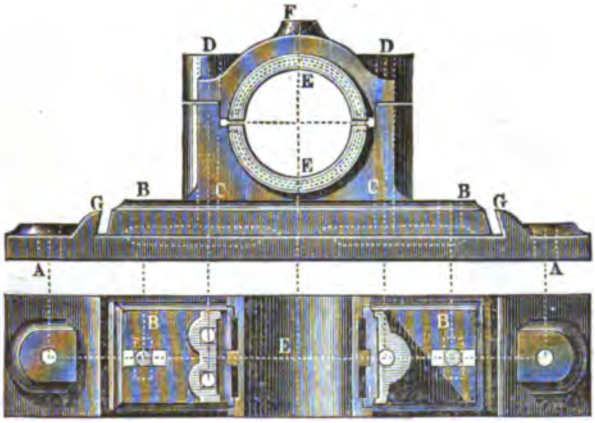
* For proportions of this bearing, see *Redtenbacher's* Resultate für den Maschinenbau.

Fig. 95.



It is especially important that the brasses *E E* be carefully fitted to the casing *C*, otherwise they will wabble or vibrate when the journal turns rapidly, and thus cause heating of the journal, with all its injurious consequences. If this fitting is done upon the lathe it will be more accurate than when done by hand, and much more quickly and easily done than by the planer. Consequently those bearings having brasses whose exteriors are surfaces of revolution are decidedly preferable to those without this advantage. The brasses of some bearings are prismatic (Fig. 96), the object being to prevent their turning in the casing, which must correspond in shape; but the fitting of such brasses is always difficult and generally less accurate than when the surfaces to be fitted can be turned. Brasses can be kept from turning in their bearings either by a pin (*s* in Fig. 97), or by a pair of lugs (*k*, Fig. 98), which must fit corresponding depressions in the casings.

FIG. 96.

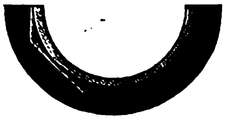


FIG. 97.

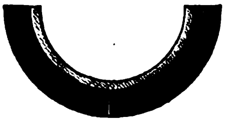
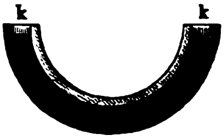


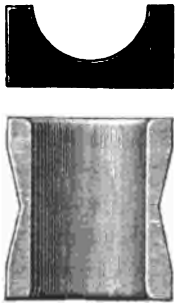
FIG. 98.



Experience shows that brasses must be thus locked if their turning is to be prevented, although the friction between the brasses and casing is greater and has a larger lever arm than the friction between journal and brasses. When, however, lignum vitæ is used for the bearing surface, the nature of the material renders the form given by *R. Werner* (Fig. 99) a desirable one.

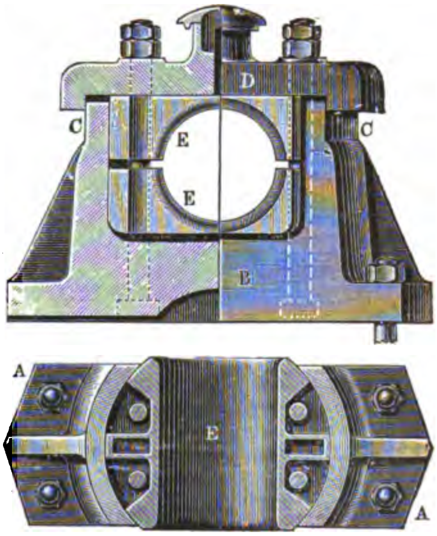
In connection with this matter of easily and accurately turning the brasses, the bearing given by *v. Reiche* deserves special mention. Fig. 100 gives a top view of this bearing with the cover removed, and another view which is partly front and partly sectional. Here the exterior of the brasses *E E* has the form of a vertical cylinder, consequently if the base *A* of the casing is fastened to the face plate of a lathe, the lower half *B* of the bearing can be accurately bored to fit the brasses. In like manner the upper half or cap of the casing can be turned, and, since little handwork is needed, the construction is comparatively a simple and accurate one. Another advantage of this form of bearing is that its lower brass is provided with a *flat* surface which rests

FIG. 99.



on the casing, thus permitting the *lining* of the bearing, *i. e.*, the putting of a thin plate under the brass, when its axis has been lowered by wear. The rest of the construction is clear from the drawing. To be sure, the large brasses in this bearing require considerable of an expensive material, bronze (or gun-metal).

FIG. 100.



§ **35.**—We have already mentioned that bearings for long high-speed shafting are sometimes made so that their brasses have a certain freedom of motion in the casing which permits them to take the direction of the journal. Such a bearing with spherical brasses is given by *Redtenbacher*,* and is shown in Fig. 101. This bearing differs from that shown in Fig. 95 chiefly in that the brasses D are surrounded by ribs or bosses $E\,E$, which are turned to the form of a sphere whose centre coincides with that of the journal. As the casing and cover are turned to correspond, the brasses have a slight motion, enabling them to support the journal uniformly through its entire length, while, with fixed and inclined brasses, the whole weight is concentrated on one edge. In any case the motion required is but slight. The exterior of the brasses will not be exactly spherical after the cover is screwed down to take up the wear.

It is an important rule in machine design to so arrange a bearing that the resultant pressure to which its journal is subjected shall be directed, if possible, against the base of the bearing, or, if that cannot be done, against its cover, but not against the joint between the brasses. The reasons for this rule are that the joint, that is the edges of the brasses, is never as firm and strong as the solid part, and because the wear which takes place in the plane of the joint cannot be taken up by bringing the brasses closer together. In some cases, placing the joint of the bearing perpendicularly to the resultant pressure inclines the joint, as, for example, in cranes, where the bearing for the axle of the drum is at right angles to the inclined chain. Inclined bearings are also found on the shafts of horizontal steam engines, because this inclined position corresponds better to the variable direction of the resultant pressure than an upright position. In this case the pressure on the journals is composed of the vertical pressure due to the weight of the fly-wheel and the horizontal pressure acting on the two sides of the piston alternately.

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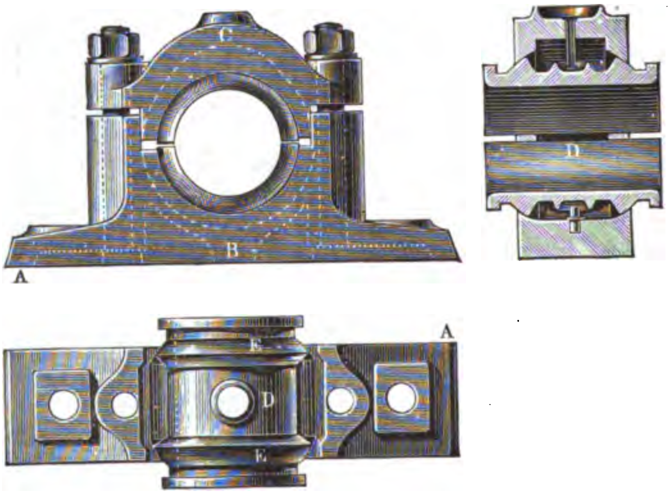
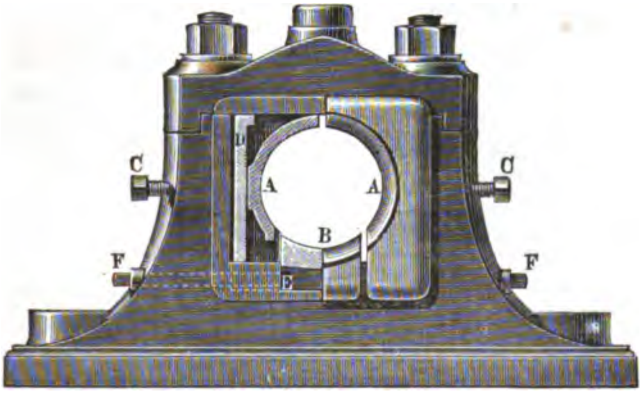
* *Redtenbacher.* Resultate für den Maschinenbau. Pl. XIV.

FIG. 101.



It frequently happens that, in cases like the last, bearings are employed which differ from those hitherto described in having more than two brasses, in order to take up the wear in several directions. Such bearings have three or four brasses; Fig. 102 representing one with three brasses, which is given by Reuleaux.* Here the journal is enclosed by three brasses, *A*, *A* and *B*, of which the two *A*, *A*, have a horizontal adjustment controlled by two set screws *C*, *C*, acting upon the plates *D*. By this means the wear in the bearing due to the horizontal pressure on the piston can be removed. Moreover the lower brass *B* rests on two parallel and adjacent wedges *E*, which are guided by grooves in the bearing. These wedges are drilled and tapped so that they act as nuts to the two set screws *F*. When these are turned they cause the wedges *E* to slide, and thus lift the lower brass *B*. In other designs the side brasses are similarly adjusted, the wedges moving vertically, however.

FIG. 102.



§ **36. Wall-boxes, Trestle-bearings, and Hangers.**— Bearings for horizontal shafts cannot always be bolted to horizontal bed-plates, but must frequently be fastened in other ways to buildings or frames. In such cases the body of the bearing is changed into forms prescribed in each case by the particular position occupied by the shaft. For example, if a shaft is placed parallel to and at a short distance from a wall, we must give to the casing of the bearing a vertical wall-plate, by means of which it can be fastened to the wall. In this way wall-boxes are obtained, of which one is shown in Fig. 103.[*] The fastening is here effected by four bolts A_1 and A_2, which pass through the wall, on the other side of which their heads press against washers, transferring the pressure to as large a surface as possible. The vertical pressure exerted upon the bearing by the shaft is resisted by the friction between the plate *B* and wall, and consequently the bolts *A* must press the plate *B* with sufficient pressure against the wall to generate the required friction. If the friction were not sufficient to resist the vertical pressure of the journal, the excess of the latter would tend to shear the bolts *A*. To avoid such an unfavorable straining action on the bolts, it is well when the pressure is considerable to provide the plate *B* with shoulders, such as are shown by the dotted lines *b b*, projecting into the masonry, which thus directly resists the vertical pressure by its compressive strength. The bolts, moreover, are protected from any shearing tendency due to lateral forces in a manner similar to that employed for ordinary bearings, Fig. 95 for example, where wedges are placed between the plate *B* and the shoulders *G*. In general, we may say that bolts should be strained only in the direction of their length, that is, subjected to tensile stress only, as they are little suited for resisting bending or shearing forces.

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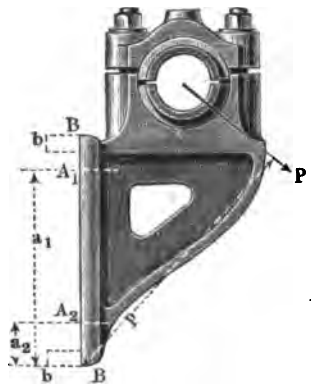
* *Reuleaux.* Der Constructeur.

FIG. 103.



The bolts *A*, Fig. 103, are subjected to tensile stress whose amount depends on the magnitude of the turning moment of the force acting on the journal, about an edge of the wall-plate B. Let P represent the total force on the journal, p the distance between the direction of this force and the edge of B, a_1 and a_2 the distances of the bolts from the same edge, and A_1 and A_2 the tensile forces on the bolts, then there will be equilibrium when

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[*] The bearings shown in Fig. 103 to Fig. 106 were taken from *Reuleaux's* Constructeur.

$$Pp = A_1 a_1 + A_2 a_2 .$$

Since a_2 is small, we may neglect the effect of the bolts A_2, and then

$$Pp = A_1 a_1 ;$$

and from this we get the *total stress in the two upper bolts* equal to

$$A_1 = \frac{Pp}{a_1}.$$

From this we see that A_1 will be smaller as a_1 is larger, and therefore, that under all circumstances it will be an advantage to place the bolts as closely as possible to the edge of the plate. For the same reasons which require the distance a_1 of the upper bolts from the lower edge of the plate to be as large as possible, require that the distance of the lower bolts A_2 from the upper edge of the plate be as large as possible when the force p acts in the opposite direction.

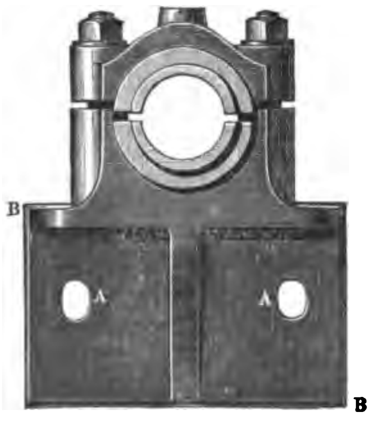
These remarks concerning the fastening of wall-boxes are applicable to all cases in which it is a question of fastening a machine part to a wall, when the part projects from the latter and is acted upon by an excentric force. The pull of the bolts can of course only be resisted by the wall, and too great care cannot be taken in fastening wall boxes and brackets.

When the moment of the force Pp is large, because of a large pressure P or considerable overhang p of the bracket, and the thickness of the wall small, it is almost unavoidable that the portion of wall grasped by the bolts will loosen itself from the rest.

This is particularly liable to occur when the pressures, as in cranks, continually change in direction and magnitude, producing vibrations which crumble the mortar. Sufficient thickness of wall and large washers, as well as the best cement, are indispensable conditions of substantial construction. The firmness of the fastening will also be much increased if the wall extends considerably above the bracket, for the superincumbent weight will increase the friction in the lower part of the wall, and thus resist any tendency to tear out a portion of the latter. The attaching of brackets for heavy shafting in the lower stories of buildings is much less hazardous than in the upper stories. If, however, one is compelled to put up heavy bracket-bearings or wall-boxes on low walls, sufficient security may in some circumstances be attained by passing vertical bolts down through the wall, and thus compelling the wall below the bracket to share the strain.

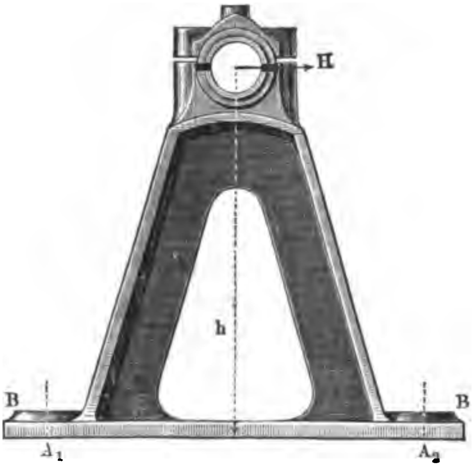
Fig. 104 likewise represents a wall-box, but it is only used when the direction of the shaft is perpendicular to the wall. The cast-iron plate *B B* is placed normal to the axis of the bearing and fastened to the wall by bolts *A* in the same manner as the wall-box previously described. If the shaft passes through the wall to which it is perpendicular, an ordinary bearing like that shown in Fig. 95 is usually preferred, because it offers more solid support. In this case a rectangular aperture is made in the wall and fitted with a corresponding cast-iron frame, whose lower plate acts as a bed-plate for the bearing, while its upper plate supports the wall above. The solidity of the bed-plate may be increased by fastening it with vertical bolts passed down through the wall.

FIG. 104.



If a shaft is to be placed at a certain height above the floor, a *trestle-bearing*, Fig. 105, may be employed. Here the foundation bolts A_1 A_2 fasten the base $B\,B$ firmly to a block of masonry, whose weight must suffice to prevent the frame from upsetting. If H represents the horizontal pressure exerted by the shaft upon the bearing, h the height of the shaft above the bearing, Q the weight of the masonry, and b its half breadth, then for stability we must at least have

FIG. 105.

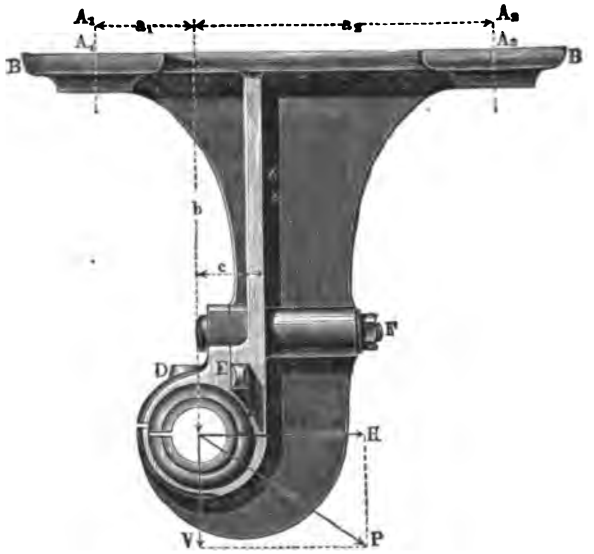


$$Q\,b = H\,h, \text{ or } Q = \frac{H\,h}{b}.$$

But it will always be well to make Q considerably larger than this, say once and a half or twice as large. In this case also the foundation bolts should be provided with sufficiently large washers to distribute the pressure evenly on the masonry, and for the same purpose, when the tipping moment is large, it is better to increase the *number* than the *size* of the bolts.

Finally, if a shaft is to be hung at a certain distance from the ceiling, *hangers* may be employed. These may be divided into one-armed and two-armed hangers. A good idea of the latter may be obtained by supposing the bearing in Fig. 105 to be inverted, and its base fastened to the ceiling. For light shafting one-armed bearings are used, a very common form being shown in Fig. 106. In this case the method

FIG. 106.



of fastening the cover D is peculiar, the cap being pressed down upon the upper brass and the journal by the key E, and then fastened by the horizontal cap-bolt F. It is evident from the figure that the vertical pressure V of the shaft subjects the arm of the hanger to a bending strain as well as to a tensile strain. Vc is the bending moment for any cross section whose centre of gravity is at the distance c from the vertical force V. A resultant stress on the arm is therefore to be determined according to the rules established for eccentric loads in Vol. I., § 271. Let a_1 and a_2 represent the distance of the bolts A_1 and A_2 from the direction of the vertical component V, then the tensile force exerted upon the bolts will be:

$$V\frac{a_2}{a_1 + a_2} \text{ for } A_1,$$

and

$$V\frac{a_1}{a_1 + a_2} \text{ for } A_2.$$

To these must be added the tensile stress due to the horizontal component H, whose moment $H h$ is to be determined in the same manner as for the bolts of the wall-box in Fig. 103.

In the examples hitherto given we have always assumed that the bearing proper and its support (bracket, frame, or arm) are cast in one piece. This design avoids inconvenience in construction or setting up only so long as the distance of the shaft from the supporting surface is small. When the bearings overhang considerably, it is customary to bolt an ordinary bearing to a separate cast-iron frame. The forms and dimensions of these frames differ widely, according to circumstances, though in general they correspond to the typical elementary forms given in Figures 103 to 106. Frequently these frames must simultaneously support the bearings for two or more parallel or intersecting shafts, consequently there is much variety in the forms of the compound bed-plates. For further examples of these forms, reference may be made to works on machine design.*

In constructing such bed-plates cast iron is almost always used, both because complicated pieces can be easily cast, and because the material is so rigid, yielding less than wrought iron. Sufficient rigidity and invariability in the supports of bearings are absolutely necessary for the safe and smooth running of machinery. In designing these frames, economy of material would require that *ribbed* cross sections be chosen, whose *moment of inertia* is as large as possible. But in frames subject to shocks and vibrations, as, for example, in the housing frames of rolling-mills, sufficient security against shocks is not to be sought in a large moment of inertia of the cross sections, but in their having as *large an area as possible*, that is the frames must be *massive*. See also Vol. I., § 348 to 351, on Resistance to Impact, etc.

§ 37. **Bearings for Upright Shafts.**—*Foot-step* bearings are used to support the pivots of upright shafts. They usually contain a plate of steel or chilled iron, which directly supports the pivot. This plate is bedded in a casing ordinarily of cast iron, either fixed, or admitting of a certain amount of adjustment in a horizontal or vertical direction. Since the lower surface of the pivot is apt to be flat or slightly curved, as is the case with the larger sized shafting, the plate is naturally flat or slightly curved also, and consequently the pivot, particularly when it is subjected to a sensible lateral pressure, must be restrained laterally by a bronze bushing or socket.

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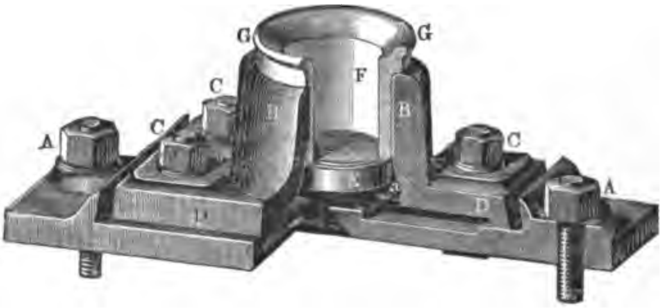
* For bed-plates, frames, etc., see *Redtenbacher*, Resultate f. der Maschinenbau ; *Wiebe*, die einfachen Maschinentheile ; *Reuleaux*, der Constructeur, etc.

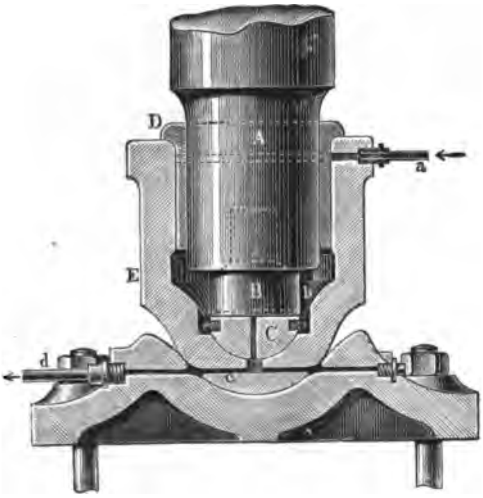
An ordinary foot-step bearing, as it is used in turbines and non-adjustable, is shown in Fig. 107. Here *E* is the extremely hard steel plate upon which rests the flat end of the pivot, the bronze socket *G G* serving to guide it. The plate *E* and the socket *G* are driven fast into the cast-iron pivot-box *B*, which, in turn, is fastened to the bed-plate *A* by the bolts *C*. The pin *a* prevents the plate *E* from turning in the casing *B*. When this bearing is not submerged it is lubricated from above, the flare of the socket *G* serving to hold the oil. When the bearing is submerged, as frequently happens in turbines, oil must be supplied by a special tube, see Vol. II., Fig. 481, 486, 487, or through a hole drilled in the vertical shaft, see Vol. II., Fig. 514.

FIG. 107.



If the shaft deviates at all from its vertical position—say on account of its upper bearing yielding—the fixed non-adjustable socket will be inevitably subjected to a binding action, and the flat pivot will exert a one-sided pressure upon its plate. When the shaft turns rapidly, as is the case in centrifugal machines, this will not only cause much wear, but also heating of the journal, thus leading to its rapid destruction. The bearing, Fig. 108, given by *Redtenbacher*,* is designed to avoid this evil. The shaft *A* has a steel pivot *B* set into its lower end and resting on the flat plate *C*. The lower part of the latter is spherical and fits a corresponding depression in the pivot-pot *E*. This gives a freedom of motion to the pivot-plate, which enables it to keep perfect contact with the pivot when there is a slight oscillation of the shaft. The lower part of the pot *E*, which contains the bronze socket *D* for guiding the pivot, is likewise spherical, so that it can follow the motion of the shaft and avoid binding of the neck *A* in the socket *D*. The oil here enters from above through the tube *a*, passes between *A* and *D* into the space *b*, and from thence to the lower surface of the pivot by means of two intersecting grooves on that surface. A central hole in the pivot plate connects with the channel *c* and the discharge pipe *d*, thus permitting continual renewal and removal of the new and old oil respectively.

Fig. 108.



In all upright shafts very much depends upon their being exactly vertical, consequently foot-steps are frequently so constructed that the pivot-plate and cup are capable of a slight adjustment by set screws. This furnishes a simple and certain means of making the shaft plumb when it is set up, and correcting any deviation from this vertical position which may occur afterward. Such arrangements have already been mentioned in Vol. II., Fig. 514. A common bearing of this kind is shown in Fig. 109; the *pivot-plate A*, which is carried by the *pivot-cup B*, is adjustable in the somewhat larger *pivot-box C*.

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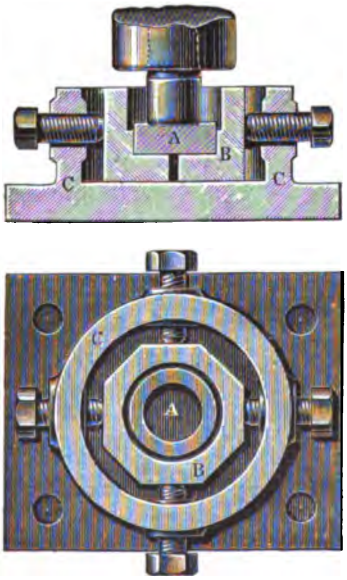
* *Redtenbacher*, Resultaten für den Maschinenbau.

When, in the course of time, the pivot and its plate become sensibly worn, the machines connected with the shaft are in danger of greater evils than mere wear. For example, if the shaft belongs to a turbine, the wear of the pivot will cause a sinking of the wheel, and the space between the wheel and the guides, in *Fontaine* and *Henschel* Turbines, will become greater, and thus increase the loss of water. On the other hand, in a grinding mill, the runner sinks, and there is danger that it will get too near the bed-stone and thus cause grinding of the stones. On this account footsteps are also constructed, so that the pivot-plate, the shaft, and the parts rigidly connected with the latter can be raised or lowered together. In this way the space between the wheel and the guides of the turbine can always be regulated, and in grinding mills it also enables us to regulate the action of the stones on the grain. Frequently pivot bearings are provided with a vertical as well as a horizontal adjustment (see Vol. II., Fig. 507, where there is a turbine bearing by *Laurent*, which has both kinds of adjustment; in Vol. II., 504, a turbine bearing by *Fourneyron*, is capable of vertical adjustment only). Fig. 110 is a design, by *Wiebe*,* of a bearing for a grinding mill,

FIG. 109.



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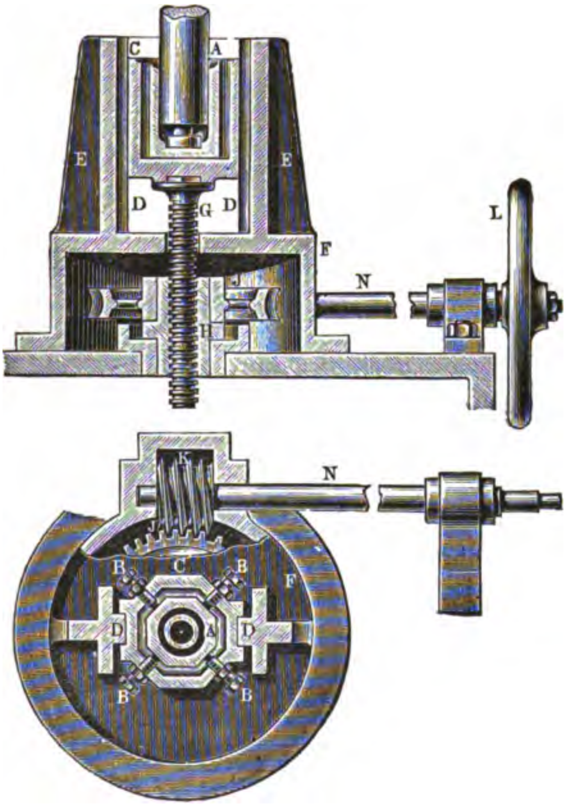
* *Wiebe*, die Lehre von den einfachen Maschinentheilen.

whose pivot-plate is adjustable horizontally and vertically. An arrangement for raising and lowering the mill-spindle and runner is called the *millstone regulator*, and its object is not only to diminish the wear on the pivot, but also to regulate the grinding process. The cup *A*, cylindrical within and octagonal without, is adjustable in the pivot-box *C*, by means of the four set screws *B*. The box *C* has two grooves, which guide it along the ribs *D* on the standards *E E* of the frame *F*. The vertical adjustment of the pivot-box and the mill-spindle with its heavy stone, is effected by the screw *G*, which moves up and down when its nut *H* is

FIG. 110.



turned. This nut is easily and safely turned by means of a worm-wheel *J* fastened to it, and worm *K* keyed to the arbor *N* of a hand wheel *L*.

Other designs for pivot bearings when the shaft is *suspended* from the bearing, as is sometimes the case in turbines, may be seen in Vol. II., Figs. 488, 512, and 521. Fig. 488 also shows how little change is made in the design when a collar-journal (see § 5) is used instead of the ordinary pivot.

Every upright shaft must have besides its pivot bearing a neck bearing to guide it, which, in general differs but little from the bearings for ordinary shafts. Any ordinary bearing, such as is shown in Fig. 95, can be used as a neck bearing, provided it is suitably set up; that is, so fastened to the support that the bore or axis of the bearing is vertical. Frequently, however, neck bearings are designed somewhat differently. Fig. 111 is an example. Here the three brasses *C* which guide the shaft, are moved in the body of the bearing by the set screws *B*. The bolts *D* fasten the three-cornered base of the bearing to its support, which is, of course, provided with a central opening for the shaft.

FIG. 111.

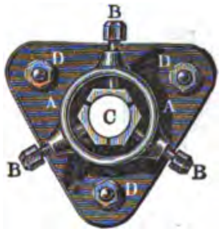


FIG. 112.

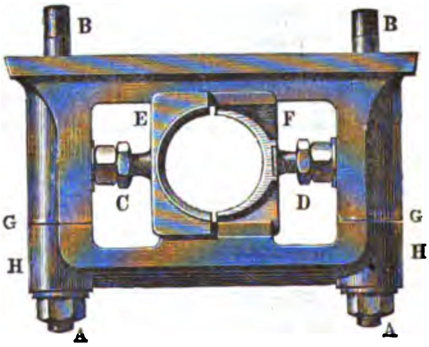
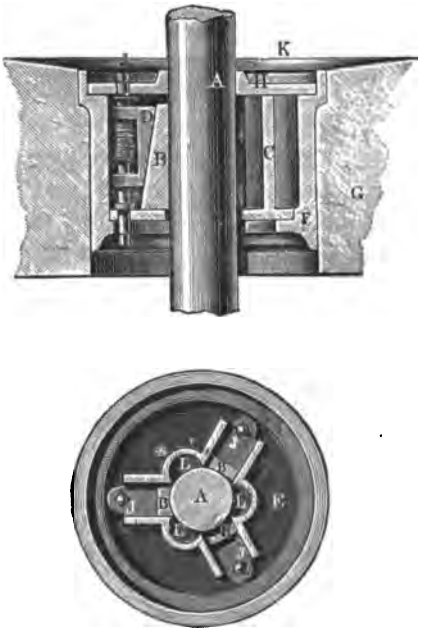


Fig. 112, taken from *Reuleaux's* Constructeur, is likewise a neck bearing for upright shafts. The two brasses *E* and *F* can be adjusted in the frame of the bearing by the bolts and their nuts *C* and *D*. The brasses can also be taken out of the frame without taking off the cover of the bearing, the openings in the frame on each side of the brasses permitting this. The bolts *A B* have threads at the two ends, and are provided with collars *G G*, which fit into corresponding recesses in the casing. By means of the nuts on *B B*, the bearing may be fastened to its support, and the nuts *A A* will unite the cap *H H* to the part *G G* of the bearing. By this arrangement, which is occasionally found in other bearings, it is possible to take off the cap without removing the part *G G* from its support.

A peculiar kind of neck journal is found in grinding mills. Here an upper portion of the mill-spindle is guided in a bearing which is set into a corresponding eye of the lower stationary millstone (*bed-stone*). Fig. 113 shows such a bearing, and is taken from *Wiebe's* Lehre von den einfachen Maschinentheilen. The spindle *A* is here guided by three brasses *B*, set between the ribs of the cast-iron holder *C*, and moved radially by keys *D* sliding on the backs of the brasses. The holder *C* is fastened by its lower rim *E* to a projection on the cylindrical pivot-box *F*, which latter is firmly cemented to the eye of the bed-stone *G*. The disk *H* serves to fix the holder *C* and to guide the upper journals of the three sets crews *J*, by the turning of which the above mentioned keys *D* can be shoved so as to press the brasses *B* against the spindle. The screws can be conveniently turned from below, thus avoiding the lifting of the runner from the bed-stone. To prevent the

FIG. 113.



grist from getting into the bearing, the disk *K* is set on the pivot-box *F*. The three recesses *L* in the holder serve as receptacles for felt, cow hair, or any other oil-absorbing substance, thus insuring smooth running of the spindle.

§ **38. Lubricants.**—The necessity of the careful lubrication of journals has already been referred to, § 33. The object of lubrication is to diminish the coefficient of friction by suitable means, for the loss of work caused by the friction of journals is proportional to this coefficient, and we can assume that the wear of the bearing, as well as the heating, is directly proportional to it. All substances which diminish friction when they are placed between rubbing surfaces, are suitable for lubricants, their action probably consisting in filling up the little depressions which exist on the surfaces of all solid bodies. For this purpose it is in general necessary that the lubricant be in a fluid or at least pasty condition. Experience shows that among fluids all animal and vegetable oils, without exception, as well as many mineral oils, possess in a very high degree the property of diminishing friction. Water also, in some circumstances, diminishes the coefficient of friction, but it is used less as a lubricant than as a means of cooling the journal and bearing by constant sprinkling. We have already mentioned its use for this purpose in the journals of rolls. On the whole, watery fluids are little used, their rusting effect rendering them undesirable. Semi-fluid and pasty substances, tars, fats, and certain soft soaps, are also in common use as lubricants; those melting at higher temperatures, as tallow, needing a certain amount of heating of the bearing before they will act. This, to be sure, is a drawback common to all solid lubricants, but there is this advantage: that the lubricant is consumed only while the machine is running, which is not the case with fluid lubricants, as they continue to flow when the machine is at rest, unless stopped by some special device. Moreover, when solid lubricants are employed, they to a certain extent regulate their flow, an increase of heat causing an increased melting of the material.

A fluid lubricant will the more completely fulfil its function of diminishing friction the thinner it is and the longer it can be kept in its thin fluid state, and consequently, for certain circumstances and machines, we can determine the value of an oil with respect to its greater or less fluidity and durability. In fact, it is upon this that the oil test given by *Nasmyth* is based. This test consists in placing drops of the oils to be compared in as many grooves on a slightly inclined iron plate 2 metres [6¼ feet] long. Owing to the slight inclination of these grooves the drops will flow very slowly and with velocities varying with their different degrees of fluidity. The distances traversed by two different oils in the same time, say a day, measure their fluidity, while the time during which the oil remains in motion furnishes a means of judging of its unchangeable character. These two properties must be considered in choosing oils for particular purposes.

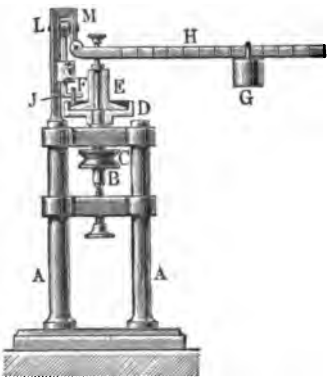
In clocks and instruments which are only occasionally oiled, it is especially important that the fluidity of the oil be as unchangeable as possible; for such purposes, therefore, a very pure neat's-foot oil is generally to be preferred. On the other hand, for swiftly running and constantly inspected arbors, such as the spindles of a spinning machine, that lubricant is best which is most fluid, and thus reduces the friction to a minimum. When a factory contains many thousand spindles, the saving of power which is effected by a good lubricant is considerable, and will well pay for itself in the saving of coal used for generating steam. For such cases the very fluid natural and artificial mineral oils are well suited, and have recently been extensively used on account of the high price of neat's-foot and rape-seed oil.

The greater or less fluidity which may be determined by *Nasmyth's* test is not always the measure of the usefulness of a lubricant for a particular purpose. Too great fluidity may even in some cases be detrimental to lasting lubrication. This happens when the surfaces to be lubricated are pressed together with much force; then there is danger that a thin oil will be rapidly forced out and lubrication cease. For this reason the axles of carts are lubricated with tar, a much more viscous material than oil. In some circumstances it is an advantage to thicken such lubricants as tallow, palm oil, etc., by adding to them solid substances like graphite, gypsum, etc.; for example, the lubricant used for the teeth of wheels contains graphite.

Consequently for the comparison of lubricants other methods and apparatus have been employed which measure the friction between two surfaces, under a given pressure, when lubricated with different materials.

The essential features of the apparatus designed by *MacNaught** for testing oils, are shown in Fig. 114. On the frame *A* is placed a vertical spindle *B*, which is driven by means of a grooved pulley *C*. On the upper end of the spindle is fastened a smoothly turned dish *D*. In this dish lies the metal disk *E*, which is likewise smoothly turned and is ground to fit *D*. By means of a movable weight *G* and a lever *H*, a definite pressure can be exerted through *E* on the dish *D*. If, now, a small quantity of the oil to be examined is placed between the disk and dish, and the latter set in motion by means of its spindle *B*, the friction between the surfaces in contact will turn the disk *E* till its pin *F* hits the vertical arm *J* of a bent lever turning about *L*. The other arm of this lever is usually horizontal and carries a sliding weight *K*. This can be fixed at such a distance *a* from the axis *L* that the lever arm *M* will remain horizontal under the influence of the weight *K* and the pressure of the pin *F* against the arm *J*. Let *b* be the length of the lever

FIG. 114.



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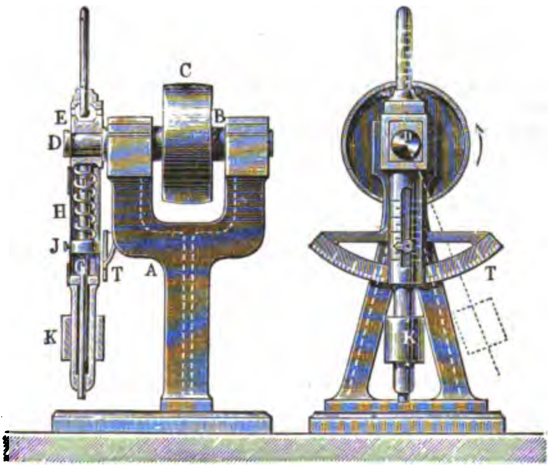
* Mechanic's Magazine, No. 774, p. 154; Dingl., Vol. 70, p. 108, 1838. See also *Wiebe*, die einfachen Maschinentheilen, and Zeitschr. Deutsch. Ing., Vol. V., p. 301, 1861.

arm J between the axis L and the pin F, and let P be the pressure exerted by the latter upon the lever J, then $P = \frac{Ka}{b}$ will evidently be a measure of the friction between the surfaces of disk and dish when reduced to the pin F, and will therefore serve to measure the value of the lubricant.

In the apparatus designed by *Thurston* (Fig. 115),* the resistance of the friction is measured by a weight, but the arrangement of the apparatus is different. Here the frame A carries the spindle B, which is uniformly turned by the belt pulley C. The front of the spindle B carries a journal D enclosed by two brasses E, which can be pressed against the journal D after oil has been admitted to the journal, the helical spring F and the screw G producing any pressure required. An index J which passes through a slit in the tube H, shows on a scale attached to H, the pressure exerted by the screw. There is besides on the arm H a weight K, which can be set by means of a screw at a definite distance from the journal D. When the spindle B turns in the direction of the arrow, the friction on the journal D will carry the rod H into a position in which the moment of the loaded arm equals that of the friction on the journal. Let Q be the total weight of

FIG. 115.



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* Jour. Frank. Inst., p. 1, July, 1873, and Dingl. polyt. Journal, Vol. 209, p. 411, 1873.

arm, spring, screw and load K; let a be the distance of the centre of gravity of all these parts from the axis B; r the radius of the journal, and F the friction at the circumference, then will

$$F = Q\frac{a \sin \alpha}{r},$$

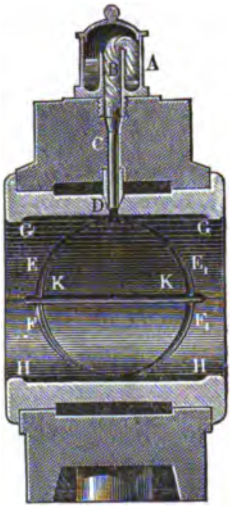
where α is the angle of deflection of the rod H from its vertical position. This angle may be easily read off from a graduated arc fastened to the frame. The upper brass inclosing the bearing contains a recess for a thermometer bulb for ascertaining the heat developed by the journal.

Lubricants have also been tested by determining the number of revolutions necessary to produce a certain rise of temperature, that lubricant being the best which requires the greatest number of revolutions for a given rise. According to Zeitschrift deutscher Ingenieure, 1871, the following relations were obtained for three kinds of oil, A, B, and C.

KIND OF OIL.	PRICE PER CWT.	REVOLUTIONS.	RELATIVE COST FOR SAME EFFICIENCY.
A. Refined rape-seed oil............	$11.25	69,975	100
B. Mineral oil.....................	7.50	41,850	111.4
C. Adulterated rape-seed oil........	9.60	26,392	225.9

§ **39. Lubricators.**—The simplest and most common means for lubricating journals with oil consists of an oil cup A, Fig. 116, placed on the cap of a bearing. A wick B placed in the cup, by its capillarity, continually furnishes oil through the hole C in the cap to the inner surface of the upper brass D. By means of the diagonal oil-grooves $E E_1$ and $F F_1$ and the longitudinal grooves $G G_1$ $H H$ and $K K$, the oil is uniformly distributed over the whole surface of the journal. This oil cup is much used because of its simplicity and satisfactory working, but is open to the objection that lubrication takes place when the journal is not running, thus causing

Fig. 116.



a waste of the lubricant. There have been many attempts to devise contrivances free from this objection. In one of these the supply of oil from the reservoir to the bearing is regulated by the periodical opening and closing of a cock, this periodical motion being imparted to the cock by the rotating shaft, by gear wheels, or by some other means. As a consequence oil can only be supplied while the shaft is in motion. Many such contrivances* have been designed and may be used to advantage in some cases, where there is a question of lubricating important parts, for example the shafts of large steam engines. They are, however, too complicated and therefore too costly to be generally used. To attain the same object by simpler means many very differently constructed mechanical lubricators have been devised, few of which have come into common use; of these we will give a few examples. In the lubricator by *Lacolonge*, Fig. 117, a little bucket is fastened to the journal *A B* by an arm *F G*, and is carried around by the journal, so that at every revolution it dips a little oil from the lower part *H* of the casing, and empties it into a trough *K*, from which it passes to the journal. With rap-

Fig. 117.

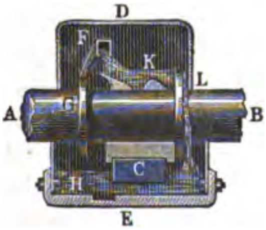
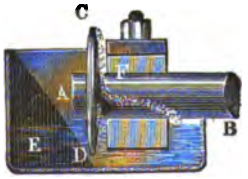


Fig. 118.



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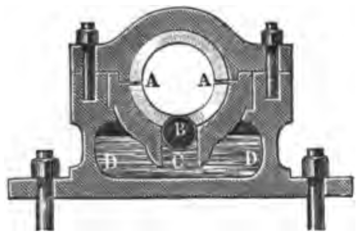
* See *Wiebe*, die Lehre von den einfachen Maschinentheilen.

idly running shafts the number of dips can be reduced by gearing, which, of course, considerably complicates the apparatus.

In the lubricator shown in Fig. 118 the journal *A B* carries a disk *C D*, which dips into the lubricant *E*, and at every revolution takes up some oil by adhesion, and delivers it to the journal as shown in the figure. To increase the adhesion, the disk is serrated at the edge, or provided with little holes.

The lubricator in Fig. 119 causes continuous lubrication of the journal by means of the hollow cylinder *B*, which floats on the oil *D D* in the lower part of the casing, and is constantly pressed up by the fluid against the journal *A*. When the latter turns, the cylinder also turns, and thus lubricates the journal. To avoid using a float, some of the designs are so arranged that the journal dips wholly or partially into the oil, in which case care must be taken that oil is not wasted by flowing off laterally.

FIG. 119.



None of these lubricators, however, are sufficiently simple for bearings in common use, such for instance as are used for shafting.

A simple arrangement, shown in Fig. 120, and first exhibited at the Paris Exposition in 1867 under the name of Pneumatic Lubricator, is, however, much used. It consists of a glass globe *A*, whose neck *B* is fitted with a wooden tube *C*. The diameter of the bore of the tube varies from 1 to 2 millimeters [0.04 to 0.08 inch]. The bore gradually increases in diameter, being widest below. The tube is placed in the cap so that its lower end *D* rests on the journal. When the shaft is at rest, the oil cannot flow out on account of the pressure of the air at the opening *D*, but when the shaft turns, the oil gradually flows out of its tube by adhering to the surface of the journal. The emptying process is of course a very slow one, but the lubrication seems to be good.

It is said to be a special advantage of this device that the supply of the lubricant is proportioned to the speed of the journal. At any rate the arrangement is very simple, and the exchange of an empty globe for a full one is easily made without spilling oil.

With all lubricators the oil finally becomes thick, sticky, and useless. This is due to the natural tendency of the oil to gum when exposed to the air, and to the absorption of dust and particles worn off from the journals and brasses. Care must therefore be taken that the oil is removed as fast as it becomes useless. In horizontal bearings this takes place of itself, the material escaping laterally and then trickling down. In pivot bearings, however, it is in many cases necessary to provide special channels for the worn-out oil. This is particularly necessary when the velocity of the shaft is great, as in centrifugal machines, for then the lubrication must be most carefully attended to. In this connection we may refer to the pivot bearings shown in Fig. 108.

FIG. 120.

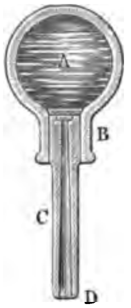
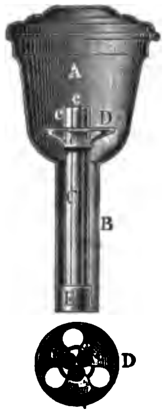


FIG. 121.



A peculiar lubricating arrangement is shown in Fig. 121, which is due to *Blandin*, of Rouen, and is intended for a lubricant of the consistency of butter. Here the tin vessel *A* and its tubes *B* are filled with the fatty mass, whose composition is kept secret by the inventor. The tube *B* is then closed below by the finger and a peculiar system of tubes *C* pressed into the mass, the system being composed of three small tubes *c* united below by a ring *E* and above by a pierced disk *D*. As in the pneumatic lubricator, the lower end of the tube in this apparatus rests on the journal. Its action may be explained as follows: the journal makes an air-tight joint at the opening *E* of the tube, and as fast as the lubricant is consumed at this point fresh material takes its place, being forced downward by the atmospheric pressure from above. From time to time the fatty mass in the vessel *A* must be compressed and renewed. Although this apparatus was reported * to have given excellent results, it does not seem to have fulfilled the expectations created at the time of its appearance.

The following works treat of lubricators: *Wiebe's* Lehre von den einfachen Maschinentheilen, Vol. II., Berlin, 1860; Armengaud, Vignola de Mécaniciens, Paris, 1863, in the article, Paliers graisseurs. The technical journals also contain many shorter communications.

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* Zeitschr. deutsch. Ingenieure, 1864, p. 70, and elsewhere.

CHAPTER II.

GEARING.

§ 40. **Wheels.**—The shafts or axles mentioned in the preceding chapter receive their rotation, unless they are directly driven by the Prime Movers, from similarly moving shafts, and can then communicate their rotation to other shafts. Wheels are universally used for this purpose. Like all other machine parts, wheels never occur singly, but always in pairs. This is not altered by the fact that frequently a wheel *A* is found working with two or more other wheels, *B*, *C*, etc., for in such a case we must distinguish as many pairs *A B*, *A C*, etc., as there are other wheels working with *A*, all the pairs having the member *A* in common.

The action of all kinds of wheels may be characterized geometrically as the moving upon one another of two bodies with certain ideal surfaces, and this motion may be either a purely rolling one without sliding, or it may be rolling combined with simultaneous sliding along the straight line of contact. From this it is evident that these surfaces have the character ascribed to *axoids* in § 26 of the Introduction.

When there is both rolling and sliding, these surfaces are hyperboloids, on each of which there is a system of straight lines, of such a nature that each straight line of one system coincides with a corresponding straight line of the other system at the instant when each of the two lines is at the line of contact. It has likewise been explained in the article just mentioned that, when there is no sliding along the line of contact, these surfaces are of such a character that they may be developed one upon another, and that they reduce to the particular case of two conical surfaces with a common vertex when the two shafts which are to be united by the wheels intersect. The point of intersection of the shafts is then the common centre of the conical surfaces, and from this it follows that these cones will become cylinders when the shafts are parallel. These conical and cylindrical surfaces are of course to be understood as having a perfectly general character, and though in practice, *normal* and *circular* cylinders and cones are usually employed, the general surfaces are not excluded, for example, in parallel shafts, elliptical, heart and spiral-shaped wheels frequently occur. The consideration of wheels as the connecting links of two shafts is simplified by treating them; as rotations were treated in the Introduction, under the three following heads:

a) Parallel axes,
b) Intersecting axes,
c) Axes which cross without intersecting.

These different positions of the shafts prescribe certain essential differences in the mode of action and arrangement of their wheels, and therefore constitute a natural geometrical basis for a corresponding classification. There is, however, another more physical point of view from which a classification may be taken, which depends on the peculiar *manner of transmitting* or *communicating the motion.* On the one hand, this communication of motion may take place directly between the two surfaces represented by material forms, these forms coming into immediate contact and fixing the distance between their shafts; on the other hand, motion may be transmitted indirectly by a flexible intermediate link, say a belt, having the ability to adapt itself to the form of the oft-mentioned surfaces, and thus permitting a certain freedom of arrangement in choosing the distance between the shafts. Each of these two cases permits of a subdivision into two classes, according as the transmission of motion from one body to another is effected by friction between the surfaces, *i. e.* their natural inequalities, or according as the surfaces are provided with *artificial* inequalities, or *teeth,* for the transmission of motion. Consequently all wheels may be brought into the following four classes:

1. Friction wheels, with direct transmission by friction;
2. Belted wheels, with indirect transmission by friction;
3. Toothed wheels, with direct transmission by teeth;
4. Chain wheels with mediate transmission by teeth.

Although this difference in the action of the wheels upon each other is of the utmost importance with respect to their construction and application, and although the hurtful resistances, friction for instance, essentially depend on this difference, it is nevertheless clear that this does not necessitate a difference in either their kinematic or dynamic connection. The changes of motion as well as of velocity, and consequently the changes of force, if we neglect hurtful resistances, are independent of the *manner* in which motion is transmitted, and therefore for the groups 1 to 4 these changes are the same.

In what follows, this kinematic and dynamic connection of the wheels will first be investigated, and it will be clear from the preceding remark, that the results obtained will be as valid for friction wheels as for toothed wheels, and as valid for belt gear as for chain gear. After this investigation of the geometrical connection, we will examine the transmission of motion from its physical side.

§ **41. Wheels for Parallel Shafts.**—The simplest and most frequently occurring arrangement of wheels is to have a shaft A, Fig. 122, turn with uniform velocity and transmit a uniform rotation to a second and parallel shaft B. Let α be the angular motion of the shaft A in a given time, and β the angle of rotation of B in the same time, then the ratio $\frac{\alpha}{\beta}$ of these two motions must always be constant, if a uniform rotation of A is to cause a uniform rotation of B. It has already been fully explained in § 27 of the Introduction, that this kind of motion is represented by two cylindrical axoids whose geometrical axes coincide with those of the shafts A and B, and whose radii a and b have the relation

$$a\,\alpha = b\,\beta,$$

or

$$a : b = \beta : \alpha.$$

The centroids of this motion are therefore two circles AO and BO described about A and B respectively, Fig. 122 and Fig. 123, and touching each other in O. This point O, found from the relation

FIG. 122.

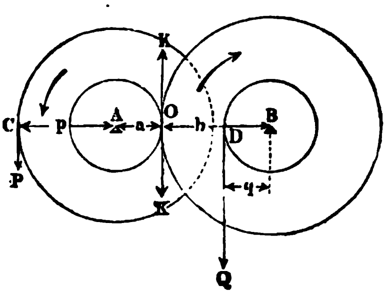
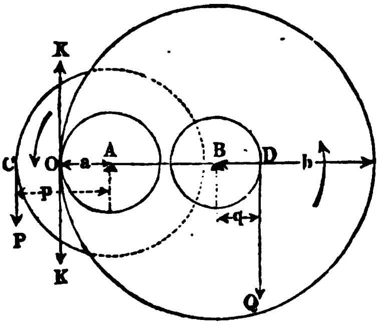


FIG. 123.



$$AO : BO = \beta : \alpha,$$

lies either between the centres A and B when the shafts have *opposite* rotations, Fig. 122, or outside of that straight line AB when the rotations are in the same direction. In the latter case it is evident that O must lie on the prolongation of AB which starts from the centre of the more rapidly running shaft. In Fig. 123 the point of contact O can only have the position shown when $\alpha > \beta$, because $BO > AO$.

In practice these axoids are cylindrical constructions, the cases represented by Fig. 122 and Fig. 123 corresponding to toothed or friction wheels in external or internal gear respectively. When these wheels effect the transmission by teeth, they are called *spur-wheels*. In indirect transmission of motion by belts or chains, the fundamental form of the axoids and wheels will not change, but only their distances from each other, and it is evident that for the case in which the direction of rotation of the two axes is the same, an *open* belt must be employed as in Fig. 124, while for opposite rotations a *crossed* belt, Fig. 125, must be used. For all these arrangements, we may deduce from the fundamental equation $a\alpha = b\beta$ the following proposition: *the number of revolutions of two shafts connected by two wheels is inversely proportional to the radii of the wheels.* The *smaller* wheel will always have the *larger* number of revolutions, the *circumference* of the two wheels moving at the *same speed.*

For the case of direct transmission of motion the distance $a + b$ between the axes may be represented by d, and if we represent the velocity ratio $\frac{\beta}{\alpha} = \frac{a}{b}$ by n, we shall have

$$a\alpha = (d - a)\beta,$$

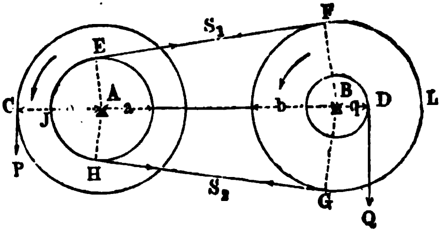
and

$$a = d\frac{\beta}{\alpha + \beta} = \frac{d}{1 + \frac{1}{n}}.$$

In like manner

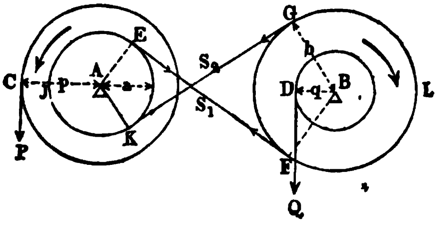
$$b = d\frac{\alpha}{\alpha + \beta} = \frac{d}{1 + n}.$$

FIG. 124.



It is evident from this that the wheels not only serve to transmit, but to modify motion, by altering its direction and changing its velocity. It is this last property in particular which causes wheels to be so extensively used, for the necessity of varying the velocity of rotations is one that meets the designer almost everywhere.

FIG. 125.



It has been sufficiently explained in the second part of this work, that motors act most efficiently when the prime movers run at certain velocities. Since these velocities are included between narrow limits, and since, on the other hand, special machinery must also have that velocity which is most advantageous for its particular work, it is evident that there must be some means of transforming one velocity into another, as it will rarely happen that the velocity most favorable for the prime mover will be the same as that most favorable for the special machinery. Every change in the velocity, of course, intimately depends on a change in the relations of the acting forces, a consequence which follows from the principle of virtual velocities (Vol. I., § 82). If a force P acts at the end of a lever arm $A C = p$ fixed to the driving shaft A, Fig. 122 and Fig. 123, its moment will be Pp, then if for the instant we neglect all hurtful resistances, as friction, etc., it will follow from the general principles of equilibrium that when uniform motion obtains, a force $K = \frac{Pp}{a}$ must be exerted at the point of contact O of the wheels. This force appears at the point of contact of the two wheels, as an action of the wheel A upon B, and as a reaction of B upon A. These two actions are equal; for example, in toothed wheels the pressure of the teeth upon one another is equal if we neglect the influence of the friction of the teeth. When the motion is uniform, the force K with the radius $B O = b$ of the wheel B for a lever arm, will be in equilibrium with the resistance Q at the end of the lever arm $B D = q$, and we shall have $Kb = Qq$. If we here substitute for K the above value $\frac{Pp}{a}$, we shall get

$$Q = P\frac{p}{q}\frac{b}{a}.$$

If the resistance Q acted directly on the shaft A with the same lever arm q, we should have the relation $Qq = Pp$, and then $Q = P\frac{p}{q}$. By inserting two wheels, and transferring the resistance Q to the shaft B, we are thus able with the same force P to overcome $\frac{b}{a}$ times as great a resistance Q, as would be possible if the resistance acted directly on the shaft A. This explains the importance of wheels for windlasses and hoisting gear whose object is to overcome a large resistance with a small force P. In general this object is attained by inserting one or more pairs of wheels between the shaft at which the force P acts, and that at which the load Q acts. Such a combination of pairs of wheels it is customary to call a *train;* we may also speak of windlasses and other machines as single, double, or triple-geared, according to the number of pairs of wheels inserted. The action of each elementary combination in the above cases is to diminish the velocity, and thus to increase the force in the ratio $b : a$, from which it follows that whenever it is desirable to increase the force as much as possible, the wheel b must be made as large as is practicable relatively to a. Practical considerations of smooth running and possible construction principally determine the upper limit to which $\frac{b}{a}$ may be extended, as will be more fully explained hereafter. The largest values given to the ratio $\frac{b}{a}$ in windlasses, seldom exceeds 8, while for shafting and machinery in general, the ratio is apt to be much smaller, not usually exceeding 4 or 5. It is evident that, if it were only a question of transmitting motion without change of velocity, a must $= b$. It is clear that increasing the resistance Q to be overcome by the force P, does not cause an increase or gain of mechanical work, for every increase of force is secured at the expense of the distance to be traversed by that force. On, the contrary, the addition of even necessary machine parts, such as wheels, brings with it new resistances, friction, etc., involving a corresponding loss of mechanical work, the amount of which will be determined later. In general, we may say that the perfection of a mechanism, from a dynamic point of view, will be greater the fewer the movable parts and the simpler the whole mechanism.

EXAMPLES. 1. The shaft A of a steam engine makes 36 revolutions per minute, and drives a shaft B making 90 revolutions per minute, by means of a toothed fly-wheel 4 meters [13.12 feet] in diameter. What is the diameter of the wheel on the shaft B?

The required diameter $2b$ of this wheel is obtained from

$$90 : 36 = 4 : 2b;\ 2b = 1.6 \text{ meters } [5.25 \text{ feet}].$$

2. The spindle of a grinding-mill carries a pulley 0.8 meter [2.62 feet] in diameter, and receives its motion from a 2-meter [6.56 feet] pulley on an upright shaft making 50 revolutions per minute. Required the speed of the stone?

The number of revolutions n of the spindle may be obtained from

$$0.8 : 2 = 50 : n;\ \text{or } n = 125.$$

3. A windlass having a drum 300 millimeters [11.81 inches] in diameter carries on the drum shaft a wheel 1.2 meters [3.94 feet] in diameter, which gears into a 180-millimeters [7.09 inches] pinion on the arbor of the crank. What force must be exerted by a laborer at the end of a 0.4-meter [1.31 feet] crank, in order to lift a load of 250 kilograms [551 pounds], hurtful resistances being neglected?

We have

$$Q = P \frac{p}{q} \frac{b}{a},$$

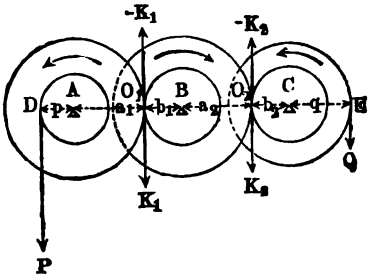
or, after substituting:

$$250 = P \frac{0.400}{0.150} \frac{1.2}{0.180},$$

from which follows $P = 14.1$ kilograms [31 pounds].

§ 42. Train of Wheel-work.—When the *velocity ratio* of two shafts is too great to be accomplished by a single pair of wheels, several pairs are employed, as shown in Fig. 126. Here the motion of the shaft A is transmitted to B by the wheels a_1 and b_1, and from B to C by the wheels a_2 and b_2. If the shaft A has an angular velocity α, we will find for the shaft B the velocity $\beta = \alpha \frac{a_1}{b_1}$, and for the axis C,

FIG. 126.



$$\gamma = \beta\frac{a_2}{b_2} = \alpha\frac{a_1}{b_1}\frac{a_2}{b_2}.$$

For the relation of the force P to the resistance Q acting at C we have:

$$Qq = Pp\frac{b_1}{a_1}\frac{b_2}{a_2},$$

while forces acting at O_1 and O_2 are

$$K_1 = P\frac{p}{a_1},$$

and

$$K_2 = K_1\frac{b_1}{a_2} = P\frac{p\,b_1}{a_1\,a_2} = Q\frac{q}{b_2}.$$

It is evident from this, that in a train of wheel-work the *velocity ratio of the whole combination is equal to* the *product of the velocity ratios of all the individual pairs.* The same may be said of the forces.

All that was said in the previous article concerning the reversal of the direction of motion, of course holds here for each individual pair, and it is also evident that a shaft like C, Fig. 126, which receives its motion from another shaft A after *two* reversals, must move in the same direction as A. The total velocity ratio of a combination of several pairs is $\frac{b}{a} = \frac{b_1\,b_2\ldots}{a_1\,a_2\ldots}$ and is evidently greater the smaller the driving wheels $a_1\,a_2\ldots$ and the larger the driven wheels $b_1\,b_2\ldots$ If we assume for the velocity ratio of a single pair, the maximum velocity ratio 8 allowed in windlasses, then will the use of two pairs of wheels give a velocity ratio 64, and three pairs a velocity ratio of about 500. More than three pairs of wheels probably do not occur in windlasses, and though more do occur in some special machinery, the object is less to obtain a great velocity ratio than to transmit motion to different shafts.

Although the arrangement shown in Fig. 126, where the shafts are placed side by side, is much used, especially in hoisting gear, the double-geared arrangement of Fig. 127 is also much used in metal-working machinery, as lathes; here the axis of the driving shaft *A* coincides with that of the driven shaft *C*. To accomplish this, of course one of the shafts must be hollow and ***turn loosely*** on the other. In this case it is the driving shaft *A* in the form of a cone pulley which turns freely on *C*. This pulley is rigidly connected to the wheel a_1, the latter driving the wheel b_1 on the shaft *B*, which in turn communicates motion to the wheel b on the spindle *C*, by means of its pinion a_2. The same law of motion holds for this arrangement as for the one in Fig. 126, consequently if the shaft *A* has a rotation equal to α, that of *C* will be $\gamma = \alpha \frac{a_1 a_2}{b_1 b_2}$. This compact arrangement is chosen principally because the wheels of the shaft *B* may be easily thrown out of gear, and the pulley *A* easily fastened to the wheel b_2. For *increasing* the velocity, a similar arrangement has been employed by *Pinet* * in the horse power, Fig. 128, in which the levers *Z* are drawn by horses, thus turning the wheel a_1 fastened to *Z* but turning freely on the cast-iron pillar *A*, in the midst of which stands the driven shaft *C*. The shaft *B* is in this case a bolt fastened to the base-plate, and carries the wheels b_1 and a_2,

FIG. 127.

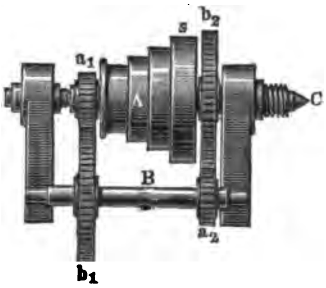
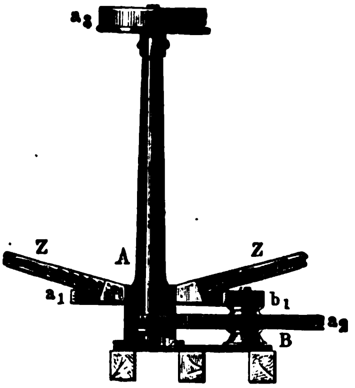


FIG. 128.



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* See Polyt. Centralblatt, p. 1158, 1856.

while the driven shaft C carries a pulley a_3 for transmitting the motion to a threshing machine.

EXAMPLE. In the horse power, Fig. 128, suppose each of the wheels a_1 and a_2 to have a radius equal to 0.8 meter [31.5 inches], each of the smaller wheels b_1 and b_2 a radius equal to 0.12 meter [4.72 inches], and the pulley a_3 a diameter of 1 meter [39.37 inches], then if the horse takes two turns per minute, and the pulley b_3 of the threshing machine makes 600 revolutions per minute, what must be the diameter of the pulley b_3?

The diameter b_3 may be found from

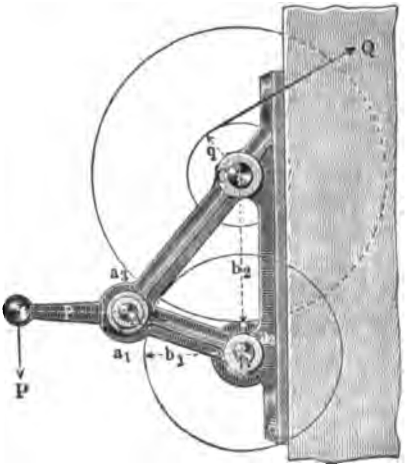
$$600 = 2 \cdot \frac{0.8}{0.12} \, \frac{0.8}{0.12} \, \frac{1.0}{b_3},$$

hence

$$b_3 = 0.148 \text{ meter } [5.83 \text{ inches}].$$

§ 43. **Variable Velocity Ratio.**—In many cases it is desirable to make the velocity ratio between two shafts variable, now greater, now less. This is the case, for example, in hoisting apparatus, where a large load requires a large velocity ratio, which, however, would unnecessarily retard the motion of a small load. To overcome this difficulty, the rotation is transmitted by one or two pairs of wheels according to circumstances, as may be seen from Fig. 129. Here A is the crank shaft and C the axle of the drum which carries the load Q. Alongside the drum, on the same axle C, is the large wheel b_2. The shaft B carries the two wheels a_2 and b_1, the latter gearing into a wheel a_1 on the crank shaft A, when this shaft, which can slide in the direction of its axis, has a suitable position. The velocity of rotation under the influence of two pairs of wheels will be diminished in the ratio $\frac{a_1 a_2}{b_1 b_2}$, consequently the force P at the crank,

FIG. 129.



will be increased to a pressure at the circumference of the drum equal to

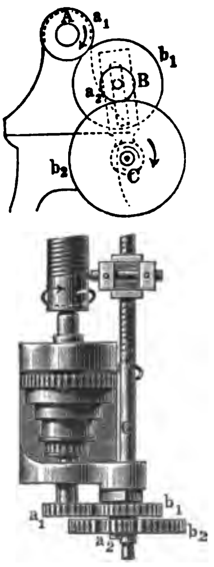
$$Q = P\frac{p}{q}\,\frac{b_1 b_2}{a_1 a_2}.$$

If a smaller load than Q is to be lifted, the crank shaft is slid in the direction of its axis until its wheel a_3 gears into b_2, when the wheels a_1 and b_1 will be thrown out of gear. The velocity ratio will now be transmitted by one pair of wheels a_3 and b_2 only, and this ratio will be measured by $\frac{a_3}{b_2}$, and therefore the resistance will equal

$$Q = P\frac{p}{q}\,\frac{b_2}{a_3}.$$

On a similar principle are based other devices, used particularly in planers for giving a quick return motion to the table.

FIG. 130.



There is an interesting arrangement, of importance to lathes and spinning machines, which is used to produce any desired velocity ratio between given limits. It is used in lathes to transmit from the spindle A, Fig. 130, the desired velocity to the lead or feed screw C, which moves the tool. It ordinarily consists of two pairs of wheels, as shown in the figure, $\frac{a_1}{b_1}\,\frac{a_2}{b_2}$, so arranged that all the wheels may be replaced by others of different diameters. For this purpose they have on hand a great number of *change wheels*, any one of which will work with any other, and from which, consequently, any four may be chosen and inserted in the mechanism. The bolt or stud B is so arranged that the distance of its axis from A and C can be changed at pleasure, this being effected by means of a slotted radial arm which turns about C. That we can, at least approximately, transmit any desired velocity ratio with such an arrangement, and with properly chosen diameters for the wheels, is evident from the following considerations. If the number of change wheels on hand is n, it is evident that if each one of them works with every other successively, $n(n-1)$ different pairs will be formed. When we take two of the wheels for one pair, say $\frac{a_1}{b_1}$, the remaining $n-2$ wheels will in like manner give $(n-2)(n-3)$ pairs, from which we may at pleasure choose any pair for $\frac{a_2}{b_2}$. Consequently the number of possible permutations of n wheels taken 4 at a time will evidently be

$$N = n(n-1)(n-2)(n-3).$$

If we assume $n = 12$, we shall have $N = 11,880$, while for $n = 20$, we shall have more than a hundred thousand permutations which can be employed for transmitting half as many different veiocity ratios. The limits of these ratios must evidently be included between $\frac{a_1 a_2}{b_1 b_2}$ and $\frac{b_1 b_2}{a_1 a}$, when $a_1 a_2$ represent the two smallest, and $b_1 b_2$ the two largest of the set. We will here remark that, since the change wheels are always constructed as toothed wheels, their circumferences and diameters must be proportional to whole numbers, and therefore it is only possible with this arrangement to transmit velocity ratios which can be expressed by whole numbers. On the other hand, irrational ratios can only be approximately realized. We can see from the great number of possible arrangements, that adjacent velocity ratios will differ but slightly from each other, and that, in practice, required ratios can be obtained with sufficient accuracy. It is customary, in practice to use a table which has been calculated for the set of the change wheels employed.

EXAMPLE. Suppose the drum of a crane to have a radius of 0.15 meter [5.9 inches] the pull on the chain, hurtful resistances being taken into account, to be 2,400 kilograms [5,291 pounds], and the power exerted by laborers at the end of a crank 0.4 meter [15.75 inches] long, to be 30 kilograms [66 pounds], what must be the velocity ratio of the gearing $\frac{a_1}{b_1}\frac{a_2}{b_2}$?

We have $Qq = Pp\frac{b_1 b_2}{a_1 a_2}$, Fig. 129, hence, in the present case:

$$2{,}400 \times 0.15 = 30 \times 0.4\frac{b_1 b_2}{a_1 a_2},$$

from which we get

$$\frac{b_1 b_2}{a_1 a_2} = 30.$$

If we take $\frac{b_1}{a_1} = 5$, by assuming $a_1 = 0.08$ meter [3.14 inches] and $b_1 = 0.4$ meter [15.75 inches], we get $\frac{b_2}{a_2} = 6$, which may be realized by assuming the radius of $a_2 = 0.1$ meter [3.94 inches], and that of the wheel b_2 on the drum $= 0.6$ meter [23.62 inches]. If we threw the pair $\frac{a_1}{b_1}$ out of gear, and allowed a wheel a_3 on the crank-axle to gear into b_2, then if $a_3 = a_2$, as is usually the case, we should get for the resistance of the chain

$$Q = P\frac{p}{q}\frac{b_2}{a_3} = 30\frac{0.4}{0.15}\frac{0.6}{0.1} = 480 \text{ kilograms [1,058 pounds].}$$

But if it were required that the hoisting gear should lift but 400 kilograms [882 pounds], we should find a_3 from

$$400 = 30\frac{0.4}{0.15}\frac{0.6}{a_3},$$

whence $a_3 = 0.12$ meter [4.72 inches].

The pressure K at the circumference of the wheels would then be

$$K = P\frac{p}{a_3} = 30\frac{0.4}{0.12} = 100 \text{ kilograms [220.5 pounds],}$$

while for a resistance $Q = 2{,}400$ kilograms [5,291 pounds], the pressures between the wheels a_1 and b_1 would be:

$$K_1 = P\frac{p}{a_1} = 0.30\frac{0.4}{0.08} = 150 \text{ kilograms [331 pounds];}$$

and between a_2 and b_2

$$K_2 = P\frac{p}{a_1}\frac{b_1}{a_2} = 30\frac{0.4}{0.08}\frac{0.4}{0.1} = 600 \text{ kilograms [1,323 pounds].}$$

§ **44. Idle Wheels.**—If the two wheels a and b on the shafts A and B do not gear together directly, but have a third intermediate wheel, c, Fig. 131, or two intermediate wheels, *c* and *d*, Fig. 132, it is evident that in the first case the two reversals at O_1 and O_2 will cause the shafts *A* and *B* to revolve in the same direction, while in the second case, Fig. 132, the three reversals will cause opposite rotations. For all such cases of repeated transmission of motion, the general rule holds, that an *even* number of reversals will leave the direction of motion unchanged, and an *uneven* number reverse this direction. Of course a transmission of motion by internal gearing or by means of an open belt cannot count as a reversal.

FIG. 131.

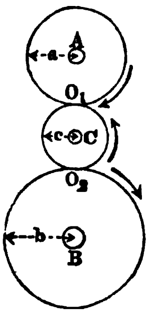
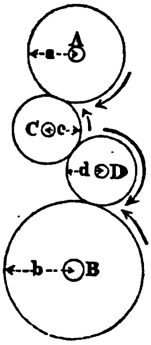


FIG. 132.



These intermediate wheels have no influence on the magnitude of the velocity, for the velocity β of the shaft *B*, Fig. 131, is given by

$$\alpha \frac{a}{c} \frac{c}{b} = \alpha \frac{a}{b},$$

and in Fig. 132 by

$$\alpha \frac{a}{c} \frac{c}{d} \frac{d}{b} = \alpha \frac{a}{b}.$$

The intermediate wheels can therefore only influence the direction of the motion, and on this account are sometimes called *idle wheels* or *idlers.*

By constructing the idler as a toothed wheel, a counter may be formed; for example, if the two wheels *a* and *b*, Fig. 131, differ but slightly, the wheel *a* having for instance 49 and *b* 50 teeth, then with toothed wheels it is possible to put the two loosely on the same shaft and connect them by a third wheel *c*, as shown in Fig. 133. Evidently the ratio of the revolutions of *a* and *b* is determined solely by their radii, that is, their number of teeth, the size of the wheel *c* being without influence in the matter. Such arrangements are employed for counters as follows. If the number of revolutions of a rapidly rotating shaft C is to be counted, we provide it with a wheel c which gears into the two wheels a and b, differing from each other by but one tooth; here b is *loose* and a *fixed* on the common shaft A. The latter also has fixed to it an index z which shows on a dial fixed to the wheel b the number of revolutions made by the shaft A. The graduation of the dial may be found as follows: When the wheel c makes one revolution, the shaft A and the index z will make $\frac{c}{a}$ revolutions, and the wheel b will make $\frac{c}{b}$ revolutions in the same direction as the wheel a. The motion of the index relatively to the dial will therefore be:

$$\frac{c}{a} - \frac{c}{b} = \frac{(b - a)c}{ab}.$$

For example, if c had 10, a 49, and b 50 teeth, we should have

$$\frac{(50 - 49)\,10}{50 \times 49} = \frac{1}{245},$$

and the dial on the wheel b will have to be divided into 245 equal parts if each division of the graduation is to indicate one revolution of the wheel C.

Strictly speaking, the wheel c is not here an idler, for when a and b have different radii and are on the same axis, they can only be connected by the wheel c when the difference between a and b is small, and when the wheels are all toothed wheels. If this arrangement is to be a correct one for all forms of gearing, two wheels must be placed on the shaft C, of which one will gear with a and the other with b, and they must be of such magnitude that the sum of the radii of each pair will be equal to the common distance of the shafts. In fact this arrangement, Fig. 134, occurs in practice, especially in boring machines, and then does not at all differ from the arrangement shown in Fig. 127. The object is here, as in the counter, Fig. 133, to make use of the slight difference in the number of revolutions of two shafts. Here *A* is a slowly turning boring bar which carries the tool *D* for boring a hollow cylinder. The slow axial motion

FIG. 133.

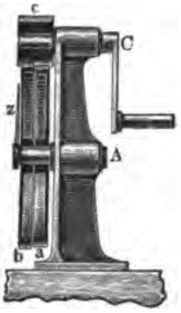
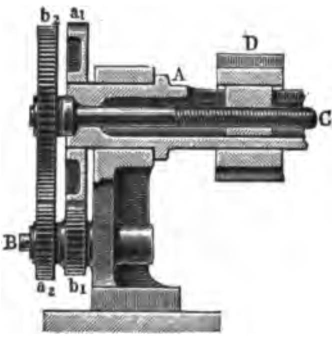


FIG. 134.



needed by the tool is communicated by the screw *C*, whose nut is the cutter-block *D*. For the purpose of communicating the motion to the screw, there is fixed to the boring bar *A* the wheel a_1, and beside it a slightly larger wheel b_2 is placed on the shaft of the screw *C*. To the stud *B* are fastened two wheels, b_1 gearing into a_1, and a_2 into b_2. It is clear from what has preceded that when the boring bar *A* makes one revolution the screw *C* will be turned an amount represented by $\frac{a_1}{b_1} \frac{a_2}{b_2}$, in consequence of which the nut or cutter-block *D*, which partakes of the rotation of the boring bar, will have an axial motion represented by

$$\left(1 - \frac{a_1 a_2}{b_1 b_2}\right) s = \frac{b_1 b_2 - a_1 a_2}{b_1 b_2} s,$$

where *s* is the pitch of the screw. By properly choosing a_1, a_2, b_1, and b_2, the axial motion or feed can be made as small as desired.

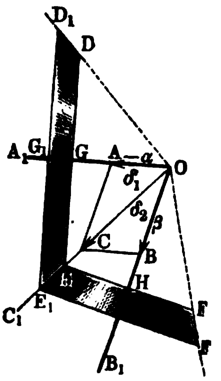
EXAMPLE. If, in the wheels of a boring machine, Fig. 134, the number of teeth for $a_1 = 100$, $b_1 = 10$, $a_2 = 11$, and $b_2 = 99$, what will be the width of cut, that is, axial motion of the tool, for one revolution of the bar, when the screw *C* has a pitch of 12 millimeters [0.48 inch]? Since the radii of the wheels are proportional to the number of teeth, we can find the axial motion from

$$\frac{10 \times 99 - 100 \times 11}{10 \times 99} 12 = 0.11 \times 12 = 1.33 \text{ millimeters } [0.053. \text{ inch}],$$

consequently to bore a cylinder 1 meter long the boring bar must make $\frac{1,000}{1\ 333} =$ 750 revolutions.

§ **45. Bevel Wheels.**—When the shafts of two wheels intersect, the arrangement of the latter may be determined as follows: Let $A A_1$ and $B B_1$, Fig. 135, be two axes intersecting at O; let us again suppose that the uniform rotation α of the shaft A is to be transformed into a uniform rotation β of the shaft B. Now if the shaft A is brought to rest (see § 27 of the Introduction) by giving to the whole system, that is, both shafts, a rotation $- \alpha$, the axoids or surfaces of instantaneous axes can easily be determined. For we are now dealing with a system which is at every instant subject to two rotations, one $- \alpha$ about the axis $A A_1$, and another β about the axis $B B_1$. Since it is only a question of the *ratio* of α to β, we can assume the latter as small as we please, and consequently may assume that they are infinitely small. According to the parallelogram of rotations (§ 22, Introduction), these two infinitesimal rotations about two intersecting axes may be combined into a single rotation about an axis $C C_1$, passing through the point of intersection O. This axis coincides with the direction of the diagonal $O C$ of a parallelogram whose sides $O A$ and $O B$ coincide with the direction of the axes. The length of the diagonal, moreover, measures the resultant rotation about the new axis, when the sides of the parallelogram measure the rotation about the original axes. The straight line $O C$ thus obtained is likewise to be regarded as the instantaneous axis of the shaft B for the instant in question.

FIG. 135.



Since the ratio of the velocities of rotation is to remain constant, the position of the instantaneous axis $C\,C_1$ relatively to the two axes must also remain constant. The desired axoids must, therefore, be the surface of the two cones which would be generated if the line $C\,C_1$ were rotated about both the axes A and B. The rotation of the axis B when A is fixed may therefore be regarded as the rolling of its conical surface $C\,O\,F$ on the fixed conical surface $C\,O\,D$.

If we now give up the assumption which brought the shaft A to rest by giving it an additional rotation $-\,\alpha$, we shall return to the state of motion which was assumed at the beginning, namely, that both shafts rotated uniformly with the angular velocities α and β respectively. It is, therefore, evident that the wheels, which are to be the embodiment of the axoids, must have the form of the conical surfaces just determined.

The relations of these conical surfaces, whose *common vertex is at the intersection of the two axes*, readily follow from what has preceded. If we represent the angle $A\,O\,C$ by δ_1 and $B\,O\,C$ by δ_2, we shall have

$$\alpha : \beta = \sin\,\delta_2 : \sin\,\delta_1,$$

or

$$\alpha \sin\,\delta_1 = \beta \sin\,\delta_2.$$

The velocities of rotation of the axes are, therefore, inversely proportional to the sines of the halves of the angles at the vertex. If from any point E of the line of contact of the cones we drop the perpendiculars $E\,G$ and $E\,H$ on the axes, we shall evidently have

$$\frac{\sin\,\delta_1}{\sin\,\delta_2} = \frac{E\,G}{E\,H} = \frac{a}{b},$$

when a and b represent these perpendiculars, or the radii of circular conic sections made through E.

If we let n represent the velocity ratio, we get

$$n = \frac{\alpha}{\beta} = \frac{\sin\,\delta_2}{\sin\,\delta_1} = \frac{b}{a},$$

and from the figure,

$$tan\ \delta_1 = \frac{\beta\ sin\ \delta}{\alpha + \beta\ cos\ \delta} = \frac{sin\ \delta}{n + cos\ \delta}$$

$$tan\ \delta_2 = \frac{\alpha\ sin\ \delta}{\beta + \alpha\ cos\ \delta} = \frac{n\ sin\ \delta}{1 + n\ cos\ \delta},$$

when

$$\delta = \delta_1 + \delta_2 = A\ O\ B$$

represents the angle included between the axes.

If through the point E_1 we pass two intersecting planes $E_1 G_1 D_1$ and $E_1 H_1 F_1$, respectively parallel to the planes $E\ G\ D$ and $E\ H\ F$, we shall get two conical disks $E\ E_1 D_1 D$ and $E\ E_1 F_1 F$, which will be the fundamental forms of the conical wheels that are to transmit motion between two intersecting axes. Since the points E and E_1 were arbitrarily chosen in forming these wheels or frustums, we can make the latter larger or smaller as we please by assuming the points E and E_1 at a greater or less distance from the point of intersection O of the axes. It follows from this, that for the same axes A and B, and for the same velocity ratio, we can obtain an unlimited number of conical wheels whose radii a and b, drawn from any two points which come in contact, will have the same ratio; that is, they will be to each other inversely as their velocities. No such freedom exists when parallel shafts are to be connected by two wheels, for the given distance between the axes and the given velocity ratio render only one pair of wheels possible.

From what has been said, we may now easily construct the required wheels for every case. In Fig. 136 and Fig. 137, let $O\ A$ and $O\ B$ be the direction of the axes, and from the point of intersection O, lay off the distances $O\ A = -\ \alpha$ and $O\ B = \beta$, observing what has been said in § 3, Introduction, concerning the representation of the direction of rotation by arrows. In every case the diagonal $O\ C$ of the parallelogram will be the generatrix of both conical surfaces, from which we may cut the wheels $E\ D$ and $E\ F$ of any desired size by means of planes perpendicular to the axes.

FIG. 136.

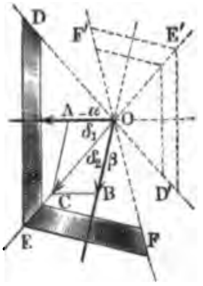


FIG. 137.

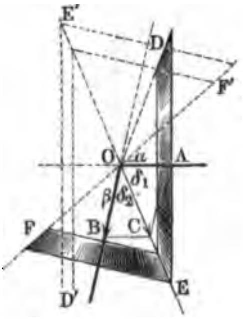
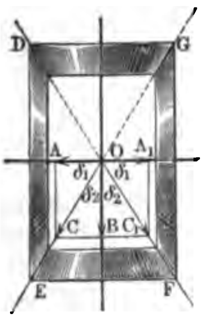


FIG. 138.



Since these conical surfaces extend indefinitely on each side of the vertex O, we can arrange the required wheels upon either side of O, for example, at $E' D'$ and $E' F'$; these will always have the assumed *direction* of rotation of the shafts. We also see that the two cases represented by the two figures differ only in having opposite rotations for the shaft A, and that the construction employed will give two different pairs of conical surfaces, even when the angle between the axes and the velocity ratio remains the same. These two pairs of axoids, however, reduce to a single pair when the axes, as in Fig. 138, form a right angle, for in this special case the generating line $O C$, corresponding to one direction of rotation of A, forms the same angles with the axes that the generating line $O C$ does for the opposite rotation of A. This shows that for one direction of rotation of A we may choose either of the pairs of wheels $E D$ and $E F$ or $G F$ and $G D$, while for the opposite rotation of A we may employ either of the pairs $F G$ and $F E$ or $D E$ and $D G$. In Fig. 136 to Fig. 138, the generating line of the cone forms acute angles δ_1 and δ_2 with the direction of the axes, and therefore the two conical surfaces lie upon *opposite sides* of the line of contact $O C$. This case corresponds to that of *external gearing* of wheels with parallel axes. In Fig. 139, however, with the angle $\delta_2 > 90°$ the two cones lie on the *same side* of the line of contact, and the contact may be called an internal one, corresponding to the internal gearing of wheels with parallel axes. We may regard the case represented by Fig. 140 as a limiting case for each of the two just mentioned, the angle δ_2 made by the generating line OC with the axis OB being equal to a right angle. As a consequence, the conical surface belonging to B reduces to a plane normal to the axis OB.

FIG. 139.

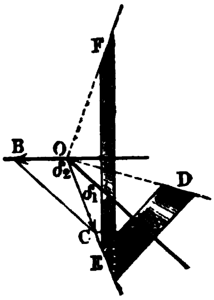
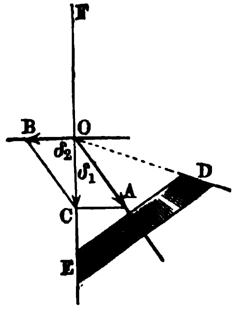


FIG. 140.



EXAMPLES: 1). The angle included between two axes is equal to 70°, the required velocity ratio $n = \frac{\alpha}{\beta} = 3$ is to be transmitted by means of conical wheels. The angles δ_1 and δ_2 may be found from

$$\tan \delta_1 = \frac{\sin 70^\circ}{3 + \cos 70^\circ} = 0.2812 ; \quad \delta_1 = 15^\circ\, 42',$$

consequently $\delta_2 = 54^\circ\, 18'$. Now, if the exterior point of contact between the two wheels is to be 1.2 meters [47.25 inches] from the intersection of the axes, we have for the radii of the wheels:

$$a = 1.2 \sin 15^\circ\, 42' = 0.325 \text{ meter } [12.8 \text{ inches}],$$
$$b = 1.2 \sin 54^\circ\, 18' = 0.975 \text{ meter } [38.4 \text{ inches}] = 3a.$$

When the length of the side of the cone is 0.1 meter [3.94 inches], the inner radii of the wheels will be:

$$a_1 = 1\ 1 \sin 15^\circ\, 42' = 0.298 \text{ meter } [11.73 \text{ inches}],$$
$$b_1 = 1.1 \sin 54^\circ\, 18' = 0.894 \text{ meter } [35.20 \text{ inches}] = 3a_1.$$

2). How large is the velocity ratio of two conical wheels whose axes make an angle of 120°, and the line of contact of the wheels an angle of 90° with one of the axes?

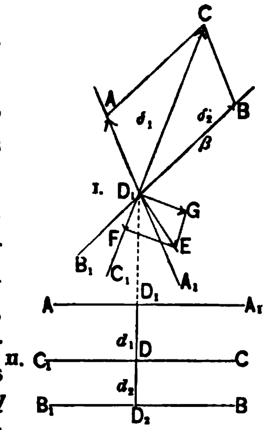
Here $\delta = 120^\circ$, $\delta_1 = 90^\circ$, and $\delta_2 = 30^\circ$; hence the velocity ratio will be

$$n = \frac{\alpha}{\beta} = \frac{\sin 30^\circ}{\sin 90^\circ} = 0.5 ;$$

the radii of the flat wheel will, therefore, be twice as large as those belonging to its partner.

§ **46. Wheels for Axes which Cross but do not Intersect.**—Let us suppose the problem given in the preceding article for intersecting axes to be now given for a case in which two axes cross but do not intersect. In Fig. 141, I and II, $A\,A_1$ and $B\,B_1$ are represented in two planes of projection I and II, which are perpendicular to each other, the plane I being parallel to the parallel planes passed through the two crossed axes. Consequently; $D_1\,D_2 = d$ represent the normal (shortest) distance between the two axes. We will again suppose one of the shafts, say $A\,A_1$, to be brought to rest by giving to the whole system, that is, to each of the axes, a rotation $-\alpha$ equal and opposite to that of $A\,A_1$. This will subject the shaft $B\,B_1$ to two rotations, namely, one β about its own axis B, and another $-\alpha$ about A, α and β being again infinitesimal in amount. The determination of the axoids for this case is easily made with the help of § 25 of the Introduction, where it was established that the motion about two crossed axes could always be replaced by a *determinate helical motion about a definite axis.* To find this helical motion, let us suppose the rotation about the axis $B\,B_1$ to be resolved (§ 4, Introduction) into a rotation about an axis parallel to $B\,B_1$ which cuts $A\,A_1$ at the foot D_1 of the common normal d, and into a translation $2\,d\,sin\frac{\beta}{2}$ perpendicular to the plane of the axes which is equal to βd when the angle β is small. This translation or sliding is represented in I by $D_1E = \beta d$ normal to $B\,B_1$. We can now combine the rotations $-\alpha$ and β according to the parallelogram of rotations as in the previous article, so as to form a resultant rotation γ about an axis $C\,C_1$ passing through D_1 and dividing the angle of the axes $A\,D_1B = \delta$ into two parts, $A\,D_1\,C = \delta_1$ and $B\,D_1\,C = \delta_2$; hence the proportion :

FIG. 141.



$$\frac{\alpha}{\sin\delta_2} = \frac{\beta}{\sin\delta_1} = \frac{\gamma}{\sin\delta} \text{ (§ 22).}$$

This resultant axis $D_1 C$ is inclined to the direction of the translation $D_1 E = \beta d$, for the latter is perpendicular to one of the resultant's components, $D_1 B$, and we can therefore resolve the translation βd into two components, $D_1 F$ parallel to $D_1 C$, and $D_1 G$ perpendicular to $D_1 C$. Since

$$C_1 D_1 E = 90^\circ - \delta_2,$$

we have for the components,

$$D_1 F = \beta d \sin\delta_2,$$

and

$$D_1 G = \beta d \cos\delta_2.$$

The latter component (§ 4) can be combined with the rotation $C C_1$, giving as a result a rotation which is likewise equal to γ, about an axis parallel to $C C_1$, and at a distance from it equal to d_1, which may be found from

$$\gamma d_1 = \beta d \cos\delta_2,$$

that is,

$$d_1 = \frac{\beta}{\gamma} d \cos\delta_2.$$

The result of these resolutions and combinations is that the two rotations $-\alpha$ and β about the crossed axes $A A_1$ and $B B_1$ may be replaced by a rotation γ about an axis $C C_1$, and a sliding or translation $\beta d \sin\delta_2 = \alpha d \sin\delta_1$ parallel to the axis $C C_1$. The latter is parallel to the two parallel planes through the axes, and passes through D at the distance d_1 from $A A^1$; the resultant axis $C C_1$ likewise makes with the original axes the angles δ_1 and δ_2 given by the equation,

$$\frac{\alpha}{\sin\delta_2} = \frac{\beta}{\sin\delta_1} = \frac{\gamma}{\sin\delta}.$$

This rotation and translation together produce a helical motion about the axis $C\,C_1$. Since the velocity ratio is constant, it follows that the axoid of the moving shaft B must be that surface of revolution which is generated when the instantaneous axis $C\,C_1$ is rotated about $B\,B_1$, its distance and inclination from the latter remaining the same. In like manner the axoid rigidly attached to $A\,A_1$ is the surface of revolution generated by revolving this same straight line $C\,C_1$ about $A\,A_1$. The surfaces of revolution thus generated are known as ***hyperboloids of revolution of one nappe.*** If we suppose the rotation $-\alpha$ which was given to the whole system, to be neutralized by a rotation $+\alpha$, the shaft $A\,A_1$ will no longer be at rest, but will assume its original rotation $+\alpha$, and the shaft $B\,B_1$ will also keep its original rotation $+\beta$, and the motion required of the two axes will be established.

The foregoing investigation shows us that this motion can be obtained by embodying the axoids in hyperboloidal wheels rigidly connected to the shafts $A\,A_1$ and $B\,B_1$, and then rolling them on each other, at the same time sliding them along the instantaneous line of contact a distance equal to

$$\alpha\, d \sin \delta_1 = \beta\, d \sin \delta_2.$$

It may be of interest to determine the ratio of the segments into which the shortest distance $d = D_1 D_2$, Fig. 142, II, is divided at D by the resultant axis or generatrix $C\,C_1$. We obtained above the expression

$$D\,D_1 = d_1 = \frac{\beta}{\gamma}\, d \cos \delta_2.$$

In like manner we could obtain for the distance $D\,D_2 = d_2$ the value:

$$D\,D_2 = d_2 = \frac{\alpha}{\gamma}\, d \cos \delta_1.$$

Dividing, we obtain

$$\frac{d_1}{d_2} = \frac{\beta}{\alpha}\,\frac{\cos \delta_2}{\cos \delta_1},$$

and substituting

$$\frac{\beta}{\alpha} = \frac{\sin \delta_1}{\sin \delta_2},$$

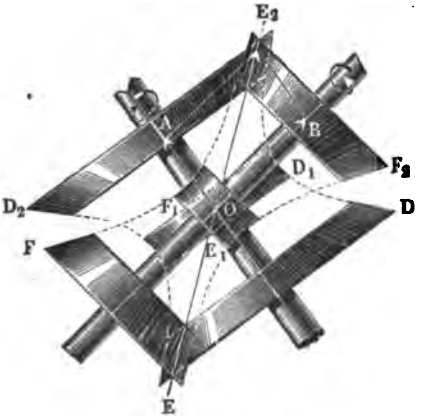
we get

$$\frac{d_1}{d_2} = \frac{\tan \delta_1}{\tan \delta_2} = \frac{D D_1}{D D_2}.$$

The distances d_1 and d_2 are evidently the radii of the gorge circles of the hyperboloids, and the results obtained can therefore be stated as follows:

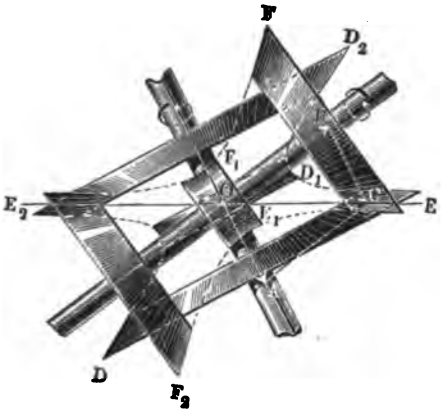
In hyperboloidal (skew-bevel) wheels *the radii of the gorge circles are directly proportional to the tangents of the angles made by the line of contact with the axes, while the velocities of rotation, and consequently the number of revolutions, are inversely proportional to the sines of these angles.* In this latter aspect hyperboloidal wheels to a certain extent correspond to conical wheels. In other respects also there are certain resemblances between the two cases of intersecting and crossing axes, as will be seen in what follows.

FIG. 142.



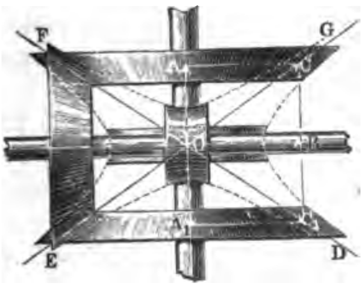
In hyperboloidal, as in conical wheels, the position of the axes and the velocity ratio by no means suffice to determine completely the size of the wheels, for by making sections normal to the axes of the two hyperboloids at arbitrarily chosen points of the line of contact $E E_1 E_2$, Fig. 142 and Fig. 143, we can cut out frustums whose radii increase with their distance from the gorge circles. The smallest wheels $E_1 F_1$ and $E_1 D_1$ are naturally those whose middle planes coincide with the gorge circles, and these frustums have also the property of deviating but little from cylinders, while all frustums which are cut out of the hyperboloids on either side of the gorge circles, like those at E and E_2, are approximately represented by frustums of cones, the inclination of whose sides is equal to that of the tangents to the hyperbolic profile at the points mentioned.

FIG. 143.



Moreover here, as in conical wheels, we recognize that the two cases represented by Fig. 142 and Fig. 143, differ only in the opposite directions of rotation of one axis A. Corresponding to these two cases, we also here get two different pairs of hyperboloids of revolution, which reduce to a single pair in the case of rectangular axes, as may be seen from Fig. 144. If we suppose the rotation about the axis B to take place in a particular direction, and construct the two instantaneous axes $O\ C$ and $O\ C_1$, which for the same velocity ratio correspond to the two directions of rotation about A, we shall find that the two hyperboloids which are generated by the revolution of generating lines $O\ C$ and $O\ C_1$ about each axis will coincide, as in the case of conical wheels. From this follows the interesting fact that a pair of hyperboloids constitute a pair of axoids which, when one axis is given a rotation in a *particular* direction, will cause a rotation about the other axis in the *one* direction as well as in the *opposite* direction, the ve-

FIG. 144.



locity ratio remaining the same with either rotation. Consequently, if in such a case the two axoids were constructed as solid hyperboloids, and pressed against each other in order to transmit the motion by friction, the revolution of one would not cause the revolution of the other. The reason for this is that the two hyperboloids simultaneously touch each other along the two straight lines $O C$ and $O C_1$, which are symmetrically situated with reference to the axis, and therefore the actions arising from these two contacts mutually neutralize each other. But if the hyperboloids are provided with teeth, they can be made to transmit motion either in one direction or the other, according as the teeth are arranged in the direction of one line of contact or the other.

Here, as in conical wheels, the axoids touch ***externally*** so long as the generating line forms acute angles with the axes. But if the angle $\delta_2 > 90°$, we shall have:

$$d_2 = \frac{d_1}{\tan \delta_1} \tan \delta_2$$

negative, that is, the two axoids will be upon the same side of the line of contact, and the contact will be ***internal.*** For the limiting case $\delta_2 = 90°$, we have $d_2 = \infty$, and the corresponding axoid will then reduce to a line coinciding with the generating line, that is, it will correspond to a rack which gears into a hyperboloidal wheel set obliquely to it, just as in parallel shafts the ordinary rack with cylindrical pinion represents the limiting case between external and internal contact.

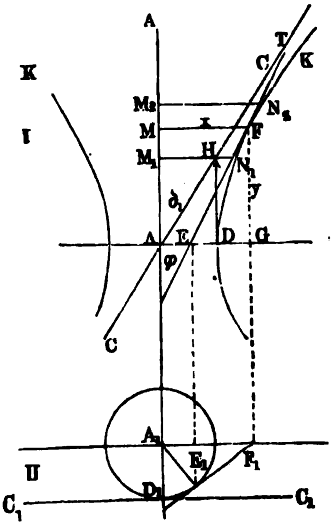
That the axoids generated by the revolution of the instantaneous axis $C C_1$ about the axes A and B are hyperboloids of revolution is readily seen from the following discussion. Let I and II, Fig. 145, be two planes of projection at right angles to each other (elevation and plan), $A A$ and A_1 the projections of the axis of rotation, $C C$ and $C_1 C_1$ the projections of the generating line, the latter making the constant angle δ_1 with $A A$ while it is rotated at the distance $A D = A_1 D_1 = d_1$ about the axis A. The surface of rotation thus generated is cut by the first plane of projection (passing through the axis $A\,A$) in a certain curve $K\,K$, which may easily be proved to be an hyperbola with reference to the axis $A\,A$. Let $E\,F$ and $E_1\,F_1$ be the projections of the generating line in any position, the point F in which the generating line pierces the first plane of projection will evidently also be a point of the curve K, and let the rectangular co-ordinates of this point, referred to A as an origin of co-ordinates and $A\,A$ as an axis of Y, be $A\,G = A_1\,F_1 = x$ and $G\,F = y$. Then from the triangle $A_1E_1F_1$ we have

$$A_1\,E_1^2 = A_1\,F_1^2 - E_1\,F_1^2,$$

or

$$d_1^2 = x^2 - y^2\,tan^2\,\delta_1.$$

FIG. 145.



Transforming we have the equation:

$$1 = \frac{x^2}{d_1^2} - \frac{y^2}{d_1^2\,cot^2\,\delta_1},$$

representing an hyperbola whose semi-axes are $d_1 = A\,D$, and $d_1\,cot\,\delta_1 = D\,H$, and for which $A\,C$ is an asymptote. Therefore, instead of obtaining this axoid by revolving $C\,C$, we can obtain it by revolving the hyperbola $K\,K$ about its principal axis $A\,A$, that is, the surface of the axoid is an hyperboloid of revolution of one nappe.

From the above equation for the hyperbola, we get

$$y = cot\,\delta_1\,\sqrt{x^2 - d_1^2}.$$

If we here assume for x so large a value that d_1 may be neglected, the value of y will approach $x \cot \delta_1$, that is, the wider the hyperboloid is made, the more does it approximate to a cone having the same axis $A A$ and a vertical angle $= 2 \delta_1$ (cone of asymptotes).

As has already been stated, we can substitute for the hyperboloidal frustums, which constitute the fundamental forms of the wheels, conical frustums whose elements coincide with the tangent to the hyperbolic profile. The inclination φ of the tangent $E F$ to the axis of Y at any point F of the hyperbola, may be found by differentiating the equation of the hyperbola.

$$\tan \varphi = \frac{dx}{dy} = \frac{y}{x \cot^2 \delta_1} = \tan \delta_1 \frac{\sqrt{x^2 - d_1^2}}{x}.$$

This inclination to the axis of revolution should therefore be given to the conical zone $N_1 N_2$, which is to be substituted for the hyperboloidal zone whose middle plane has the radius $FM = x = a$. If the axial height $M_1 M_2$ of the zone be represented by e, we shall have for its smallest and largest radius:

$$M_1 N_1 = a_1 = a - \frac{1}{2} e \tan \varphi = a - \frac{e}{2a} \tan \delta_1 \sqrt{a^2 - d_1^2},$$

$$M_2 N_2 = a_2 = a + \frac{1}{2} e \tan \varphi = a + \frac{e}{2a} \tan \delta_1 \sqrt{a^2 - d_1^2}.$$

If it is desired that these radii be expressed in terms of the distance $MA = h$ of the central plane from the gorge circle, we have, since

$$MA = h = \cot \delta_1 \sqrt{a^2 - d_1^2},$$

$$a_1 = a - \frac{e h}{2 a} \tan^2 \delta_1,$$

$$a_2 = a + \frac{e h}{2 a} \tan^2 \delta_1.$$

If, finally, we represent the distance of a point F from the gorge circle, measured along the line of contact FE, by $l = FE$, we shall get from Fig. II for the radius a of the hyperboloid at this point:

$$MF = A_1F_1 = a = \sqrt{d_1^2 + (l \sin \delta_1)^2}.$$

Remark. The most general case of motion, that about two crossed axes, must, of course include the cases of motion about intersecting and parallel axes. If we put $\delta_1 = \delta_2 = 0$, we get from

$$\frac{d_1}{d_2} = \frac{\beta}{\alpha} \frac{\cos \delta_2}{\cos \delta_1},$$

the known relation

$$\frac{d_1}{d_2} = \frac{\beta}{\alpha},$$

and for the radius at any point,

$$= \sqrt{d_1^2 + (l \sin \delta)^2} = d_1,$$

also $b = d_2$. These are the formulas for spur wheels with parallel axes. In like manner the assumption of $d = 0$, gives, since $\alpha d \sin \delta_1 = \beta d \sin \delta_2$,

$$\frac{\alpha}{\beta} = \frac{\sin \delta_2}{\sin \delta_1}$$

which corresponds to conical wheels.

Example. Required a pair of hyperboloidal wheels for two axes at right angles to each other, whose shortest distance $= 0.5$ meter [19.69 inches] and velocity ratio $\frac{\alpha}{\beta} = \frac{1}{2}$.

We have, since $\delta_1 + \delta_2 = 90°$,

$$2 = \frac{\sin \delta_1}{\sin \delta_2} = \frac{\sin \delta_1}{\cos \delta_1} = \tan \delta_1,$$

hence $\delta_1 = 63° \, 26'$, and $\delta_2 = 26° \, 34'$.

For the radii of the gorge circles we get:

$$\frac{d_1}{d_2} = \frac{\tan \delta_1}{\tan \delta_2} = \frac{2}{\frac{1}{2}} = 4:$$

consequently

$$d_1 = \frac{4}{5} 0.5 = 0.4 \text{ meter [15.75 inches]},$$

and

$$d_2 = \frac{1}{5} 0.5 = 0.1 \text{ meter [3.94 inches]}.$$

If the distance of the innermost and outermost point of contact of the wheels from the point of contact of the gorge circles is respectively $l_1 = 0.8$ meter [31.5 inches], and $l_2 = 1$ meter [39.37 inches] the corresponding radii for the one wheel will be

$$a_1 = \sqrt{d_1^2 + (l_1 \sin \delta_1)^2} = \sqrt{0.4^2 + (0.8 \sin 63^\circ\, 26')^2} = 0.82 \text{ meter } [32.28 \text{ inches}];$$

$$a_2 = \sqrt{d_1^2 + (l_2 \sin \delta_1)^2} = \sqrt{0.4^2 + (1 . \sin 63^\circ\, 26')^2} = 0.98 \text{ meter } [38.58 \text{ inches}],$$

and for the other wheel

$$b_1 = \sqrt{0.1^2 + (0.8 \sin 26^\circ\, 34')^2} = 0.372 \text{ meter } [14.65 \text{ inches}];$$

$$b_2 = \sqrt{0.1^2 + (1 . \sin 26^\circ\, 34')^2} = 0.458 \text{ meter } [18.03 \text{ inches}].$$

The average radii of the hyperboloidal wheels are :

$$a = \sqrt{0.4^2 + (0.9 \sin 63^\circ\, 26')^2} = 0.898 \text{ meter } [35.35 \text{ inches}],$$

and

$$b = \sqrt{0.1^2 + (0.9 \sin 26^\circ\, 34')^2} = 0.414 \text{ meter } [16.30 \text{ inches}],$$

consequently the average angles of inclination φ and ψ of the wheels are

$$\tan \varphi = \frac{\sqrt{a^2 - d_1^2}}{a} \tan \delta_1 = \frac{\sqrt{0.898^2 - 0.4^2}}{0.898} \tan 63^\circ\, 26' = 1.79, \text{ or } \varphi = 60^\circ\, 50'.$$

$$\tan \psi = \frac{\sqrt{b^2 - d_1^2}}{b} \tan \delta_2 = \frac{\sqrt{0.414^2 - 0.1^2}}{0.414} \tan 26^\circ\, 34' = 0.485, \text{ or } \psi = 25^\circ\, 50'.$$

If we consider the wheels to be cones of this inclination, the inner and outer radii a_1 and a_2 of one of the wheels will be given by

$$a \mp \frac{1}{2}(1 - 0.8) \cos \delta_1 \tan \varphi = 0.898 \mp 0.1 \cos 63^\circ\, 26' \tan 60^\circ\, 50',$$

or

$$a_1 = 0.818 \text{ meter } [32.2 \text{ inches}], \text{ and } a_2 = 0.978 \text{ meter } [38.5 \text{ inches}],$$

and of the second wheel by

$$b \mp 0.1 \cos \delta_2 \tan \psi = 0.414 \mp 0.1 \cos 26^\circ\, 34' \tan 25^\circ\, 50',$$

or

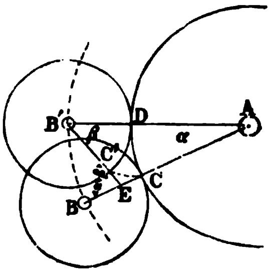
$$b_1 = 0.371 \text{ meter } [14.61 \text{ inches}], \text{ and } b_2 = 0.457 \text{ meter } [17.99 \text{ inches}].$$

The deviations from the hyperboloidal form are therefore very insignificant.

§ **47. Differential Wheels.**—In the arrangement of wheels hitherto discussed the assumption has always been made that both axes were held by fixed bearings which could not take part in the rotations. The action of the wheels was simply to transmit motion of rotation from one axis to another. This case, to be sure, occurs most frequently in practice, but cases do arise in which it is desirable either to prevent the axis from rotating or to give it a perfectly definite rotation which, in a certain sense, is independent of the rotation of the other axis. This kind of transmission of motion we will now investigate.

We will assume the simplest case, in which two parallel shafts *A* and *B*, Fig. 146, are so connected by two spur wheels *a* and *b* having external contact, that the shaft *A* and the wheel *a* which is keyed fast to it are absolutely fixed so that they take no part at all in the rotation. We can kinematically pass to this case from the ordinary one, in which both axes turn, and in which *A* has a rotation α and *B* one $\beta = -\frac{a}{b}\alpha$, by giving to each of the two shafts an additional rotation equal to $-\alpha$ about the axis *A*. This will bring *A* to rest, while the motion of *B* will be made up of two components, one rotation $\beta = -\alpha\frac{a}{b}$ about *B*, and another rotation $-\alpha$ around *A*. These two rotations were found above to give a resultant rotation $-\alpha\left(\frac{a}{b} + 1\right)$ about the instantaneous axis *C*.

FIG. 146.



Now if the shaft *B*, in addition to the revolution about its own axis, be made to rotate about *A*, it is evident that the frame of the bearing of *B* can no longer be fixed, but that it must be attached to an arm *A B*, which can turn about *A*. Let us suppose the arm *A B* and the wheel *b* to receive a rotation $+\alpha$ about the axis *A* when the wheel *a* is fixed; it will then follow that the wheel *b* will have in addition a rotation about its *own* axis in the same direction, that is, a right-handed one, which is equal to $+\alpha\frac{a}{b}$, and that the total rotation of the wheel b, or its absolute rotation in space, will be

$$\alpha\left(\frac{a}{b}+1\right)=\alpha\,\frac{a+b}{b}=\alpha\frac{d}{b},$$

where $d=a+b$ is the distance between the axes. This is also evident from the figure, for when the arm B rotates through an angle α, the point of contact will move from C to D, and the point of the wheel b which was originally at C will move to C', this point being determined by the angle

$$DB'C'=+\,\alpha\frac{a}{b},$$

consequently the radius $B'C'$ will make with its original position an angle

$$BEB'=DB'E+BAB'=\alpha\frac{a}{b}+\alpha=\alpha\frac{a+b}{b}=\alpha\frac{d}{b}.$$

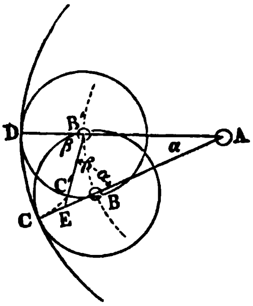
We see from this, that while a rotation α of the wheel a will cause an *opposite* rotation $\alpha\frac{a}{b}$ of the wheel b when both axes are fixed in position, a rotation α of the axis B about the fixed axis A will cause the wheel b to have a rotation $\alpha\frac{a}{b}$ in the *same* direction about its own axis. The relative rotation of the wheels, however, in all cases is:

$$\alpha\frac{a+b}{b}=\alpha\frac{d}{b}.$$

Wheels with indirect transmission of motion by means of *crossed* belts act in the same way.

In like manner we may investigate the motion of two wheels a and b having internal contact, Fig. 147. While a rotation α of a wheel a causes a motion $\alpha\frac{a}{b}$ of another wheel b in the *same direction* when the shafts are fixed, a rotation of the axis B about the fixed axis A will cause an *opposite* revolution $\alpha\frac{a}{b}$ of the wheel b about its axis. The relative motion of the two wheels, which in the latter case corresponds to the absolute motion of b in space, will be

FIG. 147.



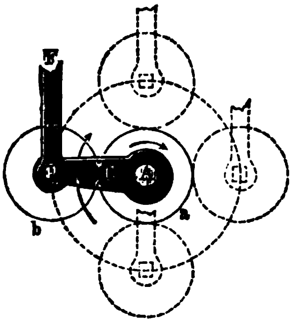
$$\alpha\frac{a-b}{b} = \alpha\frac{d}{b},$$

which holds for open belts as well as when there is internal contact of the wheels. This latter result may also be obtained from the figure, for the relative motion of b to a, or the total motion of b, will be expressed by

$$BEB' = DB'E - BAB' = \alpha\frac{a}{b} - \alpha = \alpha\frac{a-b}{b} = \alpha\frac{d}{b}.$$

These kinds of motion do occur in practice, for example: the first, with exernal contact, occurs in the turning gear of wharf cranes, in which the wheel a is firmly attached to the stationary post of the crane, while the axis B and wheel b are attached to a movable frame and can circle about the post. A similar case is the planet-wheel mechanism used by *Watt* in his steam engines. Here a movable axis A, Fig. 148, has keyed to it a toothed wheel a, the crank C turning loosely on A. This crank is driven by the connecting rod F and carries a toothed wheel b which gears into a. The wheel b is united to the connecting rod by a square bolt which prevents it from rotating about its own axis. If now we suppose the crank to be turned through an angle $+\alpha$, by the connecting rod F, there

FIG. 148.



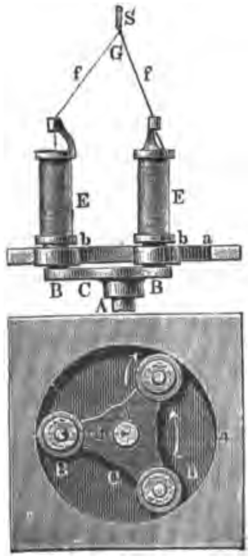
would, if the wheel a were stationary, be communicated to the wheel b an absolute rotation in space equal to $+ \alpha \frac{a+b}{b}$. But since such a rotation is not possible, we must imagine the fastening of the wheel b to have been effected by giving to it an equal and opposite rotation $- \alpha \frac{a+b}{b}$, from which it follows that the wheel a, in consequence of its gearing with b, must have a revolution

$$+ \alpha \frac{a+b}{b} \cdot \frac{b}{a} = \alpha \frac{a+b}{a}.$$

When a is taken equal to b, the rotation of the axis A will equal 2α, that is, will be twice as large as the rotation of the crank.

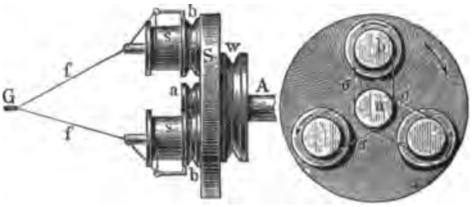
There are also applications of internal gearing, for example, in certain machines for twisting cotton yarn into strands and at the same time twisting the strands into cords. Thus, in Fig. 149, there is a fixed journal A, on which turns the three-armed cross C, carrying in its three eyes the bolts B with the wheels b. These wheels gear internally with the toothed ring a, and carry the spools (bobbins) E which hold the threads to be twisted. The threads from all these bobbins meet above at G, where they unite to form the cord S. By the rotation of the spools about their own axes B the threads are twisted into strands, while their rotation with the cross C about the axis A causes the strands to be twisted into a cord. If we let a and b represent the radii of the wheels, the rotation of a bobbin which effects the *twisting of the strand* will evidently equal $+ \alpha \frac{d}{b}$ when the cross has com-

FIG. 149.



pleted a rotation $-\alpha$ for the purpose of twisting the strands. The ratio of the number of turns of the strand in a given length of the finished cord to the number of turns of the threads in each strand, is therefore as $b : d$, the twists being in opposite directions, as is required in all rope-making processes.

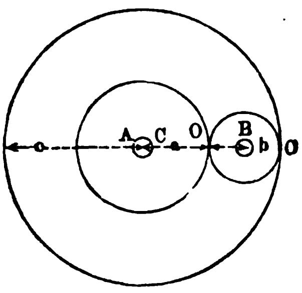
FIG. 150.



Examples may also be adduced of the use of similar mechanisms in which cords transmit the motion. Thus in making gilt cords formed of twisted strands wound with plated wire, the arrangement shown in Fig. 150 is employed. The stud *A* fixed to the frame carries a disk *S* driven by the grooved pulley *w*, the disk carrying three little bobbins *s* for holding the threads. These three threads *f*, which meet at *G*, are twisted by the rotation of the disk *S*. Now if the bobbins *s* were fixed to the disk, each turn of the disk would cause each thread to twist once, which would be injurious to the wire. To avoid twisting the wire, each bobbin is provided with a little pulley *b*, and to the fixed stud *A* is fastened a pulley *a* of the same size as *b*, but taking no part in the rotation. The cords σ connect *a* and *b*. It is plain that every revolution of the disk *S* with the bobbins *s*, say in the direction of the arrow, will cause the cord σ to give to each bobbin a left-handed rotation about its own axis, which will neutralize its right-handed rotation about *A*, for, according to the above, the total rotation of a bobbin when the disk *S* turns through α must be $\alpha \dfrac{a-b}{b}$, and consequently zero when $a = b$. Mechanisms of different arrangement but for the same purpose are employed for covering submarine cables with wire, for these cannot stand the severe twisting which takes place in ordinary rope-making.

§ **48.**—We will now combine the two cases represented in Fig. 146 and Fig. 147 in such a way, Fig. 151, that the two wheels *a* and *c* will be placed on the shaft *A*, which also acts as centre for the movable arm *A B*. This arm carries at *B* a wheel *b* which is in external contact with *a* and in internal contact with *c*. It is evident that this arrangement will permit no motion if the two wheels *a* and *c* are held fast, for a rotation of the arm *A B* through an angle φ would, on account of the external gearing of *c* with *a*, cause a rotation about *B* equal to $+ \varphi \frac{d}{b}$, and because of the internal gearing of *b* with *c*, a rotation equal to $- \varphi \frac{d}{b}$ about *B*; these rotations, being equal and opposite, would neutralize each other. For a rotation of the arm *A B* about *A*, or any other motion in the mechanism, it is necessary that either the wheel *a* or the wheel *c*, or both wheels *a* and *c*, be capable of rotation about their axes. We will assume the last mentioned general case, which may be realized by making the wheel *c* turn loosely on the shaft *A* of the wheel *a*, or we may place *c* on a hollow shaft *C* which incloses the solid shaft *A*. We will also assume that the shaft *A* and wheel *a* have a rotation α, the wheel *c* a rotation γ, and the arm *A B* a rotation β about the axis *A*, the absolute rotation of the wheel *b* in space being represented by ω. It is immaterial in what direction these rotations take place; it is only necessary that the plus sign shall stand for a particular direction of rotation, say that which corresponds to the movement of the hands of a watch; negative rotations will, of course, be in the opposite direction.

FIG. 151.



The connection between these individual rotations can now be found by finding the absolute rotation of the wheel *b* in space, which results from the working of *b* with *a* and the working of *b* with *c*. In this manner we find the absolute rotation of the wheel *b* to be :

1. equal to $-\alpha\frac{a}{b}$ because of the rotation α of the wheel a;

2. equal to $+\beta\frac{a+b}{b}$ because of the rotation β of the arm, and together these give:

$$\frac{-\alpha a+\beta a+\beta b}{b}=\omega_1;$$

the absolute rotation of the wheel b will also be:

3. equal to $+\gamma\frac{c}{b}$ because of the rotation γ of the wheel c;

4. equal to $-\beta\frac{c-b}{b}$ because of the rotation β of the arm, these two together giving

$$\frac{\gamma c-\beta c+\beta b}{b}=\omega_2.$$

Since these values ω_1 and ω_2 are equal, we have

$$-\alpha a+\beta a+\beta b=\gamma c-\beta c+\beta b,$$

or

$$\alpha a+\gamma c=\beta(a+c),$$

as the equation which represents the most general case of this mechanism. If we suppose the direction of γ to be opposite to that of α, the equation becomes

$$\alpha a-\gamma c=\beta(a+c),$$

from which it is evident that the motion of the arm $A\ B$ is proportional to the difference of the motions of the two other wheels, a and c. On this account the arrangement is called the *differential mechanism* or *differential motion*, and the arm $A\ B$, which is usually constructed in the form of a wheel, is called the *differential wheel.*

The different applications which can be made of the mechanism just examined can now be found by making particular assumptions. For example, if we make $\gamma=0$, *i.e.*, make the wheel *c* stationary, we shall get from the above fundamental equation :

$$\alpha = \beta \frac{a + c}{a} = \beta \left(1 + \frac{c}{a}\right).$$

This equation is applicable to the horse-power of *Barret, Exall* and *Andrews*,* in which, Fig. 152, the cover *B* and the wheels *b* are directly carried about by the horses, the cylindrical casing *C* having internal teeth which gear with the wheels *b*. The latter also gear into a wheel *a* fixed to the upright shaft *A*, so that it will rotate with an increased velocity in the same direction in which the horses move. For every circuit of the horses, the shaft *A* makes $\frac{c}{a} + 1$ revolutions.

On the other hand, when $\beta = 0$, we get

$$\gamma = -\alpha \frac{a}{c}$$

for the velocity of rotation of the exterior wheel. This arrangement has likewise been constructed, for instance, in the capstan, Fig. 153, in which workmen turn the shaft *A* by means of bars. A wheel on the shaft *A* then communicates motion through the

FIG. 152.

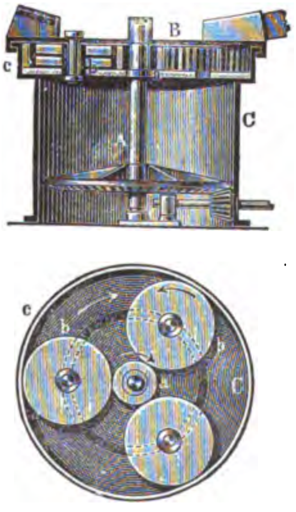
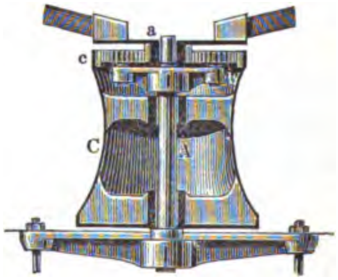


FIG. 153.



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* See *Perls*, Landwirthschaftliche Maschinen und Geräthe.

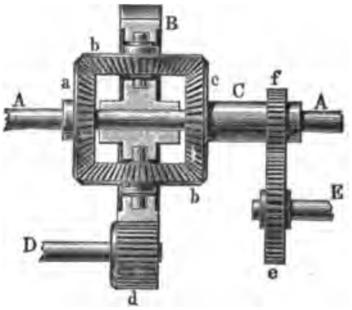
wheels *b*, whose axes are stationary, to the internally geared drum *C*, turning the latter in an opposite direction.

The differential mechanism finds another very important and interesting application in the bobbin and fly frame* for roving cotton, wool, etc., in which use is made of the property that the motion of the differential wheel is proportional to the difference of the motions of the shafts *A* and *C*. In this case the mechanism is generally given in a somewhat different form, Fig. 154, the two wheels *a* and *c* being equal, and the motion transmitted through two bevel wheels, *b b*, which are so placed that they will turn loosely in the differential wheel *B*. Two wheels *b* are here employed for the sake of symmetry in arrangement and action, being equivalent to a single wheel. It is clear that the relation above developed,

$$\alpha a \pm \gamma c = \beta (a + c),$$

is applicable here if we make $a = c$; we then get

FIG. 154.



$$\alpha a \pm \gamma a = \beta \times 2a$$

or

$$\beta = \frac{\alpha \pm \gamma}{2}.$$

This equation may also be obtained directly if we equate the motions received by the idle wheel *b* from *a* on the one hand, and from *c* on the other, when these wheels turn through α and γ respectively, the differential wheel *B*, at the same time turning through an angle β and carrying with it the idle wheels *b*. The rotation of *a* evidently causes a rotation $-\alpha\frac{a}{b}$ of the wheel *b* about its axis, while the motion of B causes a rotation $+\beta\frac{a}{b}$ of the wheel b, the sum being therefore $\frac{a}{b}(\beta-\alpha)$. In like manner the rotation γ of the wheel c causes a rotation $+\gamma\frac{c}{b}$ of b, while the rotation of the differential wheel causes the idle wheel b to rotate through $-\beta\frac{c}{b}$, so that the rotation of b about its own axis may be expressed by $\frac{c}{b}(\gamma-\beta)$. Equating these two values we get, since $a = c$,

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* See article "Baumwolle," by *Hülsse* in *Prechtl*, Technologische Encyclopädie, Supplement.

$$\beta - \alpha = \gamma - \beta,$$

or

$$\beta = \frac{\alpha + \gamma}{2}.$$

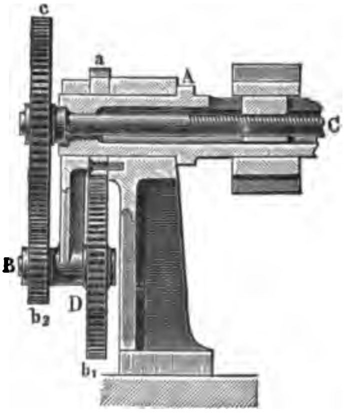
The wheels b are usually of the same size as a and c, but the size of b is immaterial, as the calculation shows, for b only acts as an idle or intermediate wheel.

The arrangement and mode of action of the mechanism under consideration is as follows: The wheel a is *keyed fast* to the shaft A, which acts as driving shaft for the spindles of the fliers on the roving machine. On these spindles turn the bobbins which hold the rovings (slightly twisted fibres), but they turn with a velocity which is greater or less than that of the spindles. The bobbins are not directly driven by A, but by the shaft E, which receives its motion from the wheel c by means of f and e, the wheel c turning loosely upon the shaft A. Consequently the spindles will be connected with a and the bobbins with c, *i.e.*, their motion will be proportional to the motion of these wheels. If we now suppose the differential wheel B to receive an independent motion β from a third axis D, it is evident that this motion β, together with the motion α of A, will produce such a rotation γ of the wheel c that β will be the difference between α and γ. But the difference between the motions of spindle and bobbin is the winding motion which effects the winding of the roving on the bobbin, and from this follows, that if the spindles are turned by *A* and the rotation of the wheel *B* is proportional to the velocity of winding, the bobbins will turn with their proper speed if they are driven by the wheel *c*. As the bobbins are filled, their diameter will increase and the number of revolutions needed to wind the roving will diminish, for the supply of roving remains constant. By giving to the differential wheel, therefore, a velocity varying with the velocity of winding, we have the means of fulfilling the conditions of perfect winding.

The differential mechanism has also been employed in self-acting mules* and for other special purposes. The use of this mechanism for dynamometers has already been mentioned in Vol. II.

The wheel *c* of the ordinary differential mechanism, Fig. 151, may be provided with external instead of internal teeth by giving to the shaft *B* two different wheels b_1 and b_2, Fig. 155, of which b_1 works with *a* and b_2 with *c*. They have been thus applied in boring machines for giving an axial or sliding motion to the tool or cutter block. Let *A* be the boring bar, *C* the screw for communicating the axial motion to the cutter block, *c* a wheel fastened to the end of the screw, and *a* a wheel which does not turn on the bar *A*, but is concentric with it, being rigidly attached to the frame. Around this fixed wheel circles the wheel b_1, whose shaft *B* is carried by an arm *D*, which turns with the boring bar. Since the wheel b_2 is keyed to *B* it is also rigidly attached to b_1, and will move with the latter, communicating to the wheel *c*, and consequently to the screw *C*, a certain rotation γ. The velocity

Fig. 155.



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* See Zeitschrift Deutsch. Ingenieure, p. 216, 1874.

ratio is as easily found as before. The absolute motion communicated to the shaft *B* is determined by the action between b_1 and the fixed wheel *a*, being equal to

$$+ \alpha \frac{a + b}{b_1} = \omega_1,$$

when the boring bar and its arm *D* rotate through the angle $+ \alpha$. From the action between the wheel *c*, which rotates through the angle γ, and the wheel b_2, whose centre has the angular motion α, we find the absolute rotation of the shaft equal to

$$- \gamma \frac{c}{b_2} + \alpha \frac{c + b_2}{b_2} = \omega_2.$$

Equating the values of ω_1 and ω_2 we get

$$- \gamma c b_1 + \alpha c b_1 = \alpha a b_2;$$

or,

$$\gamma = \alpha \left(1 - \frac{a b_2}{c b_1}\right).$$

Since the boring bar rotates with the nut or cutter-block through an angle α it will only be the relative rotation

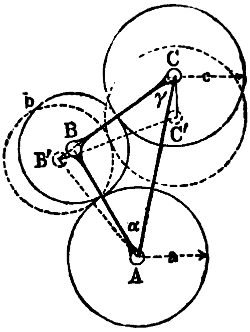
$$\alpha - \gamma = \alpha \frac{a b_2}{c b_1}$$

between the boring bar *A* and the screw *C*, which will be effective in producing a sliding of the cutter-block in the direction of the axis. This sliding will be equal to $s \frac{a b_2}{b_1 c}$ when *s* represents the pitch of the screw.

In the arrangement last considered the radii of the intermediate wheels b_1 and b_2 are of importance, for they are no longer to be considered as idle wheels, but as wheels which modify the velocity of the train of gearing $\frac{a}{b_1} \frac{b_2}{c}$. This arrangement differs from that shown in Fig. 134, § 44, only in this, that while the axis of the intermediate shaft was fixed in the former case, in the present case it is carried around the other two axes.

In many cases the motion of a shaft *A* is to be transmitted to another *C* in such a way that the moved shaft *C* can be given another motion perpendicular to its axis without interrupting the transmission of motion and without varying the velocity ratio. This is accomplished by means of the epicyclic train or ***knee-joint***, Fig. 156. Here *A* is the driving shaft supported in fixed bearings. To the shaft *A* is fixed the wheel *a*, which gears into the idle wheel *b*, the latter gearing with the wheel *c* of the shaft *C*. The wheel *c* will have the same direction of motion as *a*, the velocity ratio being $\frac{a}{c}$. Now, if the shaft *C* is to be capable of moving to *C'*, it is only necessary, as far as mere transmission of motion is concerned, to make the shaft of the idle wheel *b* part of a turning pair uniting the two links *B C* and *B A*, these links also inclosing the shafts *A* and *C* respectively. Now, if the shaft *C* shifts to *C'*, the shaft *B* will move to *B'*, so that $AB = A'B'$ and $CB = C'B'$. Since the distance between the axes *A* and *B*, as well as that between *C* and *B*, remains unchanged, the transmission of motion will still be possible. This arrangement is likewise employed in bobbin and fly-frames, the shaft *C* driving the bobbins, which are placed upon a carriage (coppin rail) that moves up and down so as to effect a uniform winding of the bobbins. It is not sufficient, however, to simply transmit the motion, the arrangement must also be such as will permit the driven shaft *C* to rise and fall without disturbing the uniform transmission of motion, since a change in the velocity ratio would cause the bobbins to turn faster or slower, according as the shaft *C* moved up or down. To ascertain the proportions of a correct knee-joint we will suppose the tri-

FIG. 15´.



angle $A\,B\,C$ to represent the original position of the axes, the links $A\,B$ and $C\,B$ making the angles α and γ with the line $A\,C$ joining the axes. Now, if the point C be moved through an infinitesimal distance to C', the shaft B will move to B', the arm $A\,B$ turning to the left through a very small angle $B\,A\,B' = \delta\,\alpha$. In the meantime the arm $B\,C$ has moved to $B'\,C'$ by turning to the right through the very small angle included between the two positions $C\,B$ and $C'\,B'$, which may be represented by $\delta\,\gamma$. The motion of the knee-joint will have no influence on the velocity ratio if, when the shaft A is at rest, the motion of the joint does not communicate any rotation whatever to the shaft C. If neither c nor a receives rotation, of course the idle wheel b cannot receive any rotation; for the two actions on b, which result from c and a acting on b in consequence of the links $A\,B$ and $C\,B$ rotating through $-\,\delta\,\alpha$ and $+\,\delta\,\gamma$ respectively, neutralize each other. By turning the link $A\,B$ through $-\,\delta\,\alpha$, the wheel b is given an absolute rotation amounting to

$$-\,\delta\,\alpha\,\frac{a+b}{b} = -\,\delta\,\alpha\,\frac{d_1}{b},$$

where $a + b = A\,B$ is represented by d_1. By turning the link $C\,B$ through $+\,\delta\,\gamma$ relatively to the stationary wheel c, the wheel b receives an absolute rotation

$$+\,\delta\,\gamma\,\frac{c+b}{b} = \delta\,\gamma\,\frac{d_2}{b},$$

where d_2 is the length of the link $C\,B = c + b$. The total rotation of b is therefore the sum of the components, which equals

$$+\,\delta\,\gamma\,\frac{d_2}{b} - \delta\,\alpha\,\frac{d_1}{b},$$

which sum, of course, has an opposite sign when C is shifted in a contrary direction. Hence, if the intermediate wheel is to receive no rotation whatever, the condition $d_1\,\delta\,\alpha = d_2\,\delta\,\gamma$ must be fulfilled.

Let us examine this condition more closely. In the triangle $A\,B\,C$ we have for every position of C,

$$A\,B : C\,B = \sin\gamma : \sin\alpha,$$

or,

$$d_1 \sin\alpha = d_2 \sin\gamma.$$

Differentiating, we get

$$d_1 \cos\alpha\,\delta\alpha = d_2 \cos\gamma\,\delta\gamma.$$

Substituting in this

$$d_1\,\delta\alpha = d_2\,\delta\gamma,$$

we get,

$$\cos\alpha = \cos\gamma,$$

or, $\alpha = \gamma$, since α and γ are smaller than π. The triangle $A\,B\,C$ formed by the axes must therefore be isosceles if the motion of the knee-joint is to be without influence on the velocity ratio of the motion transmitted. In order that

$$d_1 = d_2, \quad \text{or,} \quad a + b = c + b,$$

the two wheels a and c must be of the same size.

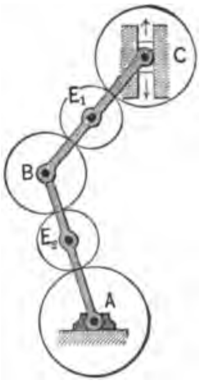
Sometimes the knee-joint is constructed with extra idlers e_1 and e_2, one on each of the links, as in Fig. 157. The only effect of this is to change the direction of rotation of b, consequently we have

$$+\,\delta\alpha\frac{a+b}{b} - \delta\gamma\frac{c+b}{b} = 0;$$

or,

$$\frac{\delta\alpha}{\delta\gamma} = \frac{c+b}{a+b}.$$

FIG. 157.



Since the lengths of the links are,

$$d_1 = a + b + 2\,e_1$$

and

$$d_2 = c + b + 2\,e_2,$$

substituting as before

$$d_1 \cos \alpha \, \delta \alpha = d_2 \cos \gamma \, \delta \gamma,$$

we also have

$$\frac{\delta \alpha}{\delta \gamma} = \frac{c + b + 2 e_2}{\alpha + b + 2 e_1} \, \frac{\cos \gamma}{\cos \alpha}.$$

These two values for $\frac{\delta \alpha}{\delta \gamma}$ can only be equal where $a = c$ and $e_1 = e_2$, hence $d_1 = d_2$ and $\alpha = \gamma$.

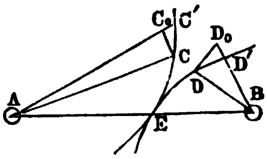
§ **49. Non-Circular Wheels.**—In the preceding investigations it was always tacitly assumed that a constant velocity ratio was to be transmitted. This case is the one which most commonly occurs in ordinary transmissions by means of wheels, and always occurs when the forces to be transmitted are large. In certain kinds of machinery, however, the peculiar character of the work requires the working piece to have a more complex motion, namely, of such a character that the velocity ratio will vary periodically, so that for a uniform motion of the driving shaft, the driven one will alternately move with greater and less velocity. Besides other means discussed further on, which can be employed for this purpose, use is made of *non-circular* wheels. These wheels, comparatively speaking, occur rarely, and then only with parallel shafts, for the difficulties of construction, which are great even for parallel shafts, would be almost insurmountable if they were to be applied to intersecting and crossing axes.

In order that such non-circular wheels may work together correctly, we must deduce the fundamental forms of the wheels in each case from certain rules which are universally applicable, and which we will now proceed to establish.

Let the two shafts A and B, and their distance d, Fig. 158, be given, also let the law according to which the velocity ratio of the two rotations is to vary be known. The perimeters of the two wheels must then be of such a character that the sum of the distances from their axes of any two points, like C and D, which meet and touch on the line of centres

$A B$, must be equal to the distance d between the axes. Two corresponding points on the perimeters must therefore satisfy this first condition: $a + b = d$. In order that the circumferences may also touch when the points C and D meet, it is necessary that the angles formed by the tangents $C' C$ and $D' D$ with the radii $A C$ and $B D$ be supplements of each other. Let a represent the length of any radius $A C$, making the angle $E A C = \alpha$ with the line of centres, let b be the radius $B D$, and β the angle $E B D$ for the corresponding point D of the other wheel, and let $C A C' = \delta \alpha$ and $D B D' = \delta \beta$. The condition for the contact of the wheels may then be expressed by

FIG. 158.



$$A C C' + B D D' = \pi,$$

i.e., if we have $C C_0 \perp AC$ and $D D_0 \perp B D$

$$C' C C_0 = - D' D D_0,$$

or $\varphi = - \psi$, if we suppose the angles formed by the tangents to the curves with the circular arcs passing through the points C and D to be represented by φ and ψ respectively.

From this follows:

$$\tan \varphi = - \tan \psi,$$

i. e.,

$$\frac{\delta a}{a \, \delta \alpha} = - \frac{\delta b}{b \, \delta \beta}.$$

Now since the condition $a + b = d$ requires $\delta a = \delta b$, the second condition is transformed into $a \, \delta \alpha = b \, \delta \beta$.

The velocity ratio, which varies from point to point, is therefore expressed by

$$n = \frac{\delta \beta}{\delta \alpha} = \frac{a}{b}.$$

If the velocity ratio n is given for any point, we shall find, as in circular wheels, from

$$a + b = d, \quad \text{and} \quad \frac{a}{b} = n,$$

for this point,

$$a = \frac{d}{1 + \frac{1}{n}} = \frac{d}{1 + \frac{\delta \alpha}{\delta \beta}},$$

and

$$b = \frac{d}{1 + n} = \frac{d}{1 + \frac{\delta \beta}{\delta \alpha}}.$$

Generally the velocity ratio n is not given for every point, but may be determined from some law $\beta = f(\alpha)$, β being the angle through which the wheel B turns, while the other wheel turns through α. When this law is known we may determine $n = \frac{\delta \beta}{\delta \alpha}$ for every instant, and if we substitute this value in the above expressions for a and b, we shall get two equations in which a and b are expressed as functions of α and β, that is, the perimeters will be expressed by polar equations, the pole in each case coinciding with the corresponding axis. A few examples may serve as illustrations:

Let us assume that $\alpha = c\,\beta$ is given, and c constant, then will $\delta \alpha = c\,\delta \beta$, and substitution in $a\,\delta \alpha = b\,\delta \beta$ will give $c = \frac{b}{a}$, *i.e.*, we have the case of circular wheels.

If we assume the law of transmission of motion to be, say,

$$\alpha = c_1 \beta + c_2 \beta^2,$$

in which c_1 and c_2 are constant, we shall get

$$\delta \alpha = c_1\,\delta \beta + 2\,c_2\,\beta\,\delta \beta,$$

and

$$\frac{\delta \alpha}{\delta \beta} = c_1 + 2\,c_2\,\beta.$$

This value substituted in

$$a = \frac{d}{1 + \frac{\delta \alpha}{\delta \beta}}$$

gives

$$a = \frac{d}{1 + c_1 + 2 c_2 \beta};$$

and in like manner we get

$$b = \frac{d (c_1 + 2 c_2 \beta)}{1 + c_1 + 2 c_2 \beta}.$$

When c_1 and c_2 are given directly we can evidently calculate β for any assumed value of α, and can calculate the corresponding values of a and b and draw the forms of the wheels. But generally the constants are not given directly, but are indirectly prescribed by certain conditions imposed upon the required motion. For example, we may assume that when the wheel *A* makes one revolution, the wheel *B* must likewise make a complete revolution, but during that time must be alternately subjected to two accelerations and two retardations, a retardation and an acceleration occurring during every 90° of rotation of the shaft *A*, and that the greatest velocity of the wheel *B* should be four times as great as its smallest velocity. From these conditions we may determine the constants as follows:

If at the beginning of the motion $\alpha = 0$, we shall also have $\beta = 0$; on the other hand, when $\alpha = \frac{\pi}{2}$, we also have $\beta = \frac{\pi}{2}$, hence

$$\frac{\pi}{2} = c_1 \frac{\pi}{2} + c_2 \left(\frac{\pi}{2}\right)^2$$

or

$$1 = c_1 + \frac{\pi}{2} c_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

The velocity ratio at the beginning, when $\alpha = \beta = 0$, is

$$\frac{\delta \alpha}{\delta \beta} = c_1;$$

on the other hand, when $\alpha = \beta = \frac{\pi}{2}$ we have

$$\frac{\delta \alpha}{\delta \beta} = c_1 + 2 c_2 \frac{\pi}{2} = c_1 + c_2 \pi.$$

The ratio of the two magnitudes must be placed equal to 4, hence

$$\frac{c_1 + c_2 \pi}{c_1} = 4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two equations (1) and (2) give

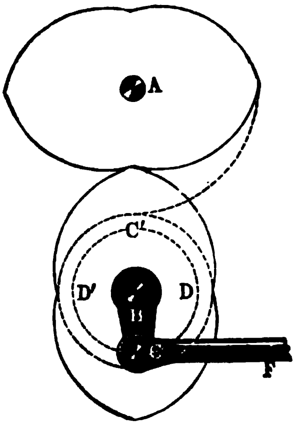
$$c_1 = 0.4, \quad \text{and } c_2 = \frac{6}{5 \pi} = 0.382.$$

The transmission of motion is therefore represented by the equation,

$$\alpha = 0.4 \beta \pm 0.382 \beta^2,$$

the positive sign corresponding to the accelerated and the negative sign to the retarded portions of the rotation. The wheels shown in Fig. 159 are analogous to those here calculated. Such wheels are used in silk winders, in which the driving shaft *A* moves uniformly, and by means of the wheels causes the shaft *B* to move now with greater, now with less velocity. In order that the crank *B C* may communicate a reciprocating motion to the guide of the threads, and cause the threads to be wound as uniformly as possible on the bobbins, the crank is given a very much more

FIG. 159.



rapid motion at the points D and D', where the rod F has only a slight motion, than in the positions $C\,C'$ at right angles to $D\,D'$.

The development of the formulas is the same, whatever equations between α and β be employed. If the wheels were to satisfy the condition expressed by $\alpha = c_1\,\beta + c_2\,sin\,\beta$, we should get from

$$\frac{\delta\,\alpha}{\delta\,\beta} = c_1 + c_2\,cos\,\beta$$

$$a = \frac{d}{1 + \frac{\delta\,\alpha}{\delta\,\beta}} = \frac{d}{1 + c_1 + c_2\,cos\,\beta},$$

and

$$b = \frac{d}{1 + \frac{\delta\,\beta}{\delta\,\alpha}} = \frac{c_1 + c_2\,cos\,\beta}{1 + c_1 + c_2\,cos\,\beta}\,d.^*$$

§ **50.**—Sometimes the conditions to be satisfied by the wheels can be stated otherwise than by establishing a relation between the velocities of rotation; for example the assumption can be made that

$$tan\,\varphi = -\,tan\,\psi = c\,;$$

that is, the tangents to the curves shall at all points make one and the same angle with the circular arcs, the tangent of the constant angle being equal to c. Then would

$$c = \frac{\delta\,a}{a\,\delta\,\alpha} = -\,\frac{\delta\,b}{b\,\delta\,\beta};$$

and from this by integration,

$$c\,\alpha = hyp\;log\,a \text{ or, } e^{c\,\alpha} = a$$

$$-\,c\,\beta = hyp\;log\,b \text{ or, } e^{-c\,\beta} = b.$$

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[*] See *Redtenbacher's* Bewegungsmechanismen.

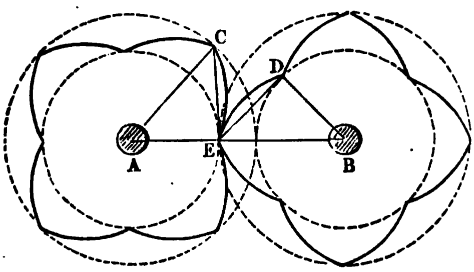
These equations represent logarithmic spirals, whose tangents it is well known everywhere make equal angles with their radii. In order that we may correctly cut pieces from these endless lines for two axes whose distance is *d*, we must assume the two radii a_1 and b_1 corresponding to the initial position, and then write

$$a = a_1 e^{c\alpha} \text{ and } b = b_1 e^{c\beta}$$

for the equations of the perimeters. In these equations α and β are to be estimated from the initial position, for $\alpha = 0$ gives $a = a_1$ and $\beta = 0$ gives $b = b_1$.

Logarithmic spirals are sometimes employed to form the perimeters of three-cornered, four-cornered and other polygonal wheels. In Fig. 160 there is shown a pair of four-toothed wheels. If the smallest radius AE of the wheel A is equal to a_1, we shall have for the greatest radius

FIG. 160.

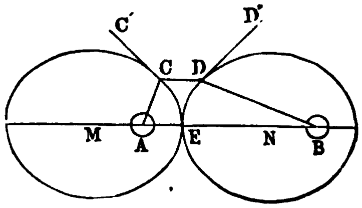


$$AC = a_1\sqrt{2} = 1.4142\, a_1,$$

for the angle of rotation

$$\alpha = \frac{\pi}{4} = 45^\circ.$$

FIG. 161.



If we substitute in the equation

$$a = a_1 e^{c\alpha} \text{ or } c\,\alpha = hyp\ log \frac{a}{a_1}$$

the values $\alpha = \frac{\pi}{4}$ and $\frac{a}{a_1} = \sqrt{2}$, we get

$$c = tan\ \varphi = \frac{4}{\pi}\ hyp\ \ log \sqrt{2} = 0.44128,$$

and consequently $\varphi = 23° 49'$. The equation of curves on the wheels is therefore: $a = a_1 e^{0.44128 \alpha}$, from which we may get values of a and $b = d - a$ corresponding to assumed values of α, ranging from 0 to $\frac{\pi}{4}$. For contact at E the velocity ratio is

$$n_1 = \frac{a_1}{b_1} = \frac{1}{\sqrt{2}},$$

and for contact between C and D_1

$$n_2 = \frac{b_1}{a_1} = \sqrt{2};$$

the ratio of greatest to least velocity is therefore

$$\frac{n_2}{n_1} = \frac{\sqrt{2}}{\sqrt{\frac{1}{2}}} = 2.$$

Among others *Bacon* and *Donkin* have made use of polygonal wheels for their printing-presses, in which the forms were placed on the sides of a rotating prism, the motion between the prism and the paper-cylinder being communicated by means of these wheels. See Nicholson's The Operative Mechanic, 1834, p. 302.

In like manner polygonal wheels may be constructed with any number of sides, and it is by no means a necessary condition that the number of sides be the same in both wheels. If one wheel has m_1 and the other m_2 sides, and if the number of revolutions for a given time are z_1 and z_2, we shall always have $m_1 z_1 = m_2 z_2$.

The elliptical wheels, Fig. 161, constitute a particular case of non-circular wheels. Two equal ellipses which turn about corresponding foci A and B, evidently fulfil the general conditions, for to any point C of one ellipse there corresponds another point D on the other ellipse, so placed that the two radii $A\ C + B\ D$ together equal the transverse axis of the ellipse, *i.e.*, equal the distance between the axes; moreover, since in every ellipse the lines drawn from any point to the foci make equal angles with the tangent at that point, it follows that in the equal ellipses the angles included between the radii and tangents of the corresponding points c and D will be supplements of each other. Let the semi-major axis $\frac{d}{2} = a$ and the semi-minor axis equal b, then we shall have for the eccentricity

$$MA = NB = e = \sqrt{a^2 - b^2},$$

and for the greatest velocity ratio

$$n_1 = \frac{a + e}{a - e}$$

and for the smallest

$$n_2 = \frac{a - e}{a + e},$$

consequently, when the driving shaft A moves uniformly, we shall have for ratio of the greatest and smallest velocity of the driven wheel

$$\nu = \frac{n_1}{n_2} = \left(\frac{a + e}{a - e}\right)^2.$$

If this ratio ν is given, and also the distance between the axes $d = 2a$, we shall get from the preceding equation

$$a + e = (a - e)\sqrt{\nu},$$

and therefore

$$e = a\frac{\sqrt{\nu} - 1}{\sqrt{\nu} + 1},$$

and for the semi-minor axis

$$b = \sqrt{a^2 - e^2} = \frac{2a\sqrt[4]{\nu}}{\sqrt{\nu} + 1}.$$

Elliptical wheels are sometimes employed in planers to give a slow advance to the table during the cutting of the tool and then a quick return, the table being given a reciprocating motion by a crank fastened to the shaft or ellipse *B*. If the greatest velocity of the crank is to be four times its smallest velocity, the ellipse *B* must have the eccentricity

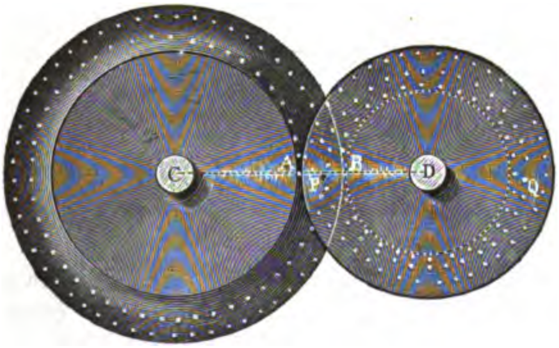
$$e = a\frac{\sqrt{4}-1}{\sqrt{4}+1} = \frac{a}{3},$$

and the semi-minor axis must be

$$b = \sqrt{a_2 - \frac{a_2}{9}} = a\sqrt{\frac{8}{9}} = 0.943\,a.$$

The wheels can also be so arranged that the point of contact will travel along a helical line. The surfaces of the wheels must be made conical when the velocity ratio is variable. Such conical wheels were first applied by the astronomer *Römer*. Fig. 162 gives an idea of such a pair of wheels.

FIG. 162.



They are constructed so that the driven wheel *B C* makes two revolutions while the driving wheel *A D* makes three, and so that at the end of the period the velocity ratio will have one-half of its original value, *i. e.*, so that $\frac{BD}{BC} = \frac{1}{2}\frac{AD}{AC}$.

If the law of variation of the motion is again represented by $\alpha = c_1 \beta + c_2 \beta^2$, we shall find the constants c_1 and c_2 from:

$$3 \times 2\pi = c_1 \times 2 \times 2\pi + c_2 (2 \times 2\pi)^2$$

and

$$\frac{c_1}{c_1 + 2c_2 \times 2 \times 2\pi} = \tfrac{1}{2},$$

hence

$$c_1 = 1 \text{ and } c_2 = \frac{1}{8\pi}.$$

For $CD = d$ we can now get for the radii of the wheels at the beginning of the motion

$$AD = a_1 = \frac{d}{1 + c_1} = \frac{d}{2}$$

and

$$AC = b_1 = \frac{d}{2},$$

and at the end of the motion

$$BD = a_2 = \frac{d}{1 + c_1 + 2c_2 \times 2 \times 2\pi} = \frac{d}{3},$$

and

$$BC = b_2 = \frac{2}{3}d.$$

The teeth are represented in the figure by points arranged in three coils on the wheel *A D* and in two coils on the wheel *B C*. The wheels first touch at *A*, and after the driving wheel has made three turns and the other two, the contact is at *B*. On the other hand, if the driven wheel has made but one turn and the driving wheel *A.D* one and a half turns, the two points *P* and *Q* will be in contact and their distances from the axes will be

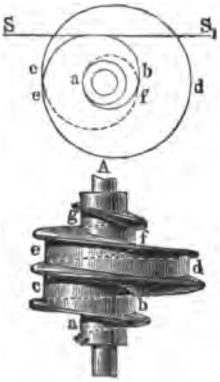
$$QD = \frac{d}{1 + c_1 + 2c_2\beta} = \frac{d}{1 + 1 + 2\frac{1}{8\pi}2\pi} = \frac{2}{5}d,$$

and

$$PC = d - \frac{2}{5}d = \frac{3}{5}d.$$

Such spiral-shaped wheels rarely or never occur as toothed wheels, but are constructed as cord pulleys for self-acting mules. Such a cord pulley or double fusee is shown in Fig. 163. Its object is to communicate motion in both directions to the carriage which holds the spindles, the motion being communicated from a uniformly rotating shaft A by means of a cord, in such a way that the carriage is gradually accelerated until it has reached the middle of its path and is then retarded to the end of its path. The result of this is to diminish the hurtful effect of shocks which occur at each end of the path, particularly when the mass of the carriage is great. For this purpose the cord winds on and unwinds from the fusee at the points a, b, c, d, e, f and g, the cord being made in two parts, S and S_{1}, which are wound on the fusee in opposite directions, one end of S being fastened at a and one end of S_1 at g. Since both cords leave the fusee at the same point (the grooves in the fusee being broad enough to hold two cords), it is evident that when the fusee turns, say in the direction of the arrow, a certain amount of the cord S will wind itself on the fusee, and an equal amount of S_1 will unwind. The cords S and S_1 are led around two pulleys and their other ends attached to the front and back of the carriage, which is not shown in the figure.*

FIG. 163.



The conical fusee ordinarily employed in watches must also be classed among spiral wheels.

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* See article by *Hülsse*, in *Prechtl's* Technolog. Encyclopädie; also *Stamm*, Der Selfactor.

§ 51. Friction Wheels.—If motion is to be communicated by *contact* of the circumferences of the wheels, the latter must be pressed together with such force that the friction at the circumferences will at least equal the force K to be transmitted. If φ is the coefficient of friction, the force with which the wheels are pressed together must be

$$R = \frac{K}{\varphi},$$

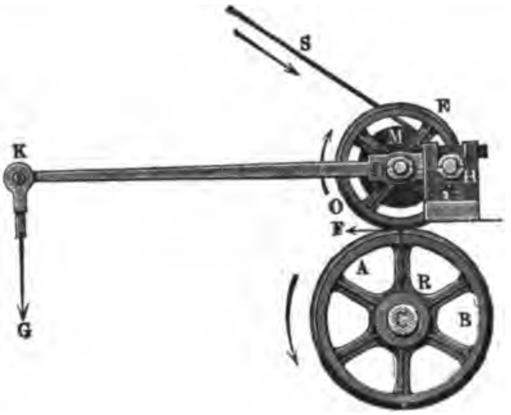
and in order that this force may not be unnecessarily large, we must try to make φ as large as possible by choosing the proper materials and employing rough surfaces of contact. On this account wood is often made to work with wood, or wood with cast iron, and sometimes the perimeters of the wheels are covered with leather. Little friction wheels are even wholly made of concentric disks of leather, and recently wheels have been made of similar disks of coarse paper, which is compressed by great hydraulic pressure. If, for these materials, we assume an average value for $\varphi = 0.4$, the necessary pressure between the wheels must be

$$R = \frac{K}{\varphi} = 2.5\,K,$$

which pressure will also cause friction at the journals of the two shafts. Another defect of friction wheels is that at least one of them cannot have a fixed support or bearing, for it is only through the bearing that the pressure R can be exerted on the wheel. On this account friction wheels are seldom used, and when they are, it is only where the load is invariable and where the machine is often thrown in and out of gear, and in furnace and other hoists.

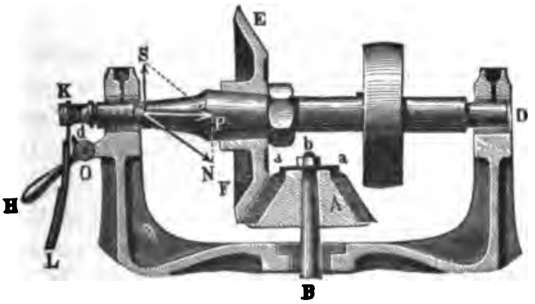
The arrangement of such a pair of wheels in a windlass is shown in Fig. 164. Here AB is the driving wheel on the shaft whose bearings are fixed, and EO is the driven wheel, which is rigidly connected with the shaft D of the drum and receives its motion from the wheel AB. To accomplish this the bearings of the journals of the shaft *D* are fixed to the lever *KDH*, which is pivoted at *H*, and is pressed down or up by a force *G* acting at *K* according as the drum is to be set in motion or stopped. When the lever is pressed down, the wheel *DEO* is pressed against the circumference of the driving wheel *ACB* with a certain force *R* which causes the friction $F = \varphi R$, and thus transfers the rotation of the shaft *C* to the drum *M* on which is wound the rope *S* that carries the load.

Fig. 164.



Friction wheels are also used in machines having rapidly running shafts which are frequently started and stopped. Since the action of friction is to produce a gradual change of velocity, its application as a transmitting force avoids the shocks which would occur if toothed gearing were employed. In this respect friction wheels act like friction couplings, and are consequently used for transmitting rapid motion in *centrifugal machines.* Conical friction wheels for a high speed turbine are shown in Fig. 165. The wheel *A* is fixed to the turbine shaft and is covered with a layer of leather rings, which are

Fig. 165.



pressed together by an annular nut *a a*, the wheel itself being fastened to the shaft *B* by means of the bolt *b*. On the horizontal shaft *C D* is fastened the smoothly turned wheel *E F*, which is pressed against the leather surface of the first wheel by a certain normal force *N*. The resulting friction φN is the greatest force which can be transmitted from one wheel to the other. A spring *K L* acts on the shaft *C D* in the direction of its axis with a certain force *P*, which may be resolved into the components *N* and *S*, the former being the normal force already mentioned, and the other a lateral force acting against the bearings. Let α represent the inclination of the side of the cone *A* to the axis; or $\alpha = D C N$, then we shall have for the components,

$$N = \frac{P}{\cos \alpha}, \quad \text{and } S = P \tan \alpha,$$

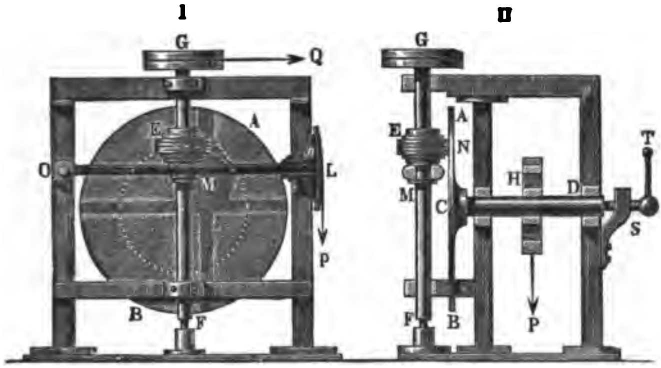
and the greatest force that can be transmitted by the wheels is

$$K = \varphi N = \frac{\varphi P}{\cos \alpha}.$$

During the motion the pressure *P* of the spring on the end of the shaft will cause pivot friction there, which is considerably reduced by rounding the end of the shaft. The bent lever *H O d* can disengage the wheels by pressing its short arm *O d* against the spring.

§ **52.**—Instead of conical friction wheels the arrangement of Fig. 166 is in many cases employed because it is a convenient means of changing the velocity ratio. The driving shaft *C D* is provided at one end with a flat disk, while the driven shaft *G F* carries a cylindrical friction disk *E*, which is made of wood or leather. The surfaces are pressed together by making the shaft *D C* movable axially, *i.e.*, without collars, the screw *S* pressing the flat disk against *E*. The shaft *D C* receives its motion from a belt pulley *H*, while the driven shaft *F G* propagates its motion by means of the wheel *G*. The lever *O L* turns about *O*, and fits into a circular grove at *M* on the disk E. It can thus shift the disk along its axis, and by this means vary the velocity ratio as desired. If the mean point of contact N of the disks is at the distance a from the centre C of the flat disk, and if b is the radius of the friction disk E, the velocity ratio will be $\frac{a}{b}$, for the two circles which pass through N, concentric to their respective axes, have the same peripheral velocities. If a_1 is the greatest distance from C that can be reached by the centre of the disk E, we shall have for the greatest velocity ratio $\frac{a_1}{b}$. By shifting the disk E to the centre C, the velocity ratio may be reduced to zero. Now if the disk E is shifted past the centre C, the velocity ratio will again increase, but it will be negative, that is, the shaft GF will rotate in an opposite direction. Little use is made in practice of this last circumstance, but the mechanism is employed when it desired to easily regulate the velocity. For example, in gang-saws it is frequently employed to regulate the feed, *i.e.*, the motion of the log toward the saws. It is also used in the bobbin and fly frames employed in flax spinning, in which the disk E is regularly shifted a certain amount as the bobbins fill.

FIG. 166.



This arrangement cannot be called an exact one, for the peripheral velocities of the two disks cannot be the same at all points of their line of contact. To accomplish this § 45 shows that it is necessary that the disks have conical surfaces whose common vertex lies at the intersection of the axes. As this condition cannot be met on account of the sliding of the disk *E*, the peripheral velocity of the two disks cannot be the same, except at one point of the line of contact, say the middle point *N*, though we cannot be absolutely sure that this point will be exactly at the middle of the line of contact. All points of the flat disk which lie beyond this point *N* of equal velocity have a greater speed, and all points lying within have a less speed than the points of the disk *E* with which they come in contact. The result is that the continual friction soon rounds off the disk *E*, as that contact only takes place on a very narrow surface. It is customary therefore to make the disk *E* quite narrow.

In centrifugal (drying) machines this mechanism is chosen, principally because the drum, which is directly attached to the shaft, *G F*, can be set in motion gradually even when the shaft *D C* is rotating rapidly, for it is only necessary to move the disk *E* slowly from the centre *C*. If in stopping the machine the disk were moved past the centre *C*, the opposite rotation would act as a brake, but preservation and durability of the parts require that this be seldom done.

A serious defect in the arrangement just discussed, as well as in all friction disks, is the occurrence of great frictional resistances at the journal. These, in the arrangement under consideration, exist partly at the pivot on the screw *S* and partly at the bearings of the shaft *G F*. The latter resistance is very troublesome in the shafts of centrifugal machines, which with their extraordinary velocity (about 1,000 revolutions per minute) soon begin to run unsteadily, and as this evil is increased by the high speed, serious accidents sometimes result. In this respect the use of a double mechanism, such as is shown in Fig. 167, may be regarded as a decided improvement.* Here two flat disks *A* and *A'* are placed upon two shafts *C D* and *C' D'* which run with equal velocity but in opposite directions. The flat disks are pressed against the disk E so that the shaft $G\,F$ is subjected to no lateral pressure whatever. The belt pulleys H and H' are of course driven by means of a crossed and an open belt.

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* See Zeitschrift deutsch. Ing. 1874, p. 531.

In order to effect the transmission of great forces by friction wheels, without employing too great a pressure, the arrangement shown in Fig. 168 may be employed, an end view $D\,A\,D$ being given of one wheel and a section through the rim $B\,A\,B$ of the other. Here the surfaces of the wheels consist of a number of V grooves so arranged that the projections on one wheel enter the grooves of the other. That this will cause an increase of friction is evident from the theory of the wedge, the force R which presses the wheels together being resolved into two components, the relation between R and N being given by:

FIG. 167.

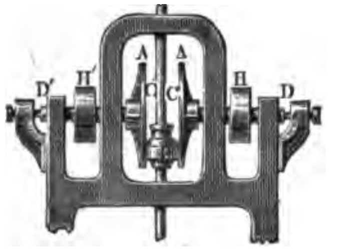
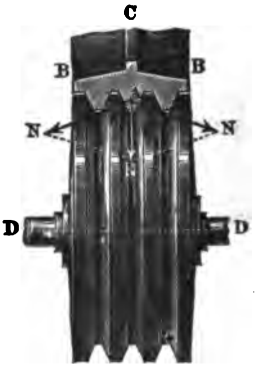


FIG. 163.



$$R = 2\,N\left(\sin\frac{\alpha}{2} + \varphi\cos\frac{\alpha}{2}\right),$$

α representing the angle included between the sides of the groove. As in former examples (§ 11 and § 31), we determine the friction which resists the sliding of the surfaces of the wheels on each other to be:

$$F = \frac{\varphi R}{\sin\frac{\alpha}{2} + \varphi\cos\frac{\alpha}{2}},$$ *

φ being the coefficient of friction.

---

* For this case, also, some text-books give $F = \frac{\varphi R}{\sin\frac{\alpha}{2}}$; the remarks of § 31 are therefore here applicable.

Theoretically the number of grooves employed is immaterial, but practically several grooves are preferable to a single one, because their depth may then be less and the pressure on the surfaces can be sufficiently small. That the depth of the grooves will act the more injuriously the greater it is, may easily be seen. If a ring and a groove, Fig. 169, touch along a line *a b*, the velocities on the circumferences can only be equal at a single point, say *c*; in all other points, between *c* and *a* as well as between *c* and *b*, unequal velocities exist, causing great friction, which will be greater the greater the ratios of the lengths *c a* and *c b* to their radii. In consequence of this friction the surfaces will soon take a form represented by Fig. 170, where the pressure is concentrated on a single point *c* or upon a very narrow surface, wherefore it is desirable that several grooves, etc., be used on each wheel.

FIG. 169.

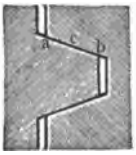
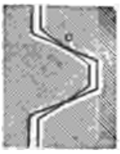


FIG. 170.



The angle α of the grooves is generally assumed equal to 30°. *Minotto* has also employed this arrangement in conical wheels, using only one groove, and that adjustable. Grooved wheels have also been recommended for locomotives for overcoming steep grades. *Reuleaux* advises their use when speeds are to be reduced.

REMARK. If we assume the coefficient of friction equal to 0.2, and the angle $\alpha = 30°$, we shall get

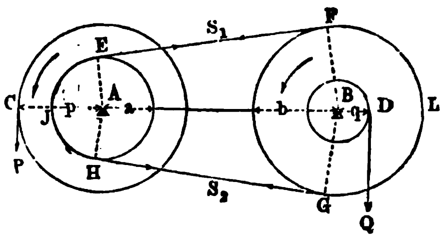
$$F = \frac{0.2\,R}{\sin 15° + 0.2 \cos 15°} = 0.43\,R,$$

while the often mentioned simple formula gives the improbable high value :

$$F = \frac{0.2\,R}{\sin 15°} = 0.77\,R.$$

§ **53. Belt Pulleys.**—Belt pulleys likewise come under the head of friction wheels, for they can only transmit motion because there are frictional resistances at the circumferences of the two wheels which are greater than the useful resistance to be overcome; since it is the smaller of two resistances which is first overcome, the useful resistance will be overcome as long as it is less than the friction which would result from the sliding of the wheel relatively to the belt. Belt pulleys differ from ordinary friction wheels in that they transmit motion indirectly by means of belts, cords, or ropes; that is, by means of bodies of sufficient flexibility to permit of their hugging the surfaces of the wheels as closely as possible. Let *A* and *B*, Fig. 171 and Fig. 172, be the shafts of two belt pulleys whose circumferences are embraced by the endless belt *EFGH*. If a resistance *Q* with a lever arm *q* acts on the shaft *B*, and hurtful resistances are neglected, we know from what has preceded that the force *P* acting with the lever arm *p* on the shaft *A*, which is necessary to overcome this resistance, may be determined from

FIG. 171.



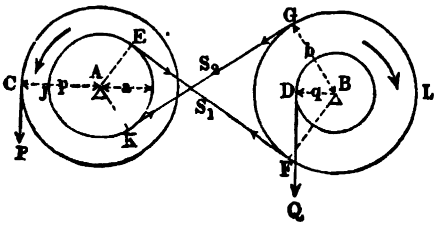


FIG. 172.

$$P = Q\frac{q}{p}\frac{a}{b}.$$

The resistance at the circumference of the pulley *B* is given by $Q\frac{q}{b}$, and of course the force *K* at the circumference of the wheel *A* must be as large in order to overcome the resistance of the shaft *B* by means of the belt. For this purpose, no sliding of the belt upon either the pulley *A* or *B* must take place; there must consequently be at each circumference a frictional resistance at least equal to

$$K = Q\frac{q}{b} = P\frac{p}{a},$$

for opposing a slipping of the belt. Now the resistance experienced by a flexible body, belt or rope, when it slides on a cylinder may easily be found from the developments of § 194, Vol. I. Let S_2 represent the tension in a portion of the belt *G H*, then in order that the belt may slide on the fixed pulley *A*, there must be exerted at *E* a force S_1, acting in the direction *E F*, equal to: $S_2 e^{\phi\gamma_1}$, γ_1 representing the arc *HJE* (having radius equal unity) embraced by the belt. The corresponding friction is therefore

$$F_1 = S_1 - S_2 = S_2(e^{\phi\gamma_1} - 1).$$

In like manner we may find, for the frictional resistance to the sliding of the belt on the pulley *B*, the expression

$$F_2 = S_1 - S_2 = S_2(e^{\phi\gamma_2} - 1),$$

γ_2 representing the arc *F L G*. Of these two values the smaller is evidently the one having the smaller enclosed arc; calling γ this smaller arc, we may conclude from the above that the force to be transmitted cannot exceed the smaller friction, *i.e.*, we have the equation:

$$K = S_1 - S_2 = S_2(e^{\phi\gamma} - 1),$$

which may also be written,

$$S_2 = \frac{K}{e^{\phi\gamma} - 1},$$

$$S_1 = \frac{e^{\phi\gamma}K}{e^{\phi\gamma} - 1}.$$

These equations show that for the transmission of a certain force *K*, the tensions of the two parts of the belts between the wheels must be of a different magnitude, and that the ***difference between the tensions must be equal to the force transmitted.*** The part of the belt which has the greater tension S_1 is called the ***driving*** or ***tight side***, while the other with the smaller tension S_2 is called the ***driven*** or ***slack side*** of the belt. When the belt is at rest, the two parts have the same tension S, and if the belt is not to slip, we must have

$$S = \frac{S_1 + S_2}{2} = \frac{e^{\phi\gamma} + 1}{e^{\phi\gamma} - 1}\,\frac{K}{2}.$$

The pressure exerted by the belt upon each of the two shafts is, therefore,

$$2S = S_1 + S_2 = \frac{e^{\phi\gamma} + 1}{e^{\phi\gamma} - 1} K,$$

that is, always greater than K. If the force K is not given in terms of Q and q it must be determined from the data ordinarily given, namely, the number of revolutions per minute u_1 and u_2 of the shafts A and B, and the work N to be transmitted expressed in horse powers. Then if the radii a and b are given in meters and K in kilograms, we have

$$K = \frac{30\,N\,75}{\pi\,a\,u_1} = \frac{30\,N\,75}{\pi\,b\,u_2},$$

$$= 716.2\frac{N}{a\,u_1} = 716.2\frac{N}{b\,u_2}.$$

When we know the speed of the belt,

$$v = \frac{2\,\pi\,a\,u_1}{60} = \frac{2\,\pi\,b\,u_2}{60},$$

we have

$$K = \frac{75\,N}{v}.$$

[When K is expressed in pounds, a and b in inches, and v in feet per second, the above formulas become

$$K = \frac{6 \times 60 \times 550 \times N}{\pi a u_1} = 63{,}025 \frac{N}{a u_1},$$

$$= \frac{6 \times 33{,}000 \times N}{\pi b u_2} = 63{,}025 \frac{N}{b u_2}.$$

$$v = 2\pi \frac{a}{12} \frac{u_1}{60} = 0.00873\, a u_1,$$

$$= 2\pi \frac{b}{12} \frac{u_2}{60} = 0.00873\, b u_2.$$

$$K = 550 \frac{N}{v}.]$$

EXAMPLE. If a belt transmits $N = 2$ horse powers [1.973 H. P.] at a velocity of the belt equal to 2 meters [6.28 ft. per sec.], and if the pulleys are so far apart that we can assume the arc of contact on each of the wheels to equal a semi-circumference, we shall have, when the coefficient of friction between belt and wheel is $\varphi = \frac{1}{2}$:

$$e^{\phi\gamma} = (2.71828)^{0.5 \times 3.1416} = 2.71828^{1.5708} = 4.81\,;$$

hence the tensions:

$$S_2 = \frac{K}{e^{\phi\gamma} - 1} = \frac{N 75}{v(e^{\phi\gamma} - 1)} = \frac{2 \times 75}{2(4.81 - 1)} = \frac{75}{3.81} = 19.68 \text{ kgs [43.38 pounds]},$$

and

$$S_1 = e^{\phi\gamma} S_2 = 4.81 \times 19.68 = 94.68 \text{ kgs [208.73 pounds]},$$

consequently the average tension before starting the machine is:

$$S = \frac{S_1 + S_2}{2} = \frac{94.68 + 19.68}{2} = 57.2 \text{ kgs [126.1 pounds]}.$$

§ **54. Belt Tensions.**—From what has preceded it is evident that, in order to calculate the tensions on the belt, it is necessary to know the coefficient of friction between belt and pulley, and the size of the arc covered by the belt. As regards the first we have, according to *Morin* (see his Aide mémoire, or his Nouvelles expériences sur le frottement, etc., Paris, 1838):

$\varphi = 0.50$ for hemp ropes on wooden drums.
$= 0.50$ for new belts " " "
$= 0.47$ for old shop belts on wooden drums.
$= 0.38$ for moist belts on turned (*i.e.*, smooth) cast-iron pulleys.
$= 0.28$ for dry belts on turned (*i.e.*, smooth) cast-iron pulleys.
$= 0.12$ for very greasy shop belts on cast-iron pulleys.

The arc covered by the belt may be determined from the radii $AC = a$ and $BD = b$ and the distance $AB = d$ between the axes, Fig. 173 and Fig. 174. We must distinguish between two cases when the belt is open and when it is crossed. When the belt is *open*, as in Fig. 173, we can determine half of the angle $DBF = CAE = \gamma_1$ by means of the formula

FIG. 173.

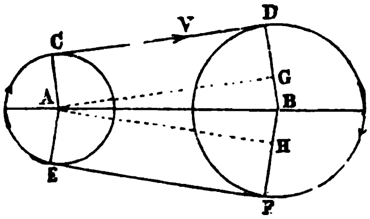
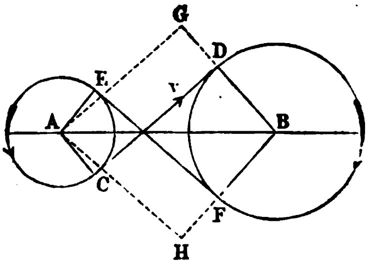


FIG. 174.



$$\cos ABD = \frac{BG}{AB} = \frac{BD - AC}{AB},$$

i.e., from

$$\cos \frac{\gamma_1}{2} = \frac{b - a}{d};$$

when the belt is crossed, Fig. 174, we find the angle

$$DBF = 2\pi - \gamma_1 = 2\pi - \gamma_2$$

from

$$\cos ABD = \frac{BG}{AB} = \frac{BD + AC}{AB},$$

that is,

$$\cos\left(\pi - \frac{\gamma}{2}\right) = \frac{a+b}{d}.$$

In the latter case the belt covers the same arc $2\pi - \gamma$ on each wheel, while in the open belt, Fig. 173, the arc γ_1 is covered on the smaller wheel and the arc $\gamma_2 = 2\pi - \gamma_1$ on the larger wheel; consequently, since there is a greater arc of contact with a crossed belt, the transmission of a given force can be effected with less tension on the belt than with an open belt.

The arc γ must be always understood to be the length of the arc on a circle whose radius is unity, if therefore γ is given in degrees (γ°) we shall have

$$\gamma = 0.017453\,\gamma^\circ \text{ or } \gamma^\circ = 57.297\,\gamma = 57.297\,\frac{\lambda}{a},$$

where λ is the length of the arc for the radius a.

The length of the whole belt when it is an open belt is

$$l = CD + EF + \text{arc}\,EC + \text{arc}\,DF = 2AG + \gamma_1 a + (2\pi - \gamma_1)b,$$

i.e.,

$$l = 2d\sin\frac{\gamma_1}{2} + \gamma_1 a + (2\pi - \gamma_1)\,b,$$

and for crossed belt:

$$l = 2d\sin\frac{\gamma}{2} + (2\pi - \gamma)(a+b).$$

For a great distance between the axes we may write

$$\cos\frac{\gamma}{2} = 0,$$

therefore

$$\frac{\gamma^\circ}{2} = 90^\circ, \text{ or } \gamma = \pi,$$

hence in both cases

$$l = 2d + \pi(a+b).$$

The exponential quantity $e^{\phi\gamma}$ can be determined from φ and γ, and then we can find the ratio of the tensions on the belt. The following *table of belt tensions* will be found convenient for this purpose:

	Values of $e^{\phi\gamma}$.					
Ratio.	New belts,	Ordinary belts,		Moist belts,	Cords on wooden wheels.	
$\frac{\gamma}{2\pi}=\frac{\gamma^\circ}{360^\circ}$	on wooden drums,	on wooden drums,	on iron pulleys,	on iron pulleys,	Rough,	Polished,
	$\phi = 0.050.$	$\phi = 0.47.$	$\phi = 0.28.$	$\phi = 0.38.$	$\phi = 0.50.$	$\phi = 0.33.$
0.2	1.87	1.80	1.42	1.61	1.87	1.51
0.3	2.57	2.43	1.69	2.05	2.57	1.86
0.4	3.51	3.26	2.02	2.60	3.51	2.29
0.5	4.81	4.38	2.41	3.30	4.81	2.82
0.6	6.59	5.88	2.87	4.19	6.59	3.47
0.7	9.00	7.90	3.43	5.32	9.00	4.27
0.8	12.34	10.62	4.09	6.75	12.34	5.25
0.9	16.90	14.27	4.87	8.57	16.90	6.46
1.0	23 14	19.16	5.81	10.89	23.14	7.95

Example. A belt on cast-iron pulleys transmits 4 horse powers [3.95 H. P.], and has a velocity of 3 meters [9.84 ft. per sec.], the radius of the driving pulley being 0.75 meter [29.28 ins.], that of the driven pulley being 0.125 meter [4.88 ins.] and the distance between the shafts 2.25 meters [87.83 ins.]: how large are the tensions on the belt?

Here $a = 0.75$, $b = 0.125$, $d = 2,25$, therefore for open belts:

$$\cos\frac{\gamma_1}{2} = \frac{0.75 - 0.125}{2.25} = \frac{5}{18} = 0.277\ldots;$$

hence

$$\frac{\gamma_1}{2} = 78^\circ\, 52\tfrac{1}{2}', \text{ and } \frac{\gamma_2}{2} = 106^\circ\, 7\tfrac{1}{2}',$$

and

$$\gamma_1 = 147^\circ\, 45', \quad \gamma_2 = 212^\circ\, 15',$$

on the other hand for crossed belt:

$$\cos\frac{\gamma}{2} = -\frac{75 + 12.5}{225} = -\frac{7}{18} = -0.388\ldots;$$

hence

$$\frac{\gamma}{2} = 180^\circ - 67^\circ\, 7' = 112^\circ\, 53',$$

and

$$\gamma = 225^\circ\, 46'.$$

In the first case we must of course assume the smaller angle $\gamma_1 = 147° 45'$ in order that the belt may not slip on either of the two circumferences. We therefore have here

$$\frac{\gamma}{2\pi} = \frac{147.75}{360} = 0.410$$

and in the second case, for crossed belt:

$$\frac{\gamma}{2\pi} = \frac{225.77}{360} = 0.627.$$

If we assume $\varphi = 0.28$ we shall get by interpolation from the table for the first case:

$$e^{\varphi\gamma} = 2.02 + 0.1\,(2.41 - 2.02) = 2.06,$$

and for the second case:

$$e^{\varphi\gamma} = 2.87 + 0.27\,(3.43 - 2.87) = 3.02,.$$

which agrees fairly with the direct calculation. The force to be transmitted is

$$K = \frac{N\,75}{v} = \frac{4 \times 75}{3} = 100 \text{ kilograms } [220.5 \text{ lbs.}];$$

therefore for the first case:

$$S_2 = \frac{100}{2.06 - 1} = \frac{100}{1.06} = 94.3 \text{ kilograms } [207.9 \text{ lbs.}],$$

$$S_1 = 2.06 \times 94.3 = 194.3 \text{ kilograms } [428.4 \text{ lbs.}], \text{ and}$$

$$S = \frac{S_1 + S_2}{2} = 144.3 \text{ kilograms } [318.1 \text{ lbs.}];$$

adding 10 per cent for safety, we get $S = 158.7$ kilograms [349.9 lbs.] as the tension on the belt when it is at rest. For the crossed belt:

$$S_2 = \frac{100}{3.02 - 1} = \frac{100}{2.02} = 49.5 \text{ kilograms } [109.1 \text{ lbs.}],$$

$$S_1 = 3.02 \times 49.5 = 149.5 \text{ kilograms } [329.6 \text{ lbs.}],$$

$$S = \frac{S_1 + S_2}{2} = 99.5 \text{ kilograms } [219.4 \text{ lbs.}];$$

or for safety

$$S = 99.5 + 10 = 109.5 \text{ kilograms } [241.4 \text{ lbs.}].$$

§ **55. Belts.**—The best belts are usually of good ox-hide tanned with oak bark. A hide will give two strips 4 to 5 millimeters [0.16 to 0.20 in.] thick, 20 centimeters [7.9 ins.] wide, and 2½ meters [98.4 ins.] long. These strips are usually joined by being laced or riveted at the ends.

According to recent experiments[*] the modulus of elasticity for neat's leather is $E = 15$ to 20 kilograms per square millimeter [21,300 to 28,400 lbs. per square inch], and the ultimate strength 2.9 kilograms [4,125 lbs. per square inch][†]; the working stress k of the belt may be taken according to *Morin* equal to 0.2 kilogram per square millimeter [285 lbs. per square inch]. ***Reuleaux*** makes the working stress dependent on the width b of the belt according to the formula

$$k = \frac{1}{200}\sqrt[4]{b^3}, \qquad [k = 80\sqrt[4]{b^3}];$$

from the bracketed formula we get the values of k for different values of b:

$b =$	2	4	6	8	Inches.
$k =$	134	226	306	380	Pounds.

The thickness of the belt varies between 4 and 6 millimeters [0.16 and 0.20 in.], consequently, for an average thickness of 5 millimeters [0.18 or $\frac{3}{16}$ in.] we shall have for every millimeter [0.04 in.] of width, a tension $S_1 = b\,\delta\,k = 1 \times 5 \times 0.2 = 1$ kilogram [2.2 lbs.], hence, for a given tension S_1 the width of the belt $b = S_1$.

Now, since the average of φ is nearly the same as when the belt covers almost half the circumference of the wheel, *i.e.*, the coefficient of friction $\varphi = 0.28$, we shall get S_1 nearly equal to $2K = 2\frac{75N}{v}$ [$2K = 66000\frac{N}{v}$ lbs., v being here feet per min.], and therefore

$b = 2K = 150\frac{N}{v}$ millimeters, v being in meters per second

---

[*] According to the experiments of Bevan (see Dingler, Vol. XVI.) the modulus of elasticity is only 10,050 lbs. per sq. inch (7.33 kilograms per sq. millimeter).

[†] Experiments of Henry R. Towne on belts give 3,086 lbs. per sq. inch for the ultimate strength of ordinary belts $\frac{7}{32}$ of an inch thick (see Journal Franklin Institute, Jan. 1868).

[$b = 1100 \frac{N}{v}$ inches, v being here in ft. per min.].

According to *Armengaud* the velocity v of the belt should not be taken less than $0.5\,(N + 1)$ meters [about $100\,(N + 1)$ ft. per min.].

The last formula for b shows that the width of the belt varies directly as the work to be transmitted and inversely as the velocity. When $\frac{N}{v} = \frac{5}{4}\left[\frac{N}{v} = 0.0065\right]$, for example, when $N = 10$ horse powers and $v = 8$ meters [1,580 ft. per min.], or $N = 5$ horse powers and $v = 4$ meters [790 ft. per min.], we get $b = 0.19$ meter [7.4 ins.]. From this we see that if we wish to avoid great width in a belt, the work transmitted must be small and the velocity great. When very wide belts are used there is danger, unless they are very well made, of their stretching unequally, and consequently of their not hugging the face of the wheel uniformly. For very large forces $\left(\frac{N}{v}\right)$ two belts are employed, or a chain, or, better still, toothed gearing.

In order to keep the belts flexible and make them durable, they must be kept greased with tallow or castor oil.

Gutta percha belts have nearly the same properties as leather belts; their density is nearly the same (spec. grav. $= 0.84$) and, according to experiments made by *Feistmantel*, on a small scale to be sure, the strength is nearly the same, but the modulus of elasticity is about 30 per cent. less than for leather. For the same thickness, therefore, gutta percha belts may be given the same width as leather belts. Gutta percha belts are not much used, because they have the defect of stretching when slightly heated, and thus diminish their tension. On the other hand, vulcanized rubber belts have given satisfaction, especially where they are exposed to moisture, as in distilleries and in the open air (with agricultural and hoisting machinery). When leather belts are exposed to moisture they stretch at the expense of their width, and require frequent tightening. Rubber belts, however, are impervious to moisture. These belts do not consist of vulcanized rubber alone, but also of strips of strong hempen cloth, so enclosed and united by the rubber as to form a continuous whole. Quite recently, instead of leather, *hemp* and *flax*, as well as woven *hair*, have been employed for belts.

Example. In the last example (§ 54) on belt gearing, the maximum tension of the crossed belt was found to be

$$S = \frac{e^{\phi\gamma}K}{e^{\phi\gamma} - 1} = 150 \text{ kilograms [330 lbs.] nearly};$$

the width of the belt should therefore be

$$b = S_1 = 150 \text{ millimeters [6 ins.]}.$$

Its length may be found from the formula:

$$l = 2d \sin\frac{\gamma}{2} + (2\pi - \gamma)(a + b);$$

$$l = 2 \times 2.25 \sin 112^\circ 53' + (0.75 + 0.125) \operatorname{arc} 225^\circ 46'$$

$$= 4.146 + 3.448 = 7.594 \text{ meters [299 ins.]}.$$

§ **56. Tightening-pulleys.**—Since belts, particularly when they are new, stretch considerably, they must be tightened from time to time. If the ends of the belt are fastened as in Fig. 175 or Fig. 176, this tightening may be easily done, but when they are laced, as in Fig 177, or are glued, the separating and reuniting of the ends of the belt are troublesome, on which account a joint is employed which may be easily broken, belt screws, Fig. 178, being used for this purpose. In some cases it may be of advantage to tighten the belt by a special device called a *tightening-pulley*, or by moving one of the shafts so as to increase the distance between the axes. The latter method is rarely employed,

Fig. 176.

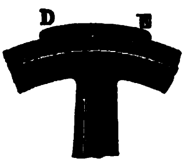


Fig. 175.

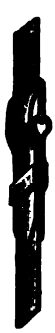


Fig. 177.

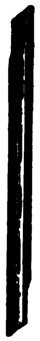
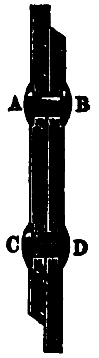


Fig. 178.



except in the hoisting gear of grinding-mills, because in general shafts must be rigidly supported. But there is frequent occasion to employ tightening-pulleys, especially where it is also desirable to have a convenient arrangement for throwing in and out of gear.

FIG. 179.

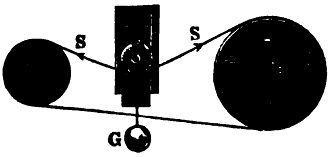
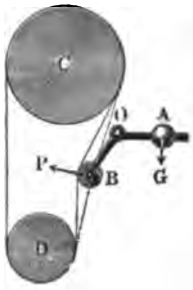


FIG. 180.

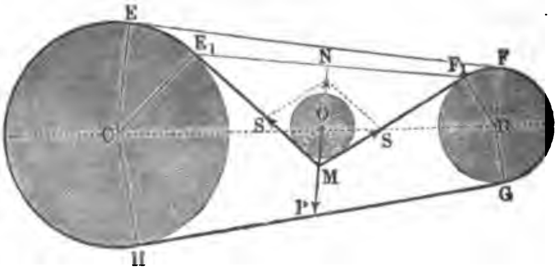


Two such tightening-pulleys are shown in Fig. 179 and Fig. 180. In Fig. 179 the pulley B is carried by a slide which is drawn down by a weight, on the other hand, in Fig. 180 the pulley is placed on the bent lever $A\,O\,B$, and is pressed against the belt by a weight hanging from the arm $O\,A$. If, in the latter case, a represents the lever arm $O\,A$ of the weight G, and b the lever arm $O\,B$ of the tightening force P, we have:

$$P = \frac{a}{b}G, \quad \text{also} \quad G = \frac{b}{a}P.$$

The tightening force P bisects the angle $E_1MF_1 = \delta$, Fig. 181, included between the two directions of the belt

FIG. 181.



ME and MF_1, and if S is the tension on the belt we have, as for a loose pulley (Vol. I., § 163):

$$1)\quad P = 2\,S\cos\frac{\delta}{2}.$$

To determine the tension S we must know whether the tightening-pulley presses against the tight or the slack side of the belt, in the first case S_1 and in the second S_2 must be substituted in the above formula. Moreover, the arc of contact FG is increased by an amount equal to FF_1 by the action of the tightening-pulley; if, as above, we make the angle $ECH = FDG = \gamma$, and the increment $ECE_1 = FDF_1 = \gamma_1$, we shall have:

$$2)\quad \begin{cases} S_1 = \dfrac{e^{\phi(\gamma + \gamma')}\,k}{e^{\phi(\gamma+\gamma')} - 1}, & \text{and} \\[2ex] S_2 = \dfrac{k}{e^{\phi(\gamma+\gamma')} - 1}. \end{cases}$$

The angle included between the two directions ME_1 and MF_1 is $E_1MF_1 = SMS = \delta$, and may be determined as follows: In consequence of the tightening by the pulley MO, the portion of the belt $EF = d\sin\frac{\gamma}{2}$ is transformed into EE_1MF_1F, which is approximately equal to $\dfrac{EF}{\sin\frac{\delta}{2}}$, and consequently increases in length by

$$\lambda \doteq \frac{EF}{\sin\frac{\delta}{2}} - EF = d\sin\frac{\gamma}{2}\left(\frac{1}{\sin\frac{\delta}{2}} - 1\right),$$

and the ratio of this increment to the whole length l is

$$\frac{\lambda}{l} = \frac{d}{l}\sin\frac{\gamma}{2}\left(\frac{1}{\sin\frac{\delta}{2}} - 1\right) = \frac{d\sin\frac{\gamma}{2}\left(1 - \sin\frac{\delta}{2}\right)}{l\sin\frac{\delta}{2}}.$$

But we must also assume $\frac{\lambda}{l} = \frac{k}{E}$, where k represents the working tension and E the modulus of elasticity of the belts; we can therefore place:

$$\frac{1}{\sin\frac{\delta}{2}} - 1 = \frac{lk}{dE\,\sin\frac{\gamma}{2}},$$

that is,

$$3)\quad \sin\frac{\delta}{2} = \frac{1}{1 + \dfrac{lk}{dE\,\sin\frac{\gamma}{2}}} = \frac{\sin\frac{\gamma}{2}}{\sin\frac{\gamma}{2} + \dfrac{lk}{dE}}.$$

If we make the assumption that the increment of the arc of contact γ, caused by applying the tightening-pulley, is the same for each of the two belt pulleys CE and DF, that is, $ECE_1 = FDF_1 = \gamma'$, we can place:

$$\delta = SMS = CDF_1 + DCE_1 = \frac{\gamma}{2} - \gamma' + 180^\circ - \left(\frac{\gamma}{2} + \gamma'\right)$$
$$= 2\,(90^\circ - \gamma'),$$

hence,
$$\frac{\delta}{2} = 90^\circ - \gamma',$$
and
$$4)\quad \gamma' = 90^\circ - \frac{\delta}{2}.$$

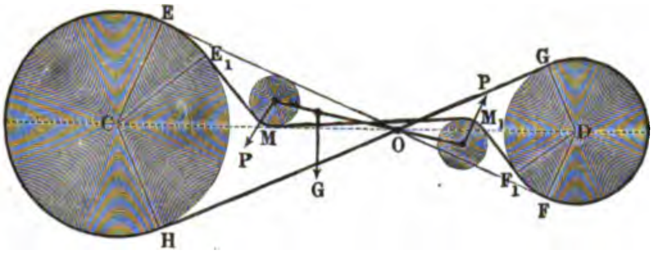
If we substitute this value γ' in the above formulas (2) for S_1 and S_2, we shall find the required belt tensions, from which we may then find the tightening force from formula (1):

$$P = 2\,S\cos\frac{\delta}{2},$$

care being taken to substitute S_1 or S_2 for S in the calculation according as the tightening-pulley presses against the tight or slack side of the belt. Since the force P is directly proportional to S, it will generally be well to let the tightening-pulley act against the slack side of the belt, because this will make P less, and thus diminish the friction on the journals of the pulley.

The crossed belt $EFGH$, Fig. 182, needs two tightening pulleys M, M_1, acting in opposite directions. Since the arcs of contact are here greater than a semi-circumference, and equal, they are a more perfect means of transmission. The two tightening-pulleys may here be fastened to the single lever MOM_1 and can be pressed against the belts by the weight G, but tightening-pulleys are seldom employed for crossed belts.

FIG. 182.



EXAMPLE. Suppose that for the belt-gear, Fig. 180, with open belt, the force to be transmitted $K = 130$ kilograms [290 lbs.], the radius of one wheel $b = 0.75$ meter [29.53 ins.], the radius of the other $a = 0.125$ meter [4.92 ins.], and the distance between the axes $d = 2.25$ meters [88.58 ins.], we shall have, as in the example of § 54, for the arc of contact of the smaller wheel:

$$\gamma = 147° \, 45', \text{ and } \sin\frac{\gamma}{2} = 0.961,$$

and for the total length of the belt,

$$l = 2d \sin\frac{\gamma}{2} + \gamma a + (2\pi - \gamma)\, b,$$

$$= 4.5 \times 0.961 + 2.579 \times 0.125 + 3.706 \times 0.75,$$

$$= 7.424 \text{ meters [24.35 feet]},$$

hence the ratio,

$$\frac{l}{d} = \frac{7.424}{2.25} = 3.30.$$

If we also assume $\frac{\lambda}{l} = \frac{k}{E} = \frac{1}{80}$, we get

$$sin\frac{\delta}{2} = \frac{sin\frac{\gamma}{2}}{sin\frac{\gamma}{2} + \frac{l}{d}\frac{k}{E}} = \frac{0.961}{0.961 + 0.0412} = 0.959;$$

hence $\frac{\delta}{2} = 73° 33'$, that is, the arc of contact of the belt on the tightening-pulley is

$$\delta = 147° 6',$$

and the increment of the arc of contact on the belt pulleys which is caused by tightening the belt is

$$\gamma' = 90° - \frac{\delta}{2} = 16° 27'.$$

Now, since

$$\gamma + \gamma' = 147° 45' + 16° 27' = 164° 12',$$

also

$$\varphi(\gamma + \gamma') = 0.28\,\frac{164.2}{180}\,\pi = 0.801,$$

and

$$e^{\phi(\gamma + \gamma')} = 2.7183^{0.801} = 2.222,$$

the tension of the slack side of the belt, against which the tightening-pulley here acts, must be

$$S_2 = \frac{K}{e^{\phi(\gamma+\gamma')} - 1} = \frac{130}{2.222 - 1} = 106.4 \text{ kilograms [234.6 lbs.]},$$

and the force of the tightening-pulley:

$$P = 2\,S_2\,cos\frac{\delta}{2} = 212.8 \times 0.283 = 60 \text{ kilograms [132.3 lbs.]}.$$

The use of the tightening-pulley is not always an advantage, for the friction of the shafts and the resistance due to the stiffness of the belt when it is bent about the pulley, add new hurtful resistances to the gearing. Although this apparatus prevents the belt from being excessively tightened, and thus reduces the friction of the shafts which would otherwise occur, this advantage is partly neutralized by the resistances of the tightening-pulley.

§ 57. **The Guiding of Belts.**—If a belt *A*, Fig. 183, is guided to a smooth cylindrical pulley *B C* in a direction normal to the axis, *i.e.*, in a direction which coincides with the middle plane of the pulley, it will remain on the surface of the pulley during its rotation without needing special means for guiding it. But if a belt, Fig. 184, approaches its pulley in a direction inclined at a certain angle α to the middle plane of the pulley, the belt will no longer stay upon the pulley, but will fall to the side of it, because the inclination α tends to make the belt describe a helical path having the pitch-angle α on the face of the pulley. A very small angle

FIG. 183.

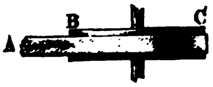


FIG. 184.

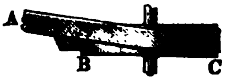
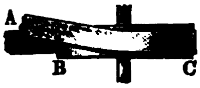


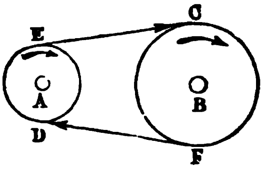
FIG. 185.



α suffices to cause the belt to fall off, and consequently, in hanging a pulley, special care must be taken that the belt is exactly guided into the plane of the pulley.

On the other hand, in leading the belt off the pulley, the receding portion *A* of the belt, Fig. 185, may, experience shows, make quite a large angle (up to 25° and more) with the middle plane *B C* without danger of the belt falling off. The reason of this is that the force which must be exerted upon the receding portion of the belt to pull it off must be sufficiently large to overcome the considerable friction existing between pulley and belt, while with the advancing portion of the belt a suitable direction only need be given to have it leave the pulley with the helical motion already described. This action may be readily seen on any belt communicating motion between two pulleys. For, if a lateral pressure be exerted upon the belt while it is in motion, a slight force will be sufficient to push it off if applied near the point *C* or *D*, Fig. 186, where the belt runs on to the pulleys, while it will require considerable exertion at the points *E* and *F*, at which the belt is delivered, to pull it from the pulleys. It follows from this, that in cases where engagement and disengagement are effected by shifting the belt alternately on to a fixed and loose pulley, the shifting lever must be applied to the belt near the points *C* and *D*.

FIG. 186..

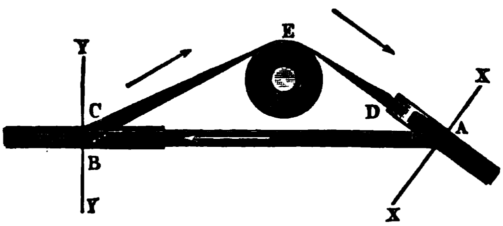


These explanations make evident the conditions that must be fulfilled if the belt is to run properly, that is, without special means being employed to keep it from running off.

Since it is here only a question of *guiding* the belt to its pulley in the middle plane of the latter, we have for the arrangement of every pair of pulleys the fundamental condition that *the point at which the belt is delivered from each pulley must be in the middle plane of the other pulley.* This condition is perfectly general, applying not only to two, but to any number of belt or guide pulleys. Moreover, this condition that the belt shall *advance properly* to its pulley is the only one to be fulfilled, provided the belt is not delivered at too oblique an angle, say not more than 25° or 30°, with the planes of the pulleys.

If the above condition is fulfilled, that the point at which each belt is delivered must be in the plane of the other pulley, the belt will run properly for the direction of motion assumed, but it does not necessarily follow that it will suffice for an opposite direction of rotation of the pulleys. If the latter motion takes place, the points at which the belts before ran on to the pulleys will now become the points at which the belt is delivered from the pulleys, consequently it is plain that the above fundamental condition must also be fulfilled for this opposite direction of rotation. For example, if three pulleys are connected by a belt in the manner shown in Fig. 187, the belt will run properly in the direction indicated by the arrows, for the point of delivery *A* lies in the plane of the pulley *B*, the point of delivery *C* in the plane of the pulley *E*, and the point of delivery *E* in the plane of the pulley *A D*. But this arrangement is not suitable for the opposite direction of motion, for then the point *B* will become the point of delivery of the pulley *B C*, and should lie in the plane *A D*, which is not the case, neither does the point *E* lie in the plane of the pulley *B C*. This shows that if a belt is correctly placed for one direction of rotation, it does not follow that it will run properly for the opposite

FIG. 187.



direction; but the above rule enables us to judge readily in any given case whether the arrangement is such as will allow the belt to run correctly in one or both directions.

The simplest case of belt transmission is when two shafts are directly connected by means of a belt—an open belt, Fig. 188, being employed when the two shafts have the same direction of rotation, and a crossed belt, Fig. 189, when they have opposite directions of rotation. As the shafts are parallel, the above condition will be fulfilled and the belt will run properly when the middle planes of the pulleys coincide, and it is also evident that the belt will then run as well in one direction as the other.

When a crossed belt is employed, each of the two parts between the pulleys is twisted 180°, thus enabling them to pass each other at the place where they cross. It has already been stated that when the belts are crossed the arcs of contact are greater, consequently the belts do not need to be tightened as much to produce a given amount of friction. In cases where the direction of rotation of the driven shaft is unimportant, crossing of the belt is employed to overcome a resistance which would cause an open belt to slip. It has this disadvantage, however, that the belt is unequally strained and therefore tears more readily. If l_1 is the average length of a straight portion of the belt measured from point of contact to point of contact, and b the width of the belt, the corresponding length of the helical edge caused by the twisting will be:

FIG. 188.

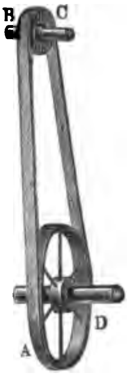
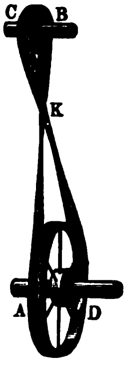


FIG. 189.



$$l_2 = \sqrt{l_1^2 + \left(\frac{\pi b}{2}\right)^2}\text{, or approximately } = l_1\left[1 + \frac{1}{8}\left(\frac{\pi b}{l_1}\right)^2\right],$$

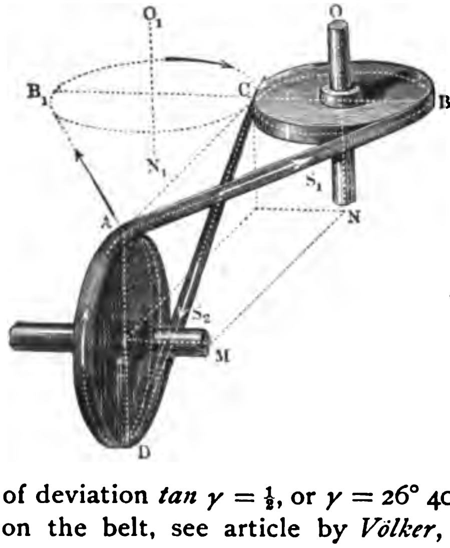
hence the elongation of each edge:

$$\lambda = l_2 - l_1 = \tfrac{1}{8}\frac{(\pi b)^2}{l_1} = 1.234\frac{b^2}{l_1},$$

from which we can easily judge of the force causing the elongation, that is, of the diminution of strength caused by the twisting.

If the two shafts are not parallel, and consequently the planes of the pulleys do not coincide, we may still satisfy the condition for proper running by suitably placing the pulleys. For this purpose the wheels, Fig. 190, must be so placed with reference to each other, and the belt so placed upon them, that the *straight line A C connecting the two points of delivery will coincide with the intersection of the two planes of the pulleys*, for then the point of delivery of each wheel will lie in the plane of the other wheel. This line $A\,C$ is moreover equal and parallel to the shortest distance $M\,N$ between the shafts. The advancing portions of the belt are tangent at the points B and D, while the receding portions are oblique to the planes of the pulleys at A and C. This oblique delivery causes a lateral pressure and a corresponding lateral friction between belt and pulley as well as in the bearings of the shafts, but the velocity ratio is not changed. To diminish this friction as much as possible, as well as to prevent the belt from running off, we must observe what has already been said concerning the amount of deviation of the receding belt. According to *Redtenbacher* the distance $A\,C$ between the pulleys must be taken equal to at least double the diameter of the larger pulley, which would give for the angle of deviation $tan\ \gamma = \frac{1}{2}$, or $\gamma = 26° 40'$. Concerning the stress on the belt, see article by *Völker*, Zeitschrift deutsch. Ing. 1860, p. 115.

FIG. 190.



We know from what has preceded that transmission of motion can take place in only one direction, as represented in the figure by arrows, for an opposite rotation would make of the points B and D points of delivery not in the planes of the other pulleys. But there is no difficulty in so arranging that transmission of motion may be possible for an opposite rotation of one of the pulleys; this is effected, as may be seen from the figure, by shifting the pulley $A\ D$ upon its shaft an amount equal to the diameter of the pulley B, the new position being shown in the figure by dotted lines B_1. In both cases the pulley A turns in the same direction, and it is clear that for an opposite rotation of this pulley A the belts may be arranged so as to transmit motion whatever the rotation of B; we have simply to shift B on its own axis through a distance equal to the diameter of A. To show this, let MM and NN, Fig. 191, be the projections of the axes upon a plane perpendicular to their shortest distance, we shall then have the four possible positions of the pulleys corresponding to the four different motions above mentioned, which can be obtained by uniting each direction of rotation of one pulley with each direction of rotation of the other. Evidently these four different arrangements are represented by $A_1 B_1$, $A_1 B_2$, $A_2 B_1$ and $A_2 B_2$, the points of delivery corresponding to these positions being projected in C_1, C_2, C_3 and C_4 respectively. If the shafts, as is here assumed, cross at right angles, a reversal of the rotation of *one* pulley only requires the shifting of the *other* pulley upon its axis. But if the shafts are not at right angles, *two* pulleys, according to Fig. 192, must be suitably shifted upon their shafts even though but one of them is to receive an opposite rotation. Here $A_1 B_1$,

FIG. 191.

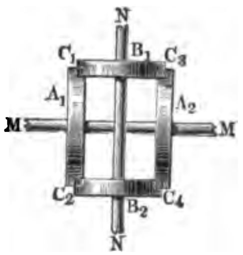
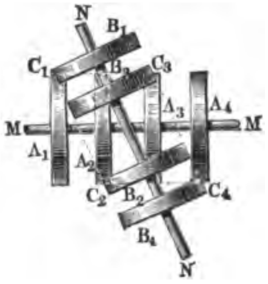


FIG. 192.

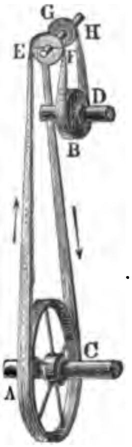


A_2B_2, A_3B_3 and A_4B_4 clearly show the relative positions of the pulleys, corresponding to the four cases of motion before mentioned.

The foregoing discussion also shows that transmission of motion by belts is possible for any direction of rotation when the shafts cross each other without intersecting, and in this respect they do not differ from parallel shafts in which the different directions of rotation are effected by open and crossed belts. To distinguish the crossed belts of parallel shafts in which each half of the belt is twisted 180° from the belts employed for two shafts which cross at right angles without intersecting, the latter are called *quarter-twist* belts, each half of the belt receiving a twist of 90°. We may also speak of one eighth twist, a twelfth twist, etc., according as the angle made by the shafts is $\frac{360^\circ}{8}$, $\frac{360^\circ}{12}$, etc.

§ **58. Guide Pulleys.**—If the condition developed above for belt transmissions cannot be satisfied, we must have recourse to *guide pulleys*. This, for example, is always the case when two shafts intersect, for then although the point of delivery of one pulley *A* can be brought into the plane of the other *B*, yet the point of delivery of the pulley *B* cannot be brought into the plane of the first pulley *A*, unless the pulleys are allowed to touch so that on each pulley the point of contact will be the point at which the belt is both received and delivered, that is unless the arc of contact on each pulley is equal to the whole periphery. Although such an arrangement is perhaps possible with a small cord, it would not be possible with a belt of any width. Guide pulleys must also be employed with parallel shafts when for any reason the two pulleys cannot be brought into the same plane. This case is represented in Fig. 193, where both guide pulleys *E* and *G* have a diameter *EF* equal to the perpendicular distance of the middle planes of *A* and *B*. Now if both pulleys *E* and *G* be placed upon a common shaft parallel to the planes of the pulleys *A* and *B*, the belt will allow of only *one* direction of rotation. For the direction of rotation indicated by the arrows, the pulley *E F* must lie in a plane which will touch *A C* at the point of delivery *A*, while the pulley *G H* must lie in a plane which is tangential to the pulley at its point of delivery *D*. For the opposite rotation, the pulleys *E F* and *G H* must be so shifted on their shafts that *G H* will lie in a plane which is tangent to *A C* at *C* and *E F* in a plane which is tangent to *B D* at *B*. Generally in such an arrangement it is only necessary to transmit motion in one direction, but if motion in both directions is required, guide pulleys must be placed upon different shafts, so that the plane of the pulley *E F* will contain the two points *A* and *B* and the pulley *G H* the points *C* and *D*. The shafts of the guide pulleys *E* and *G* will in this case only coincide when *A C* and *B D* are of equal size.

FIG. 193.



If in this case *B D* be regarded as a guide pulley, and *E F* and *G H* as driving pulleys which are to communicate motion to their respective shafts, this arrangement will permit a driving pulley *A C* to transmit motion in opposite directions to two other shafts, say counter-shafts for looms, etc. Such arrangements, frequently found in spinning-mills, etc., economize space; but considering the wear of belt and friction due to belt pressures, probably transmission by two separate belts is preferable.

Again, when the pulleys of two shafts which cross but do not intersect cannot be placed so that the line of intersection of their planes will touch both pulleys (see § 57), two guide pulleys must be employed. For this purpose suppose that from any two points *E* and *F* of the line of intersection *O Z*, Fig. 194, tangents *E A* and *E B* as well as *F D* and *F C* are drawn to their respective wheels. Now, if in each of the two planes *A E B* and *D F C*, we so place a guide pulley, *E* in one plane and *F* in the other, that they will be touched by the tangents, we shall have an arrangement which will permit rotations in both directions, for the plane of each guide pulley contains the points at which the belt is received and delivered. Consequently, these points may interchange their functions, and transmission of motion may take place in opposite directions. The two guide pulleys *E* and *F* are to be placed upon separate shafts.

FIG. 194.

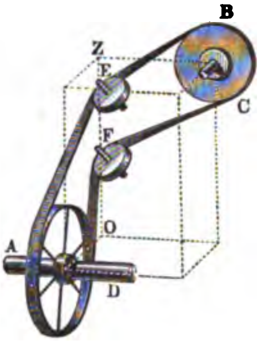
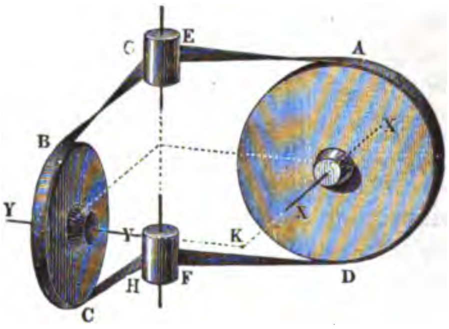


FIG. 195.



For two intersecting shafts *X* and *Y*, Fig. 195, this arrangement can be simplified when the transmission is to take place in but one direction, by placing the two guide pulleys upon one and the same shaft. To do this, suppose two planes to be passed through the points of delivery *A* and *C*, perpendicular to the line of intersection of the middle planes of the pulleys, and in these two parallel planes, suppose equal pulleys *E* and *F*, placed so that the straight line *A E* will be tangent at *E* and *C H* tangent at *F*. The common shaft *E F* will then be parallel to the line of intersection of the middle planes of the pulleys. Rotation can take place in only one direction, because the belt leaves the guide pulleys obliquely at *G* and *F*; if rotation in the opposite direction is desired, the perpendicular planes must be passed through *B* and *D*, and in these planes the pulley *E* must be placed tangent to *B G* and *F* tangent to *D F*; *i. e.*, the pulleys *E* and *F* must be shifted on their shaft.

In this last arrangement the belt is delivered obliquely only from the guide pulleys, while the portions of the belt which advance toward and recede from the belt pulleys both lie in the planes of the latter. If the belt is also allowed to leave the pulleys *A D* and *B C* obliquely, we may still further simplify this arrangement for intersecting shafts by employing but one guide pulley *E*, Fig. 196. To do this *B* is placed in such a position that its middle plane will pass through the point of delivery of *A D* and the guide pulley *E*, so that its plane will contain the point of delivery *C* of the pulley *B*, and that it will itself be touched by the middle plane of *A D*, thus satisfying the above fundamental condition that each pulley must contain the point of delivery of the preceding pulley.

Fig. 196.

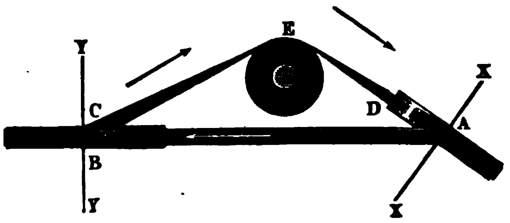
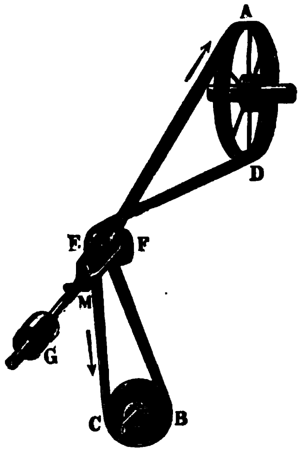


Fig. 197.



Sometimes guide pulleys are employed only to pull the belt to one side to economize space. In other cases they are employed as guide and tightening-pulleys simultaneously, as in Fig. 197, where the belt runs over two pulleys *E* and *F*, carried by a lever which turns at *M* and is pressed up against the belt by the weight *G*.*

Guide pulleys are also applied in cases where the driven pulley is carried on a movable frame or carriage and partakes of the motion of the latter, for example, in spinning machines and traveling cranes. In these cases cotton cords are generally used instead of belts, and run at extraordinarily high speeds; in traveling cranes up to 25 meters [82 ft.] per second. Fig. 198

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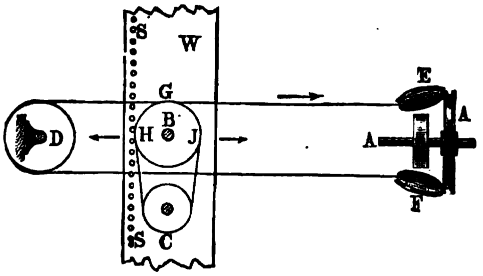
* For interesting application of guide pulleys for belt transmissions see *Reuleaux's* Constructeur.

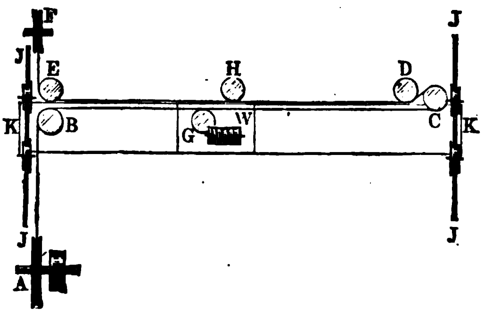
represents the rope gearing of a spinning machine (mule). Here *W* is a long carriage containing several hundred spindles and moving back and forth in a direction perpendicular to its length, while the thread is alternately twisted and wound. It is therefore a question of so guiding the driving cord from the driving pulley *A* to the pulley *B* on the carriage (which communicates motion to the spindles by suitable pulleys and cords) that the pulley *B* will receive the proper rotation in every position of the carriage. For this, one guide pulley, *C*, is placed on the carriage and another, *D*, is placed beyond the path of the carriage on a bearing attached to the foundation. Two other guide pulleys, *E* and *F*, are placed close to the driving pulley *A* to change the direction of the cord from vertical to horizontal. The cord passes from the driving pulley *A* around *E* to the pulley *B* on the carriage, which is provided with two grooves, is in contact with one groove of this pulley along the arc *G H*, then passes around *C*, is a second time in contact with the pulley *B* along the arc *J G* of the other groove, and finally gets back to the pulley *A* by passing over the fixed guide pulleys *D* and *F*. The cord is carried twice to the double-grooved pulley *B* simply in order that the arc of contact may be sufficiently large. It is clear that with this arrangement, if the direction of the straight lines between *D* and *E* coincides with the direction of motion of the carriage, the length of the cord will be the same in all positions of the carriage, and consequently motion will be transmitted from *A* to *B* and from the latter to the spindles in all positions. In this connection we must remark that the motion of the carriage itself causes a rotation of the pulley *B*, the latter rolling on the tight cord. If *w* is the path of

FIG. 198.



the carriage and b the radius of the pulley B, the latter will make $\frac{w}{2\pi b}$ revolutions for every outward motion of the carriage, and as many revolutions in the opposite direction for every inward motion of the carriage, the motion received by the pulley B from the rotation of A being increased or diminished by this amount.

FIG. 199.



The arrangement of rope gearing for a traveling crane is shown in Fig. 199. Here $K\,K$ is a bridge, movable on the rails $J\,J$. On the bridge is the hoisting gear or crab W, which can be moved in the direction $K\,K$. The motion is received from the rope pulley A, the rope being guided along the horizontal pulleys B, C, D, and E, which are all carried by the bridge $K\,K$, and after passing around the pulley F, which has an arrangement for tightening the rope, returns to A. We see that in this arrangement the pulley E, which moves the bridge along its track, receives motion from A in every position of the bridge. Since the two pulleys G and H, carried by the crab, can be brought into contact with the ropes $B\,C$ and $D\,E$ by means of pressure pulleys, as desired, it follows that for every position of the crab W on the bridge, the motion from A can be transmitted to G and H, the pulley G driving the hoisting gear for lifting the load, and H moving the crab along the bridge. What was said in the preceding example concerning the rotation of the pulleys in consequence of their motion along the rope, applies equally well to this example. Hence, if b and h represent the radii of the pulleys B and H, w_1 a motion of the bridge and w_2 a simultaneous motion of the crab on the bridge in one direction or the other, the pulley B will receive an additional rotation amounting to $\pm \frac{w_1}{2\pi b}$, and *H* a rotation amounting to $\pm \frac{w_1}{2\pi h} \pm \frac{w_2}{2\pi h}$.

The rope of such traveling cranes is usually allowed to run at the great velocity of 25 meters [82 ft.] per second, in order that it may perform considerable work, even when the force transmitted is slight.*

§ 59. **Wire Rope Gearing.**†—Wire rope was first used by *Hirn* Brothers in Alsace (1850) for transmitting power over great distances (up to 1,000 meters [3,280 ft.]); it differs from belt gearing only in employing wire rope instead of belts. The distance between the pulleys is always considerable, and must never be taken less than 15 to 20 meters [50 to 65 ft.], consequently the wire rope will sag sufficiently to prevent its being torn by its own weight. The tension on the rope caused by its own weight generates the friction on the pulleys necessary for transmission. Strictly speaking, this is also the case with horizontal belts, the nearness of the pulleys preventing the deflection from being as perceptible as in long wire ropes.

For transmitting power wire ropes are employed which are composed of strands, each containing six or more wires, and a hemp strand in the centre, Fig. 200. The diameter of the individual wires varies between 0.5 and 2 millimeters [0.02 and 0.08 in.], the diameter and transmitting capacity of the rope depending essentially upon the number of wires which it contains. On account of the stiffness of the larger sizes of wire it is advisable, when the force to be transmitted is large, to increase the number ν of the wires, but to keep their diameter δ as small as possible.

FIG. 200.

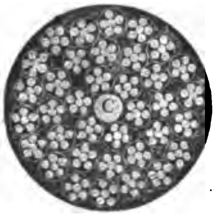
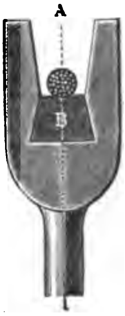


FIG. 201.



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* See article by *Lentz*, Zeitschr. deutsch. Ing., p. 289.

† Concerning wire rope gearing see *Reuleaux*, Der Constructeur.

Wire rope transmission takes place only between parallel shafts, and generally between pulleys of the same size. The wire ropes bear against the bottom of a groove, Fig. 201, on the pulley, which is lined with gutta percha or cork, and transmit the motion by means of friction from pulley to pulley just as belts do. Therefore the laws determined for belt gearing apply equally to wire rope gearing and we have (see § 53),

$$S_1 = \frac{e^{\phi\gamma} K}{e^{\phi\gamma} - 1}; \quad S_2 = \frac{K}{e^{\phi\gamma} - 1}; \quad S = \frac{S_1 + S_2}{2}.$$

If we assume the coefficient of friction between rope and pulley to be $\varphi = 0.24$ and with equal sized pulleys $\gamma = \pi$ we have

$$S_1 = \frac{e^{0.753} K}{e^{.753} - 1} = \frac{2.11\ K}{1.11} = 1.90 K,$$

$$S_2 = \frac{K}{e^{0.753} - 1} = \frac{K}{1.11} = 0.90\ K,$$

$$S = \frac{0.90 + 1.90}{2} = 1.40 K.$$

In round numbers these values become

$$S_1 = 2\ K, \qquad S_2 = K, \qquad S = 1.5\ K,$$

which we will assume, as they also give somewhat greater safety.

If we know the force K acting with the radius R at the circumference of a pulley, we shall have for the effective cross section of the wire rope

$$\frac{\pi \delta^2 \nu}{4} = \frac{S_1}{k_1} = \frac{2\ K}{k_1}$$

where ν represents the number, δ the diameter and k_1 the working tension of the wires. To determine this permissible tension k_1 correctly, we must remember that the wires are here subject to two different stresses, the tension k_1 caused by S_1 and the bending stress k_2 which results from bending the wires around the pulley having the radius R. This bending stress k_2 may be easily determined from the modulus of elasticity E and the elongation λ of the wire while bending. If we assume that the length of the middle layer of fibres in a wire remains unchanged while the wire bends about the pulley whose radius is R, and that the inner layers shorten and the outer ones lengthen, then will the length of the outermost layer whose radius is $R + \frac{\delta}{2}$ be $\pi\left(R + \frac{\delta}{2}\right)$, while its original length will be πR, the same as that of the invariable layer. The elongation per unit of length will therefore be

$$\frac{\pi\left(R + \frac{\delta}{2}\right) - \pi R}{\pi R} = \frac{\delta}{2R},$$

and the stress corresponding to this rate of elongation will be (see Vol. I, § 204),

$$\frac{k_2}{E} = \frac{\delta}{2R}, \text{ or } k_2 = \frac{E\delta}{2R}.$$

Assuming the modulus of elasticity of iron to be 20,000 kilograms per square millimeter [28,500,000 lbs. per square inch], we get for the bending stress

$$k_2 = \frac{10{,}000\,\delta}{R}, \quad \left[k_2 = \frac{14{,}250{,}000\,\delta}{R}\right].$$

Now if k represents the total permissible stress in the wires, and if we assume for the excellent material of which small wires must be made $k = 18$ kilograms per square millimeter [25,600 lbs. per square inch], we have

$$k = k_1 + k_2 = 18, \quad [25{,}600],$$

therefore

$$k_1 = 18 - k_2 = 18 - \frac{10{,}000\,\delta}{R}\left[k_1 = 25{,}600 - \frac{14{,}250{,}000\,\delta}{R}\right],$$

which must be introduced into the above formula for the cross section of the wire rope.

If according to *Reuleaux* we assume $\frac{k_1}{k_2} = \frac{1}{2}$ as the ratio for which, other things being equal, the radius of the pulley becomes a minimum, we shall get $k_1 = 6$ kilograms per square millimeter [8,500 lbs. per square inch], and

$$\frac{\pi \delta^2 \nu}{4} = \frac{2K}{6}, \text{ or } \nu\delta^2 = 0.425\,K\;[\nu\delta^2 = 0.0003\,K].$$

If instead of the resistance K at the circumference, we have given the number of horse powers N to be transmitted and the velocity of the rope v in meters per second [feet per second], we shall have

$$\nu\delta^2 = 0.425\,K = 0.425\,\frac{75\,N}{v} = 32\,\frac{N}{v},$$

$$\left[\nu\,\delta^2 = 0.0003\;K = 0.0003\;\frac{550\,N}{v} = 0.165\,\frac{N}{v}\right].$$

These formulas, which result from assuming $k_1 = \frac{1}{2}k_2 = 6$ kilograms [8,500 lbs. per square inch], of course suppose that the bending stress in the wires will not exceed $k_2 =$ 12 kilograms [17,000 lbs. per square inch], consequently the radius R of the pulleys must at least have the value

$$R = \frac{10{,}000\,\delta}{k_2} = \frac{10{,}000\,\delta}{12} = 833\,\delta,$$

$$\left[R = \frac{14{,}250{,}000\,\delta}{k_2} = \frac{14.250{,}000\,\delta}{17{,}000} = 838\,\delta\right].$$

When a different radius R is employed, we must find

$$k_2 = \frac{10{,}000\,\delta}{R}.\;\left[k_2 = \frac{14{,}250{,}000\,\delta}{R}\right],$$

and then substitute $18 - k_2 [25{,}600 - k_2]$ for k_1, in the formula for $\nu \delta^2$.

Since the tensions on the rope depend on its sag or deflection, it is important to determine the latter in order to give to the rope the above calculated tensions, which prevent it from slipping yet are not excessive. Let d be the distance between two shafts C and D, which have the same height, Fig. 202, and let $h = EK$ be the deflection of the rope, the curve of the latter being assumed to be approximately a parabola, also let $EAT = \alpha$ be the inclination of the ends A and B of the rope to the horizon, we shall then have

$$\sin \alpha = \frac{ET}{AT} = \frac{2h}{\sqrt{\left(\frac{d}{2}\right)^2 + (2h)^2}} = \frac{4h}{\sqrt{d^2 + (4h)^2}};$$

and hence, when V is the vertical component of the tension S of the rope, and $G = 2V$, the weight of the portion AKB of the rope, we shall have

$$S = \frac{V}{\sin \alpha} = \frac{V\sqrt{d^2 + (4h)^2}}{4h} = \frac{G}{8h}\sqrt{d^2 + (4h)^2}.$$

Approximately this becomes

$$S \doteq \frac{G}{8h} d;$$

hence the deflection of the rope at the centre is

$$h = \frac{Gd}{8S}.$$

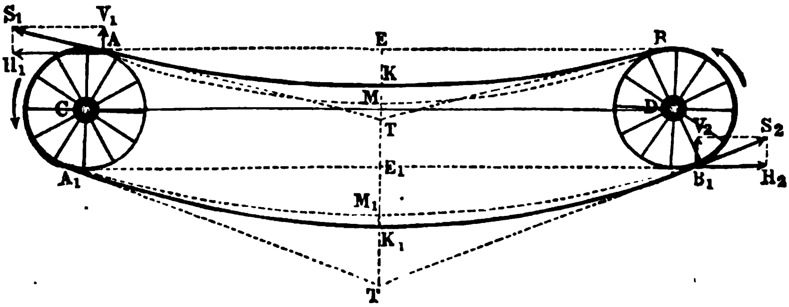
From this equation we may obtain the deflection of the driving or driven portions of the rope, and also the deflection of the rope when it is at rest, by substituting for S

$$S_1 = 2K, \quad S_2 = K, \quad S_0 = 1.5\,K,$$

and therefore get

$$h_1 = \frac{G\,d}{16\,K};\quad h_2 = \frac{G\,d}{8\,K};\quad \text{and } h_0 = \frac{G\,d}{12\,K}.$$

FIG. 202.



The weight of the wire rope per running meter, when the specific gravity of iron wire is 7.7, and the shortening of the wire by twisting is taken at 10 per cent., is

$$q = \frac{10}{9}\;\frac{\pi\,\delta_2\,\nu \times 7.7}{4 \times 1000} = 0.00672\;\nu\,\delta^2$$

$$\left[q = \frac{10}{9}\;\frac{\pi\,\delta^2\,\nu \times 7.7 \times 12}{4 \times 27.7} = 2.91\;\nu\,\delta^2 \text{ lbs. per running foot}\right],$$

consequently under the assumption ($k_1 = 6$ kilograms [$k_1 =$ 8,500 lbs.]) which gave $\nu\,\delta^2 = 0.425\,K$ [$\nu\,\delta^2 = 0.0003\,K$] we have

$$G = 2\,q\sqrt{\left(\frac{d}{2}\right)^2 + (2\,h)^2} = 0.00286\,K\sqrt{d^2 + (4\,h)^2}$$

$$\left[G = 2\,q\sqrt{\left(\frac{d}{2}\right)^2 + (2\,h)^2} = 0.000873\,K\sqrt{d^2 + (4\,h)^2}\right.$$

when d and h are expressed in feet], or approximately:

$$G = 0.00286\,K\,d,$$

$$[G = 0.000873\,K\,d].$$

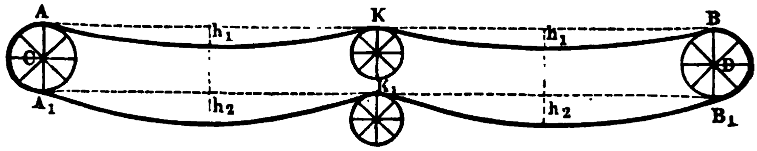
Substituting we get

$$h_1 = \frac{0.00286}{16} d^2 = 0.000179 d^2; \quad h_2 = 0.000358 d^2;$$

$$h_0 = 0.000238 d^2.$$

$$[h_1 = 0.0000545 d^2; \quad h_2 = 0.000109 d^2; \quad h_0 = 0.0000726 d^2].$$

FIG. 203.



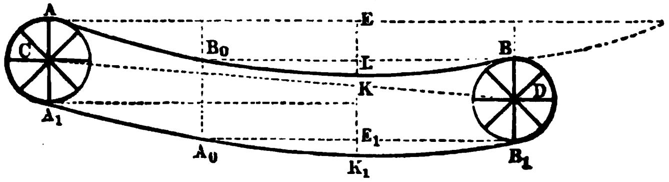
Hence, when the distance between the shafts equals 100 meters [328 ft.], the deflections are as follows: $h_1 = 1.79$ meters [5.87 ft.]; $h_2 = 3.58$ meters [11.74 ft.], and $h_0 = 2.38$ meters [7.80 ft.].

In order that the deflections may not become too great, guide or supporting pulleys K K_1 may be placed, as in Fig. 203, from 60 to 120 meters [about 200 to 400 ft.] apart. But instead of these it is better to employ a pulley with two grooves, which receives motion from A by means of one rope and communicates motion to B by means of another rope running in the second groove. This arrangement of rope gearing with intermediate stations has the advantage that short ropes may be used instead of long ones, thus making it more convenient to hang the ropes and to repair them in case of breakage.

In cases where the height under the ropes is limited, the deflection or sag may be diminished by *tightening* the rope, *i. e.*, giving a greater tension to it than would be necessary to prevent sliding. As a consequence, the rope must be of larger size, and the friction on the journals is greater, but these circumstances are of little account, for the diameters of the ropes are generally small and the resistance of friction is comparatively small. (See *Reuleaux*, on tightening of wire ropes.)

If the axes of the pulleys C and D do not lie in the same horizontal plane, we have the case shown in Fig. 204, where the curve of the rope $A\,K\,B$ is made of two unequal parts, $A\,K$ and $K\,B$, whose deflections $K\,E$ and $K\,L$ are represented by h' and h'', and their horizontal projections $A\,E$ and $B\,L$ by d' and d''. The given difference in height $E\,L$ of the two pulleys is $h = h' - h''$, and their horizontal distance $d = d' + d''$. Now, according to the above we may place $h' = \mu d'^2$ and $h'' = \mu d''^2$, where μ is a coefficient depending on the tension of the rope (for example, if for the driving portion of the rope we have $\mu = 4 \times 0.000179$, for the driven portion we have $\mu = 4 \times 0.000358$, etc.); hence we also have

FIG. 204.



$$h = \mu(d'^2 - d''^2) = \mu(d' - d'')(d' + d'') = \mu d(d' - d''),$$

and therefore

$$d' - d'' = \frac{h}{\mu d},$$

consequently the horizontal distances are

$$d' = \frac{h + \mu d^2}{2\mu d}, \quad d'' = \frac{\mu d^2 - h}{2\mu d},$$

and the height of the arcs

$$h' = \mu\left(\frac{h + \mu d^2}{2\mu d}\right)^2, \text{ and } h'' = \mu\left(\frac{\mu d^2 - h}{2\mu d}\right)^2$$

REMARK. Concerning wire rope transmission, see *Reuleaux's* Constructeur, also Le Vignole de Mecaniciens, par Armengaud ainé, as well as different articles in technical journals—among others, Zeitschrift deutsch. Ingenieure, 1862, p. 212, 1866, pp. 480, 731, and 1870, p. 14; Dingler's Polyt. Journal, Vol. 172, No. 3.

§ 60. Resistances of Belt and Rope Gearing.—In the preceding investigations we neglected the influence of the hurtful resistances which accompany belt and rope transmissions. Since in most cases these resistances are not insignificant, we will now discuss them in order to ascertain what corrections should be applied to the foregoing results. The principal resistance of every belt transmission is the friction of the journals of the pulleys, that due to stiffness of the belt when bent around the pulleys being generally smaller. In very quickly running belts and ropes the resistance of the air also reaches a measurable amount, but cannot be accurately calculated, therefore, if necessary to take it into account, its amount must be estimated approximately.

The friction on the journal is directly dependent on the pressure with which the shaft is pressed against its bearing by the pull of the belt or rope. This pressure consists not only of the weight of the pulley and its shaft, but also of the tensions of the driving and driven portions of the belt. Since the directions of these two tensions, especially when the pulleys differ but little in diameter and are far apart, may be assumed as parallel, we have for the friction of the journals:

$$\varphi_1 (S_1 + S_2) r,$$

r representing the radius of the journal and φ_1 the coefficient of friction. This expression when reduced to the circumference of a pulley whose radius is R, becomes:

$$F = \varphi_1 (S_1 + S_2) \frac{r}{R}.$$

The resistance due to the stiffness can be assumed, as in ropes, to be directly proportional to the tension S and the square of the thickness δ, and inversely proportional to the radius of curvature R, so that the coefficient of stiffness can be taken equal to:

$$S = \sigma \frac{\delta^2}{R}.$$

If we now designate by $f = \varphi_1 \frac{r}{R}$ the coefficient of journal friction reduced to circumference of pulley, we shall have for the sum of the two resistances

$$u = f + s = \varphi_1 \frac{r}{R} + \sigma \frac{\delta^2}{R}.$$

If now S_1 and S_2 again represent the tension of the driving and driven portions of the belt, we shall have the tension S_1 resisted not only by the transmitted force K at the circumference and the tension S_2, but also by the resistances $u\ S_1$ and $u\ S_2$ reduced to the circumference. We therefore have

$$S_1 = S_2 + K + u\,S_1 + u\,S_2,$$

or

$$S_1 (1 - u) = S_2 (1 + u) + K.$$

If we assume the normal condition to exist, in which S_2 is not unnecessarily large but possesses the value needed to transmit the force K, we may from what has preceded place

$$S_1 = S_2 e^{\phi\gamma},$$

and then get

$$S_2 e^{\phi\gamma} (1 - u) = S_2 (1 + u) + K,$$

or

$$S_2 = \frac{K}{e^{\phi\gamma}(1 - u) - (1 + u)},$$

and

$$S_1 = \frac{e^{\phi\gamma} K}{e^{\phi\gamma}(1 - u) - (1 + u)}.$$

The sum of all the resistances due to friction and stiffness on *one* pulley reduced to its circumference, is therefore

$$W = (S_1 + S_2)\,u = S_2 (e^{\phi\gamma} + 1)\,u,$$

consequently for both pulleys when they are equal:

$$2 S_2 (e^{\phi\gamma} + 1)\, u .$$

The ratio of these resistances to the transmitted or effective force K, and therefore the ratio of the corresponding quantities of work, may be obtained from

$$\frac{W}{K} = \frac{2(e^{\phi\gamma} + 1)u}{e^{\phi\gamma}(1 - u) - (1 + u)} .$$

If we assume for the coefficient of friction $\varphi_1 = 0.08$, we shall have

$\frac{r}{R} =$	0.20	0.15	0.10	0.08	0.05
$f = 0.08 \frac{r}{R}$,	0.016	0.012	0.008	0.006	0.004

Moreover, if we assume the stiffness of a belt to be equal to that of an equally thick rope whose diameter $\delta = 5$ millimeters [0.2 in.], we shall have, according to *Eytelwein* (Vol. I., § 197), for every winding and unwinding of the belt,

$$s = 0.0186 \frac{\delta^2}{2R} = 0.009 \frac{\delta^2}{R},$$

when δ and R are both expressed in millimeters. This gives the following table:

$R =$	100	200	300	500	800	1,000
$s =$	0.0023	0.0012	0.0008	0.0004	0.0003	0.0002

[$s = 0.23 \frac{\delta^2}{R}$, when δ and R are expressed in inches also the following table:

$R =$	4	8	12	20	32	40
$s =$	0.0023	0.0012	0.0008	0.0004	0.0003	0.0002]

From this we get, for example, when $R = 300$ millimeters [12 ins.] and $r = 0.1\ R = 30$ millimeters [1.2 ins.]:

$$u = f + s = 0.0088 = \text{about } 0.009.$$

If we also assume the coefficient of friction of the belt on iron pulleys to be $\varphi = 0.28$ and the arc of contact equal to a semi-circumference, that is $e^{\phi \gamma} = 2.41$ (see § 54), we shall get for the average value of $u = 0.009$:

$$\frac{W}{K} = \frac{2(e^{\phi \gamma} + 1)u}{e^{\phi \gamma}(1 - u) - (1 + u)} = \frac{0.0614}{1.379} = 0.045.$$

Therefore to overcome the resistances of this belt gearing will require an amount of work equal to 4.5 per cent. of that transmitted.

From the foregoing developments we may easily judge of the influence which the velocity of the belt or of the driven shaft exerts on the resistances. For if we assume in a particular case that the velocity of the belt is equal to v when N meter-kilograms per second [N foot-pounds per second] are transmitted, we shall have $K = \frac{N}{v}$, and since the resistance W at the circumference of the pulley is directly proportional to K, we have

$$W = \alpha K = \alpha \frac{N}{v},$$

where

$$\alpha = \frac{2(e^{\phi \gamma} + 1)u}{e^{\phi \gamma}(1 - u) - (1 + u)}.$$

The work per second consumed by this resistance is

$$Wv = \alpha \frac{N}{v} v = \alpha N,$$

and we see from this that the loss of work from resistances would be independent of the velocity v if α were independent of v; but α is not independent, for as the velocity v increases, the diameter of the shaft diminishes, because it is subject to a smaller twisting moment, and thus reduces the moment of the friction.

In general the value of u will vary between the limits

$$u_1 = 0.016 + 0.0023 = 0.0183$$

and

$$u_2 = 0.004 + 0.0002 = 0.0042,$$

the extreme values of f and s given in the tables being assumed. In determining the resistances we may therefore make use of the following table, when $e^{\phi\gamma} = e^{0.28\pi} = 2.41$:

$U =$	0.020	0.015	0.010	0.006	0.004
$\frac{W}{K} =$	0.102	0.076	0.049	0.029	0.020

It should be remembered that these values are applicable only so long as the tension on the belt is *not unnecessarily large*, *i.e.*, so long as the relations between S_1, S_2 and K can be expressed by the formulas of § 53, one of which is

$$K = S_2 (e^{\phi\gamma} - 1).$$

When this is not the case, when, for example, the driving shaft has to overcome a smaller resistance K', while the belt has a tension S corresponding to a much greater resistance K, the tensions S_1 and S_2 for overcoming the small resistance K' will no longer be given by the relation

$$S_1 = S_2 e^{\phi\gamma},$$

but must be found from

$$S_1 - S_2 = K', \quad \text{and} \quad S_1 + S_2 = 2S.$$

It is evident in this case that because of the dependence of the resistances of friction and stiffness on S, the ratio of these resistances to the smaller force K' will be much greater than their ratio to the larger force K, and therefore these resistances will consume a much higher per cent. of the useful work actually transmitted; the per cent. being greater the smaller the ratio of the transmitted force to the greatest force that could be transmitted with the existing tension S. We see from this *the disadvantage of making the belts too tight*, and also that *when the useful resistances vary considerably belt gearing is not very economical.*

The conclusions arrived at for belt gearing apply equally to wire ropes; we can, however, neglect the resistance due to stiffness, because the small diameter of the wires and the large diameter of the pulleys make this resistance extremely small. The friction of the journal is also less for wire ropes, for the ratio $\frac{r}{R}$ is always much smaller than in belt gearing. If in wire ropes we assume the average value $\frac{r}{R} = 0.03$, we shall find:

$$f = 0.08 \frac{r}{R} = 0.0024,$$

and by neglecting s and taking $e^{\phi\gamma} = e^{0.24\pi} = 2.11$ we can find the loss of work from

$$\frac{W}{K} = \frac{2\,(2.11 + 1)\,0.0024}{2.11 \times 0.9976 - 1.0024} = 0.014,$$

that is, the loss will be only 1.4 per cent. of the useful work transmitted. This agrees with experience, for the hurtful resistances in wire rope transmissions have everywhere been found very small. It seems that when wire ropes run with great velocity, a greater loss is occasioned by the resistance of the air, at least this agrees with a statement of *Guilleaume*,* according to which the loss is proportional to the length of rope, amounting to ⅓ of one per cent. of the useful work transmitted for every 100 feet [31 meters]. For the wire rope transmission at *Oberursal*, there was a loss of 8 horse powers, 100 horse powers being transmitted by means of 8 ropes each 125 meters [820 feet] long.

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* Zeitschrift deutsch. Ingenieure, 1870, p. 35.

The relations just determined apply only to driving and driven pulleys, and not to the guide pulleys in belt gearing, nor to those at the intermediate stations of wire rope transmissions. Now if in guide pulleys we represent by S_2 the tension of the portion of the rope advancing toward the pulley, Fig. 205, and by γ the arc of contact, we shall have the tension S_1 of the receding portion of the rope increased by the resistances:

$$S_1 = S_2 \frac{1 + s + f \sin \frac{\gamma}{2}}{1 - s - f \sin \frac{\gamma}{2}}.$$

At an intermediate station of a wire rope transmission, Fig. 206, the friction of the journals is principally due to the vertical pressure, for the nearly equal horizontal tensions of the ropes on each side neutralize each other. Therefore if G represents the weight of the rope between two stations, the friction of the journals at the intermediate station will equal $\varphi_1 G \frac{r}{R}$, when reduced to the circumference of the pulley.

FIG. 205.

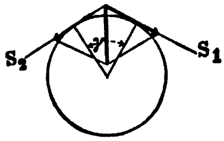
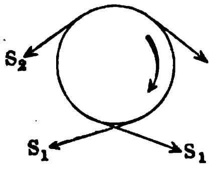


FIG. 206.



REMARK. The loss of work occasioned by the stiffness of leather belts is determined by *Autenheimer** from the work necessary to bend a prismatic bar, the tension of whose fibres per unit of area is k, and whose modulus of elasticity is E. The work A required to bend such a bar of a length l about a circle whose radius is R, is expressed by

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* See Schweiz. polyt. Zeitschrift, 1861, No. 5, or Ztsch. deutsch. Ingenieure, 1861, p. 302.

$$A = \frac{E}{24}\left(1 + \frac{k}{E}\right)^2 \frac{lb\delta^3}{R^2},$$

where b is the width and δ the thickness. For the belt in which, according to § 55,

$$\frac{k}{E} = \frac{0.2}{15} \text{ to } \frac{0.2}{20} = 0.013 \text{ to } 0.01,$$

we can neglect $\left(\frac{k}{E}\right)$ in comparison with 1, therefore the loss of work at one pulley is

$$A = \frac{E}{24}\,\frac{lb\delta^2}{R^2}.$$

Now if the belt transmits a force p for every unit of area of its cross section, so that $K = b\,\delta\,p$, and $Kl = lb\delta p$ equal to the useful work transmitted through the distance l, we shall have the ratio:

$$\frac{A}{Kl} = \frac{A}{lb\delta p} = \frac{E}{24p}\,\frac{\delta^2}{R^2}$$

Now, since

$$K = S_1\,\frac{e^{\phi\gamma} - 1}{e^{\phi\gamma}},$$

we have

$$b\,\delta\,p = b\,\delta\,k\,\frac{e^{\phi\gamma} - 1}{e^{\phi\gamma}},$$

or

$$p = k\,\frac{e^{\phi\gamma} - 1}{e^{\phi\gamma}},$$

and for an average value when $e^{\phi\gamma} = e^{0.28\pi} = 2.41$:

$p = \frac{1.41}{2.41}\,k = 0.58 \times 0.2 = 0.116$ kilograms [$0.58 \times 285 = 165$ lbs. per sq. in]

This value for p and $E = 20$ [$E = 28,400$] give,

$$\frac{A}{Kl} = \frac{20}{24 \times 0.116}\,\frac{25}{R^2} = \frac{180}{R^2}\left[\frac{A}{Kl} = \frac{0.288}{R^2}\right].$$

For example, if we assume $R = 200$ millimeters [7.87 ins.], we have

$$\frac{A}{Kl} = \frac{180}{200 \times 200} = 0.0045;$$

that is, the stiffness of the belt causes a loss of work when the belt is bent, which is equal to 0.45 of one per cent. of the useful work Kl; this, however, is under the supposition that the straightening of the belt requires no additional work. If we suppose the second pulley to be of the same size as the first, then the total loss of work arising from the stiffness of the belt will be twice that given above, or 0.9 of one per cent., in round numbers 1 per cent. of the useful work transmitted.

§ **61. Loss due to Creeping of Belt.**—In consequence of the different tensions S_1 and S_2 of the driving and driven portions of the belt, there arises a loss due to the so-called creeping of the belt on the pulleys, the result of which is that the circumference of the driven pulley runs a little slower than that of the driving one (the circumferences always being measured at the middle of the belt, the corresponding radius being $R + \frac{\delta}{2}$). To determine this effect, let us take any piece of belt whose length is l before it is stretched. This piece, when it is part of the driving (tight) side of the belt, which is subject to the tension S_1, will be stretched by the specific force $k_1 = \frac{S_1}{b\delta}$. When this piece is part of the driven (slack) side, it will be subject to a stress per unit of section equal to $k_2 = \frac{S_2}{b\delta}$. By these tensions k_1 and k_2, the piece under consideration will be (Vol. I., § 204) stretched

$$l\frac{k_1}{E} \text{ and } l\frac{k_2}{E} \text{ respectively,}$$

so that the lengths of the piece while in the driving and driven portions of the belt will be respectively,

$$l\left(1 + \frac{k_1}{E}\right) \text{ and } l\left(1 + \frac{k_2}{E}\right).$$

These lengths also give the velocities of the circumferences of the driving and driven pulleys, for it is evident that every piece of belt of the length $l\left(1 + \frac{k_1}{E}\right)$ which the driving pulley draws to itself is afterwards given off by it as a shorter length $l\left(1+\frac{k_2}{E}\right)$, and that this latter length must also be the circumferential motion of the driven pulley. The loss of motion which is thus caused bears to the motion of the driving pulley the following ratio:

$$\frac{l\left(1+\frac{k_1}{E}\right)-l\left(1+\frac{k_2}{E}\right)}{l\left(1+\frac{k_1}{E}\right)}=\frac{1-\frac{k_2}{k_1}}{1+\frac{E}{k_1}}.$$

Now we have

$$\frac{k_2}{k_1}=\frac{S_2}{S_1}=\frac{1}{e^{\phi\gamma}},$$

whose average value is

$$=\frac{1}{e^{0.28\pi}}=\frac{1}{2.41}=0.415.$$

From this value, and from $E = 20$, $k_1 = 0.2$ [$E = 28{,}400$ and $k_1 = 275$ lbs. per sq. in.], we get:

$$\frac{1-\frac{k_2}{k_1}}{1+\frac{E}{k_1}}=\frac{1-0.415}{1+100}=0.0058.$$

The loss due to slipping or creeping is therefore, on an average, about ½ of one per cent.

In wire ropes the loss due to creeping of rope is insignificant, because the material stretches but little; when

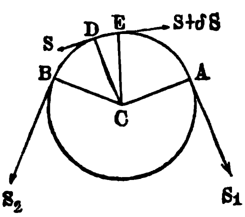
$E = 20{,}000$ [28,400,000 lbs. per sq. in.],
$k_1 = 6$ [8,500 lbs. per sq. in.], and
$k_2 = \frac{1}{2} k_1$,

the loss due to creeping is:

$$\frac{1 - \frac{1}{2}}{1 + \frac{20,000}{6}} = 0.00016.$$

Influence of Centrifugal Force.—When belts and ropes run at high velocities, the centrifugal force exerts considerable influence on them by diminishing the pressure with which the belt is pressed against the circumference of the pulley, thus reducing the friction between belt and pulley. When, therefore, a belt has to transmit a given circumferential force K, it must be tightened more the greater the centrifugal force. To ascertain the influence more exactly, let us suppose $A\,C\,B$, Fig. 207, to be the arc of contact of a belt on a pulley, and that S is the tension of the belt at any point D. The tension at the point E, whose angular distance from D is $\delta\,\gamma$, will then be expressed by $S + \delta\,S$, and if we let F represent the friction on the element $D\,E$, we shall have for the limit at which sliding will take place $\delta\,S = F$. But the friction is determined by the normal pressure N, which is the resultant of the three forces S, $S + \delta\,S$, and centrifugal force, which act on the piece of belt $D\,E$. The resultant of the two tensions S and $S + \delta\,S$ is

FIG. 207.



$$S \sin\frac{\delta\gamma}{2} + (S + \delta\,S)\sin\frac{\delta\gamma}{2},$$

which may be taken equal to $S\,\delta\,\gamma$ on account of the smallness of $\delta\,\gamma$ and $\delta\,S$. The centrifugal force of the piece of belt is equal and opposite to the resultant of the two tensions, and we therefore have:

$$q R \delta\,\gamma \frac{v^2}{R g} = q\,\delta\,\gamma\,\frac{v^2}{g},$$

q representing the weight of a unit of length of the belt, v its velocity, and R the radius of the pulley measured to the middle of the belt.

The friction between pulley and belt for the element of the arc is therefore;

$$\delta S = F = \varphi\left(S\,\delta\gamma - q\,\delta\gamma\,\frac{v^2}{g}\right),$$

and hence we have

$$\varphi\,\delta\gamma = \frac{\delta S}{S - q\frac{v^2}{g}}.$$

This gives, for the limits γ and 0, or S_1 and S_2, the expression:

$$\varphi\int_0^\gamma \delta\gamma = \int_{S_2}^{S_1} \frac{\delta S}{S - q\frac{v^2}{g}},$$

that is

$$\varphi\gamma = hyp\ log\ \frac{S_1 - q\frac{v^2}{g}}{S_2 - q\frac{v^2}{g}},$$

or

$$S_1 = S_2 e^{\varphi\gamma} - q\frac{v^2}{g}(e^{\varphi\gamma} - 1),$$

hence

$$K = S_1 - S_2 = \left(S_2 - q\frac{v^2}{g}\right)(e^{\varphi\gamma} - 1),$$

also

$$S_2 = \frac{K}{e^{\varphi\gamma} - 1} + q\frac{v^2}{g},$$

and

$$S = \frac{e^{\varphi\gamma} K}{e^{\varphi\gamma} - 1} + q\frac{v^2}{g}.$$

The force K to be transmitted therefore diminishes when the centrifugal force increases, and varies directly as the difference between the tension S_2 and the measure of the centrifugal force $q\frac{v^2}{g}$. If we substitute in the formula found above

$$S_2 = q\frac{v^2}{g}, \text{ we shall get } K = 0, \text{ and } S_1 = S_2.$$

This gives us the limit of belt velocity at which, supposing a permissible value to be assumed for S_2 or S_1, transfer of force is no longer possible. If we let f represent the cross section of a belt or rope, k the greatest permissible pressure per unit of cross section and ε the heaviness of the material per unit of volume, we shall have for the above limit $S_2 = S_1 = fk$, and $q = f\varepsilon$, and the greatest possible velocity v may be determined from

$$fk = f\varepsilon\frac{v^2}{g},$$

when the same unit of weight and linear unit are employed throughout. But when f is in square millimeters, k in kilograms per square millimeter, ε = milligrams per cubic millimeter, g = acceleration of gravity, and v = velocity in millimeters per second, we have

$$fk = \frac{f\varepsilon}{1{,}000{,}000}\,\frac{v^2}{g},$$

or

$$v = \sqrt{\frac{kg}{\varepsilon}\,1{,}000{,}000} = 1{,}000\sqrt{\frac{kg}{\varepsilon}} \text{ millimeters} = \sqrt{\frac{kg}{\varepsilon}} \text{ meters.}$$

[when f is in square inches, k in pounds per square inch, ε in pounds per cubic inch, g and v in feet per second, we have

$$v = \sqrt{\frac{gk}{12\,\varepsilon}} \text{ feet].}$$

Again assuming for leather belts $k = 0.2$ [285 lbs. per square inch] and $\varepsilon = 0.9$ [0.0325 lbs. per cubic inch], we have

$$v = \sqrt{\frac{0.2 \times 9{,}810}{0.9}} = 46.8 \text{ meters } [153.5 \text{ ft.}];$$

for wire ropes, when $k = 6$ kilograms [8,500 lbs.] and $\varepsilon = 7.7$ [0.278 lbs. per cubic inch], we shall have for the greatest velocity that can be transmitted

$$v = \sqrt{\frac{6 \times 9{,}810}{7.7}} = 87.5 \text{ meters } [287 \text{ ft.}].$$

EXAMPLE. If a circular saw makes 1,200 revolutions per minute, and transmits 6 horse powers, how tight must the belt be drawn when the driving pulley has a diameter of 0.24 meter [9.45 inches] ?

Here

$$v = \frac{1{,}200}{60} \times 0.24 \times 3.14 = 15.08 \text{ meters } [49.05 \text{ feet}],$$

and the force at the circumference

$$K = \frac{6.75}{15.08} = 29.8 \text{ kilograms } [657 \text{ lbs.}];$$

hence when the arc of contact is 0.45 of the circumference we have

$$e^{\phi\gamma} = e^{0.28 \times 0.45 \times 2\pi} = 2.2,$$

and

$$S_1 = \frac{e^{\phi\gamma} K}{e^{\phi\gamma} - 1} + q\frac{v^2}{g} = \frac{2.2 \times 29.8}{1.2} + \frac{f \times 0.9}{1{,}000 \times 1{,}000} \frac{15.08^2}{9.810}$$

$$= 54.6 + 0.021\, f.$$

Now if the belt has a width b and a thickness equal to 5 millimeters [0.2 in.], we can get from

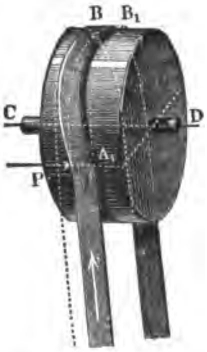
$$54.6 + 0.021 \times 5b = 5b \times 0.2$$

$$b = \frac{54.6}{0.895} = 61 \text{ millimeters } [2.4 \text{ ins.}].$$

The effect of the centrifugal force in this case, is to require an increase in the tension and consequently in the width of the belt $100 - 89.5 = 10.5$ per cent. The influence of the centrifugal force is appreciable only when the velocities are exceptionally large.

§ 62. **Construction of Belt Pulleys.**—The wheels, sheaves, pulleys and drums employed in belt gearing are generally made of iron, occasionally of wood. The width of the face of the wheel is usually from a fifth to a fourth greater than the width of the belt, and the rim is slightly rounded, being highest in the centre, thus facilitating the putting on of the belt and diminishing its tendency to run off. As rounding renders shifting difficult, it is omitted when it is necessary to shift the belt frequently for the purpose of imparting and arresting motion. From what has preceded we know that shifting itself needs only a slight lateral pressure at the point *A*, Fig. 208, where the belt runs on to the pulley, in consequence of which pressure the obliquely advancing belt will shift on the pulley in the direction of the arrow as long as the pressure lasts. Frequent use is made of this to set in motion or stop a machine by placing two equal pulleys close together upon its shaft, one being keyed and the other turning loosely on the shaft. As the belt is driven by a constantly running pulley of double width, the machine will be set in motion or not according as the belt runs over the fixed or the loose pulley. Care must also be taken to bore the pulley so that it will run true on its shaft, otherwise the belt will be apt to run off; the conditions stated in § 58 concerning the arrangement of pulleys should be carefully observed also. If lateral forces act upon the belt which tend to run it off—for example, the weight of the belt for upright shafts, or when the axis of the belt does not lie exactly in the plane of rotation, as is the case in crossed belts—the wheel must be provided with a projecting edge to prevent the running off of the belt.

Fig. 208.



Only small wooden pulleys, say to ¼ meter [about 10 ins.] diameter, can be turned from a single piece, larger pulleys must be composed of several pieces. One of the larger pulleys is shown in Fig. 209, two views being given. Here the intermediate pieces D and D_1 are rabbeted or mortised to the side pieces A and B, and firmly connected by means of bolts $E F$, $E F$; the outside of the bolts is covered with wood and an eye C is left for the reception of the shaft. Another arrangement of wooden pulleys is shown in Fig. 210, the adjacent pieces being so placed that their fibres cross. These wheels have also two disks composed of wooden sectors which are united to the body of the wheel by means of wooden pins and iron bolts $F G$, $H K$. As these disks are larger than the body of the wheel, they form projecting edges which prevent the belt from running off. Wheels of still larger diameter are made up of arms and rims, the rims being usually of wood and the arms of cast iron.

FIG. 209.

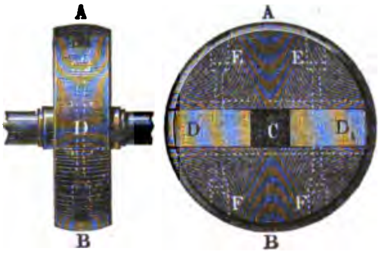


FIG. 210.

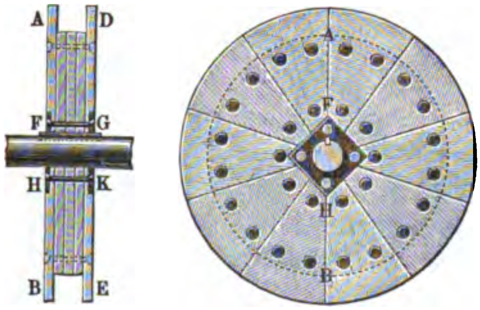
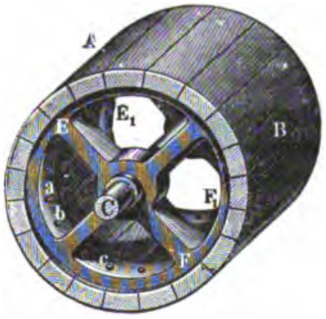


FIG. 211.



Long drums carrying several belts are covered with strips of wood, the arms being of iron. Such a drum is shown in Fig. 211; the strips $A B$ are fastened by small screws a, b, c, to the rings of the arms $E F$, $E_1 F_1$.

On account of their unchangeableness, cast-iron pulleys are always preferable to wooden ones, although the greater smoothness of the former requires a tighter belt, and consequently causes more friction on the shaft, than the latter. Small wheels up to $\frac{1}{4}$ meter [about 10 ins.] diameter have the form of a disk $A\,B$, Fig. 212, $F\,G$ being the hub and $D\,E$ the projecting edge. A section of an ordinary cast-iron pulley is shown in Fig, 213, I, side views of two styles being shown in Fig. 213, II and III. The width $A\,A = B\,B$ of the rim is usually $b_1 = 1.25\,b$, where b is the width of the belt; the thickness of the outer edge of the rim, being

$$e_1 = 0.03\,b + 0.005\,R,$$

and the thickness at the middle

$$e_2 = 0.12\,b + 0.005\,R,$$

the rise at the crown being $h = 0.03\,b$, and the depth of the inner surface $h_1 = 0.06\,b$, as shown in Fig. 214.

FIG. 212.

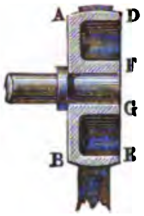
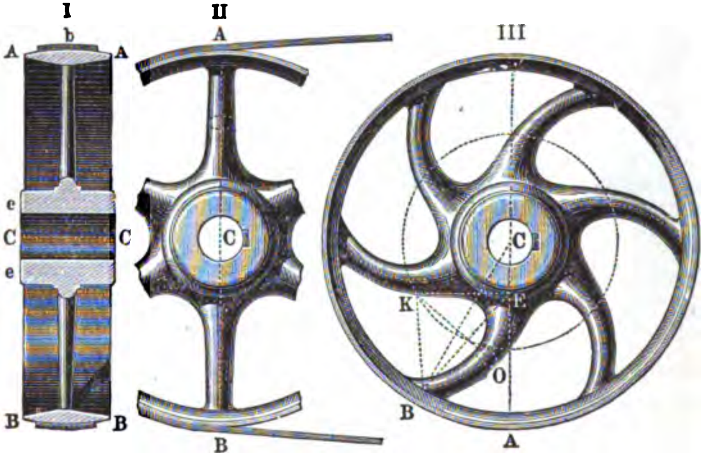
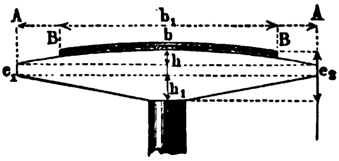


FIG. 213.



The arms of cast-iron pulleys are often made of the curved shape shown in Fig. 213, III, in order that they may contract while cooling without causing too great a tension in the arms. This is liable to occur when the

FIG. 214.



arms are straight, because of the slower cooling of the heavier rim and nave. To diminish as much as possible the resistance of the air, the arms are given an elliptical cross section, the axes being $\mathfrak{a}$ and $\mathfrak{b}$, the larger axis $\mathfrak{a}$ lying in the middle plane of the pulley. The width of the arm $\mathfrak{a}_1$ at the nave should be, according to *Reuleaux*,

$$\mathfrak{a}_1 = 0.4\,b + \frac{R}{40},$$

and the width at the rim,

$$\mathfrak{a}_2 = \tfrac{2}{3}\,\mathfrak{a}_1$$

when the number of arms is

$$m = \text{about } \frac{1}{2}\left(5 + \frac{R}{b}\right).$$

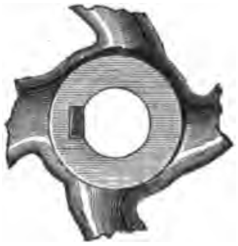
Generally, the number of arms lies between 4 and 8. A calculation of dimensions of the arms according to the principles of flexure, the arms being regarded as fastened to the nave, and bent by the force acting at the circumference, is of little practical value, because there is no certainty as to the part which the individual arms take in transmitting the force. When such a calculation is made, it will be well to make each arm sufficiently strong to bear the whole force.

The nave is given a length $l = 1.4\,b$, and a thickness $e = 0.3\,b$. The diameter of the eye, of course, depends upon the size of the shaft d, which may be found from § 15 to be

$$d = 120\sqrt[4]{\frac{N}{n}}\ \left[d = 4.8\sqrt[4]{\frac{N}{n}}\right].$$

It is not customary to strengthen the naves of belt pulleys, and for average sizes and forces, flat keys, Fig. 215, having no key-way in the shaft, may be employed.

FIG. 215.



EXAMPLE. A cast-iron pulley 1 meter [39.37 ins.] in diameter makes 60 revolutions per minute, and transmits 3 horse powers [2.96 H. P.]; what should its dimensions be?

The velocity of the belt is $v = 3.14$ meters [10.3 ft.], hence the force transmitted is

$$K = \frac{3 \times 75}{3.14} = 71.6 \text{ kilograms } [157.9 \text{ lbs.}].$$

Under ordinary circumstances, $\varphi = 0.28$, $\gamma = \pi$, we can assume, § 55, the width of the belt

$$b = 2K = 144 \text{ millimeters } [5.67 \text{ ins.}].$$

From this follows the width of the pulley

$$b_1 = 1.25\, b = 180 \text{ millimeters } [7.2 \text{ ins.}].$$

The thickness of the rim at the edge $e_1 = 0.03 \times 144 + 0.005 \times 500 =$ about 7 millimeters [about 0.28 in.], and at the middle: $e_2 = 0.12 \times 144 + 2.5 =$ about 20 millimeters [about 0.8 in.], the rise or swelling of the face of the pulley being 4.3 millimeters [0.17 in.].

The number of arms is:

$$m = \frac{1}{2}\left(5 + \frac{500}{144}\right) = 5,$$

and the width of the arms,

$$\text{at the centre: } a_1 = 0.4 \times 144 + \frac{500}{40} = 70 \text{ millimeters } [2.8 \text{ ins.}],$$

$$\text{at the edge: } a_2 = \tfrac{2}{3} \times 70 = 46 \text{ millimeters } [1.84 \text{ ins.}].$$

The thickness of the arms may be assumed at, say, 35 and 24 millimeters [1.4 and 0.96 ins.].

As the diameter of the shaft in this case is

$$d = 120 \sqrt[4]{\frac{3}{60}} = 56.7 = \text{about } 60 \text{ millimeters [about } 2.4 \text{ ins.}],$$

we have for the external diameter of the nave,

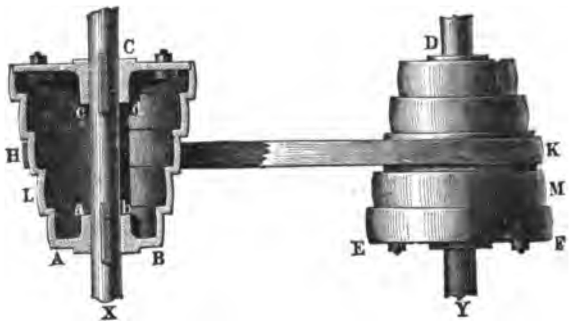
$$D = 60 + 2 \times 0.3 \times 144 = 146.4 = \text{about } 150 \text{ millimeters [about } 6 \text{ ins.}];$$

and for the length,

$$l = 1.4 \times 144 = \text{about } 200 \text{ millimeters [about } 8 \text{ ins.}].$$

§ **63. Stepped Cones.**—In many cases in practice it is necessary to drive the shaft of a machine at different times with different speeds, while the countershaft, or main line of shafting from which the motion is received, rotates with a constant velocity. This is always the case in lathes and boring machines, for in these, for a given material, the tool must have a constant speed relatively to the work, and consequently the number of revolutions must be inversely as the diameters of the different pieces of work. In such cases stepped cones are used, which, as *A B C* and *D E F*, Fig. 216, consist essentially of a series of pulleys placed side by side on the same shaft. Generally these pulleys are cast in one piece with the disk *A B* and the nave *a b*. In the smaller pulleys the nave projects into the interior of the stepped cone, but when the pulleys are large they are supported by an additional nave *c d* and disk *C*. It is evident that when the driving shaft *C X* turns with constant speed, it can communicate a smaller number of revolutions to the driven shaft *D Y*, if the belt is shifted from the position *H K* to the position *L M*, and that this number will become smaller yet if the belt is placed upon the still smaller driving wheel *A B* and the still larger driven wheel *E F*.

FIG. 216.



In order that the length of the belt may remain unchanged when shifted from one pair of pulleys to another. the diameters of the pulleys must be determined according to a certain law. If the belts are crossed, as in Fig. 217, we have, according to § 54, for the arc of contact γ of the belt:

$$\cos\frac{\gamma}{2} = -\frac{a_1 + b_1}{d},$$

and the length of the belt:

$$l = 2\, d \sin\frac{\gamma}{2} + \gamma\,(a_1 + b_1);$$

a_1 and b_1 being the radii of the pulleys and d the distance between the axes. If we let a_1 diminish as rapidly as b_1 increases, we shall have the same sum:

FIG. 217.

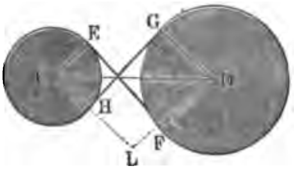
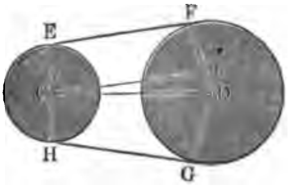


FIG. 218.



$$a_1 + b_1 = a_2 + b_2 = a_3 + b_3 \;.\;.\;.\;.$$

for every belted pair, consequently, neither the arc of contact nor the length of the belt will change when the latter is shifted from one pair of pulleys to another.

On the other hand, with open belts, Fig. 218, we have:

$$\cos\frac{\gamma}{2} = \pm\frac{b_1 - a_1}{d},$$

and the length of the belt

$$l = 2\, d \sin\frac{\gamma}{2} + \gamma\, a_1 + (2\,\pi - \gamma)\, b_1.$$

If we assume

$$\sin\frac{\gamma}{2} = 1 - \frac{1}{2}\left(\cos\frac{\gamma}{2}\right)^2 = 1 - \frac{1}{2}\left(\frac{b_1 - a_1}{d}\right)^2,$$

approximately, and

$$\gamma = \pi - (\pi - \gamma) = \pi - 2\,sin\frac{\pi - \gamma}{2} = \pi - 2\,cos\frac{\gamma}{2}$$

$$= \pi - 2\frac{b_1 - a_1}{d},$$

substituting, we shall have

$$l = 2d - \frac{(b_1 - a_1)^2}{d} + \left(\pi - 2\frac{b_1 - a_1}{d}\right)a_1 + \left(\pi + 2\frac{b_1 - a_1}{d}\right)b_1$$

$$= 2d + \pi(a_1 + b_1) + \frac{(a_1 - b_1)^2}{d}.$$

If ν_1 represents the velocity ratio,

$$\nu_1 = \frac{a_1}{b_1},$$

we shall also have

$$l = 2\,d + \pi(\nu_1 + 1)b_1 + \frac{(\nu_1 - 1)^2}{d}b_1^2,$$

from which

$$b_1 = \frac{-\pi(\nu_1 + 1)d + \sqrt{4(\nu_1 - 1)^2(l - 2d)d + \pi^2(\nu_1 + 1)^2 d^2}}{2(\nu_1 - 1)^2}$$

may also be obtained. By means of this formula, therefore, when the length l of the belt, the distance d between the axes, and the velocity ratio ν_1 are given, we find the radius b_1 and then the corresponding radius from $a_1 = \nu_1 b_1$. When the distance d between the shafts is great, the arc of contact will differ but little from a semi-circumference, and we shall have

$$l = 2\,d + \pi\,(a_1 + b_1) = 2\,d + \pi\,(\nu_1 + 1)\,b_1;$$

hence

$$b_1 = \frac{l - 2\,d}{\pi(\nu_1 + 1)},$$

which expression, when $\nu = 1$, gives the exact result:

$$b_1 = \frac{l - 2d}{2\pi} = a_1.$$

In stepped cones the distance between the shafts is generally given, and, assuming that the length of the belt remains constant, it is required to so determine the radii of the individual pulleys that the velocity ratios will satisfy certain given conditions. In lathes and boring machines, for example, the requirements are best met by making the velocities which can be attained by the separate pairs of pulleys the terms of a geometrical series. If, therefore, ν is the velocity of the first pair of pulleys a_1 and b_1, the velocity ratios of the rest must be

$$\nu\varepsilon, \quad \nu\varepsilon^2, \quad \nu\varepsilon^3 \quad \ldots \quad \nu\varepsilon^{n-1},$$

where ε is a constant number which may be determined from

$$\varepsilon = \sqrt[n-1]{\frac{V}{v}},$$

V representing the greatest and v the least velocity of the driven shaft. In such cases it is always possible to find stepped cones which will satisfy the requirements, but the two pulleys will then generally be unlike, for with a given distance d between the shafts, the length of belt must remain the same for all pairs. Now, it is often desirable in practice to let two stepped cones work together which have been cast from the same pattern and are therefore exactly alike. For this case we may employ the following calculation. If, as in Fig. 216, the number of steps is, say, $n = 5$, and if the ratio of the greatest to the least velocity $\frac{V}{v} = \alpha$ is given, and the above condition be retained that the velocities shall form a geometrical series, we shall have

$$\varepsilon = \sqrt[4]{\alpha}.$$

Now, if ν is the velocity ratio of the first pair of pulleys, and

a_1, a_2, a_3, a_4 and a_5 the radii of the pulleys, we shall have for the velocity ratios of the pairs:

No. of the pair.	1	2	3	4	5
Ratio of the radii.	$\frac{a_1}{a_5}$	$\frac{a_2}{a_4}$	$\frac{a_3}{a_3}$	$\frac{a_4}{a_2}$	$\frac{a_5}{a_1}$
Velocity ratio.	v	$v\varepsilon$	$v\varepsilon^2$	$v\varepsilon^3$	$v\varepsilon^4$

It is also clear that since the stepped cones are exactly alike the velocity ratio of the middle pair is equal to unity, and consequently we have

$$v = \frac{1}{\varepsilon^2} = \frac{1}{(\sqrt[4]{\alpha})^2} = \frac{1}{\sqrt{\alpha}},$$

so that the velocity ratios of the individual pairs form the series

$$\sqrt{\frac{1}{\alpha}}, \quad \sqrt[4]{\frac{1}{\alpha}}, \quad 1, \quad \sqrt[4]{\alpha}, \quad \sqrt{\alpha}.$$

If d is the distance between the shafts, the middle pulley a_3 may be taken of any desired size, and the exact length of belt will be

$$l = 2d + 2\pi a_3.$$

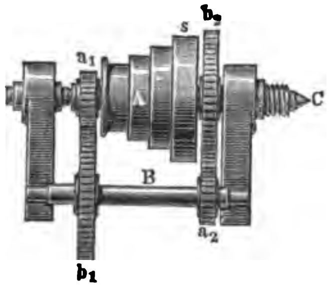
Now, from this length l, the distance d and the velocity ratios

$$v = \sqrt{\frac{1}{\alpha}} \text{ and } v\varepsilon = \sqrt[4]{\frac{1}{\alpha}}$$

we can find the radii a_1, a_2, a_4, and a_5 from the above formulas.

Very often when stepped cones are used the number of possible velocity ratios is increased by employing an arrangement which permits the shaft of the machine to be driven directly by one of the stepped cones, or indirectly by placing toothed gearing between the stepped cone and the shaft of the machine. The arrangement shown in Fig. 219 is often thus used in lathes, the driven cone being placed on a sleeve *A* and, as was explained in § 42, either rigidly attached to the wheel b_2, turning the latter and the spindle, or turning loosely on the spindle and indirectly turning the latter by means of the gearing $\frac{a_1}{b_1}\frac{a_2}{b_2}$. This enables us to obtain $2n$ different velocity ratios when the number of pulleys on each cone is n. If it is also required that these ratios form a geometrical series whose factor is ε, we shall have

FIG. 219.



$$\varepsilon = \sqrt[2n-1]{\alpha},$$

when $\alpha = \frac{V}{v}$ again represents the ratio of the greatest velocity to the least. The velocity ratios form the series:

$$\nu,\ \nu\varepsilon,\ \nu\varepsilon^2 \ldots \nu\varepsilon^{n-1},\ \nu\mu,\ \nu\mu\varepsilon,\ \nu\mu\varepsilon^2 \ldots \nu\mu\varepsilon^{n-1},$$

μ being the velocity ratio of the toothed gearing $\frac{a_1}{b_1}\frac{a_2}{b_2}$. From this follows that, since $\nu\mu = \nu\varepsilon^{n-1}\varepsilon$, we must have $\mu = \varepsilon^n$.

EXAMPLE. For a lathe with back gears, it is required to obtain two equal four-stepped cones, so that the velocities will form the terms of a geometrical series, the ratio of the extremes being 1 : 24. What dimensions must be given to the individual pulleys and the wheels, if the smallest pulley is to have a radius of 80 millimeters [3.2 ins.], the distance between the shafts of the cones being 2 meters [78.74 ins.]? We have here

$$\varepsilon = \sqrt[7]{\frac{1}{24}} = 0.635,$$

and for the velocity ratio of the two pairs of toothed wheels

$$\varepsilon^4 = 0.635^4 = 0.162.$$

If we designate the radii of the pulleys or steps by $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ and $\mathfrak{a}_4$, we have

$$\frac{\mathfrak{a}_1}{\mathfrak{a}_4}\,\varepsilon^3 = \frac{\mathfrak{a}_4}{\mathfrak{a}_1},$$

and therefore

$$\mathfrak{a}_1 = \frac{\mathfrak{a}_4}{\sqrt{\varepsilon^3}} = \frac{80}{0.506} = 158 \text{ millimeters [6.32 ins.]};$$

hence for the length of belt we have, approximately,

$$l = 2d + \pi(158 + 80) = 4{,}000 + 747 = 4{,}747 \text{ millimeters [187 ins.]}.$$

The velocity ratio of the first pair of pulleys is:

$$\nu = \frac{\mathfrak{a}_1}{\mathfrak{a}_4} = \frac{158}{80} = 1.975,$$

hence that of the second is:

$$\nu_1 = \frac{\mathfrak{a}_2}{\mathfrak{a}_3} = \frac{\mathfrak{a}_1}{\mathfrak{a}_4}\,\varepsilon = 1.975 \times 0.635 = 1.255\,;$$

consequently, for $\mathfrak{a}_3$ we have

$$\mathfrak{a}_3 = \frac{l - 2d}{\pi(\nu + 1)} = \frac{4.747 - 4{,}000}{3.14 \times 2.255} = 105.5 \text{ millimeters [4.15 ins.]};$$

and

$$\mathfrak{a}_2 = \mathfrak{a}_3 \times \nu_1 = 105.5 \times 1.255 = 132.4 \text{ millimeters [5.21 ins.]}.$$

REMARK. In practice the radii of the stepped cones often increase by equal increments, so that they form the terms of an arithmetical series. It is easy to see that in this case, if there are only 3 steps on a cone, the velocities will increase in the same ratio, *i.e.*, they will form terms of a geometrical series. For if a represents the radius of the smallest, $a + e$ the radius of the middle, and $a + 2e$ the radius of the largest pulley, the velocity ratios will be determined by:

$$\nu_1 = \frac{a}{a + 2e}\,;\quad \nu_2 = \frac{a + e}{a + e} = 1, \text{ and } \nu_3 = \frac{a + 2e}{a},$$

hence

$$\frac{\nu_1}{\nu_2} = \frac{a}{a + 2e} = \frac{\nu_2}{\nu_3}.$$

When there are more than three pulleys or steps on a cone, this relation only exists approximately, the approximation being closer the smaller the ratio of the increment e to a. For example, when the cones have 4 pulleys we have

Radii	a	$a + e$	$a + 2e$	$a + 3e$
Velocity ratios	$\nu_1 = \frac{a}{a + 3e}$	$\nu_2 = \frac{a + e}{a + 2e}$	$\nu_3 = \frac{a + 2e}{a + e}$	$\nu_4 = \frac{a + 3e}{a}$

hence the ratio of the velocities:

$$\frac{v_1}{v_2} = \frac{a}{a+3e} \cdot \frac{a+2e}{a+e} = \frac{a^2+2ae}{a^2+4ae+3e^2},$$

$$\frac{v_2}{v_3} = \frac{a+e}{a+2e} \cdot \frac{a+e}{a+2e} = \frac{a^2+2ae+e^2}{a^2+4ae+4e^2},$$

$$\frac{v_3}{v_4} = \frac{a+2e}{a+e} \cdot \frac{a}{a+3e} = \frac{a^2+2ae}{a^2+4ae+3e^2} = \frac{v_1}{v_2}.$$

Consequently,

$$\frac{v_1}{v_2} = \frac{v_3}{v_4},$$

$\frac{v_2}{v_3}$ deviating from these ratios by a quantity which is smaller the smaller the ratio of e to a. In the above example we found the radii 80, 105.5, 132.4 and 158 millimeters, that is, they differed but little from the terms of an arithmetical series having the common difference $e = 26$ millimeters [1.02 ins.].

§ **64. Conical Drums.**—Instead of stepped cones use is sometimes made of conical drums *A* and *B* tapering opposite ways, Fig. 220, the position of the belt being controlled by the belt shifter *D C E*, which encloses the belt at the points *D* and *E*, where it runs on to the pulleys. If the belt were not thus guided, it would be compelled by the conical shape of the drum to change its position. The belt tends to take the position shown in Fig. 221, and its contact with the drum is not along a circular arc but an ellipse *A B H*, whose plane is perpendicular to the element *E F*. It is evident that when it has this position the direction of the belt at *A* tends to make it move from the smaller to the larger end of the drum. This is also the

FIG. 220.

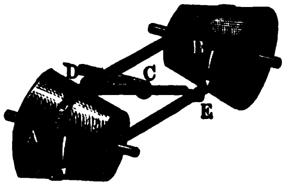
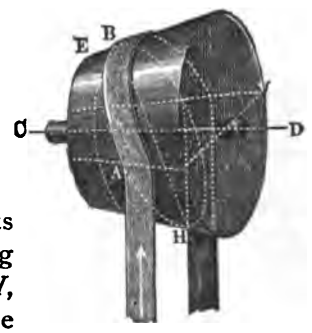


FIG. 221.



reason why a belt always tends to move toward and stay at the middle of a rounded pulley.

Conical drums have the advantage of allowing the velocity ratio to be gradually changed within certain limits, nevertheless it is not easy to obtain an exact velocity ratio unless narrow belts or cords are employed; for if we imagine the belt *CDEF* to touch the conical surfaces *A* and *B*, Fig. 222, it is evident that some portions of the belt must slip on the drums, for the ratio of the two radii *CH* and *KF* is different from that of *DG* and *IE*. As a consequence the true velocity ratio of the motion will lie somewhere between those just given, and may perhaps be equal to the ratio of *LM* and *NO*, but with wide belts it cannot be given with accuracy on account of the obliquity of the action on the conical drums. For, as may be seen from the figure, the pull of the belts will principally pass through the points *D* and *F*. Another disadvantage connected with this is that the belt is very unequally strained by this oblique pull, and this combined with the slipping causes rapid wear. These evils, of course, increase with the width of the belt and are less noticeable with cords. On this account belts have been made of two thin leather strips enclosing a cord, as in Fig. 223, the small surface of contact thus obtained greatly diminishing these evils and giving a more central pull. On the other hand, they have greater stiffness, causing them to wear rapidly.

FIG. 222.

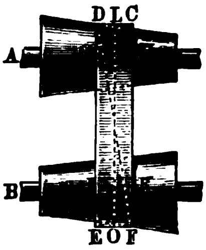
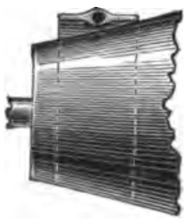
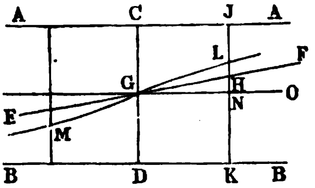


FIG. 223.



In order that the belts of conical drums may always have the same tension, the length of the belt must be the same in every position. This is the case with crossed belts when the sum of the circles connected by the belt is everywhere equal, and from what has already been said we can assume this to be approximately true for open belts also, when the ratio of the distance between the shafts to the radii of the drum is sufficiently large, *i.e.*, when the arcs of contact differ but little from semi-circumferences.

FIG. 224.



Let $A\,A$ and $B\,B$, Fig. 224, be two parallel shafts of cones placed so close together that their normal distance $C\,D$ is equal to the constant sum of their radii, it is then evident that any straight line $E\,F$, oblique to the axes, may be regarded as the generatrix of two cones which satisfy the above condition of constant length of belt. For evidently in every position, for example at H, the sum of the radii $HJ + HK$ is equal to the distance CD, the middle point G of CD corresponding to that position in which the velocity ratio of the cones is unity. If the belt is shifted out of this middle position CD through the distance $\lambda = CJ$ to a position JK, the velocity ratio will be

$$\nu = \frac{JH}{KH} = \frac{a - \lambda \tan \varphi}{a + \lambda \tan \varphi} = \frac{l - \lambda}{l + \lambda},$$

where $a = CG$ represents half the distance between the axes, φ the angle of inclination of the axes to the generatrix, and $l = \dfrac{a}{\tan \varphi}$ the length of an element of the cone measured from G to the vertex. We then get:

$$\lambda = \frac{a}{\tan \varphi}\, \frac{1 - \nu}{1 + \nu} = l \frac{1 - \nu}{1 + \nu}.$$

This equation shows that equal variations in the position of the belt λ are not accompanied by equal variations of the velocity ratio ν. For example, we get for $\nu = 0.1, 0.2, 0.3$,

$$\lambda = 0.82l,\ 0.67l,\ 0.54l, \text{ etc.}$$

If the cones are to be so formed that the velocity ratios for equal variations in the position of the belt will form a geometrical series, the profile of the cone must be made like the curve MGL, which when referred to GO as an axis of X and to GC as an axis of Y, is represented by the equation:

$$v = \frac{LJ}{LK} = \frac{a - y}{a + y} = c^x,$$

where c is a certain constant number which is determined by assigning to a certain value x, for example GN, a given value v. From this equation we see that the numerical value of y is the same for equal and opposite values of x. Drums with such curved profiles are seldom used; they may, however, be found in roving machines. Occasionally a cone is used with a cylinder, their axes being parallel, and then the belt is kept tight by a tightening pulley, which must likewise be employed in all cases where the above condition of constant length of belt cannot be satisfied.

Sometimes expanding pulleys are employed for varying the velocity ratio, namely, when it is necessary to regulate the motion accurately, as in machines for manufacturing endless paper. There are many kinds of these pulleys, but they consist essentially of a rim composed of several sectors which can be moved in a radial direction, by means of screws or spiral guides. By thus varying the distance of the sectors from the centre, the circumference of the pulley may be regulated within certain narrow limits. One of the simplest pulleys of this kind is shown

FIG. 225.

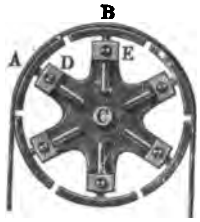
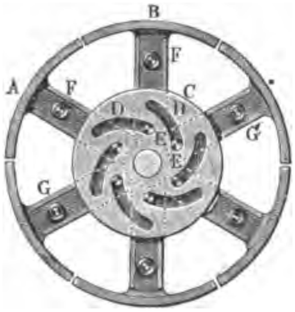
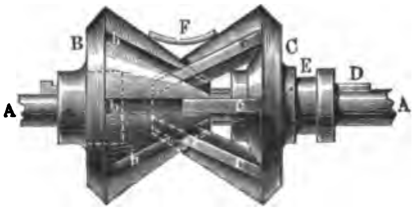


FIG. 226.



in Fig. 225. Here the six sectors $A, B, \ldots$, which constitute the rim, are provided with studs which pass through sockets D, E of the star-shaped nave C and are fastened to the latter by means of set screws. In this case each arm must be separately and equally moved, but there are other pulleys in which all the arms $A, B, \ldots$, Fig. 226, are moved simultaneously through equal distances. They consist of a disk C which can turn about the shaft and has for each arm a spiral-shaped groove D, into which fits a pin or projection of the corresponding arm E. The set screws F fasten the arms to the star-shaped centre-piece G which guides them. The arrangement shown in Fig. 227 is used for the same purpose, for example, in flax spinning machines. Here a cone B is keyed to the shaft A and an opposite cone C is placed so that it can slide along a feather D, a fork-shaped lever fitting into the neck groove E, serving to move C a certain distance along the shaft A. The cone B is provided with recesses b, into which enter the ribs c on the other cone C, in such a way that there is a sort of mutual penetration of the two cones. From this it is evident that the diameter of the neck or gorge of the two cones, in which the belt F runs, is variable and dependent upon the position of the two cones.

FIG. 227.



For fuller information concerning expansion pulleys, see Verhandlungen zur Beförderung des Gewerbfleisses in Preussen, 1843. See also *Redtenbacher*: Die Bewegungsmechanismen.

§ 65. Cords and Ropes.—When the force to be transmitted is small, ropes or cords of hemp, cotton or catgut are substituted for belts. Cords have the advantage of staying on their pulleys, and of being easily deflected, and, as already mentioned, are better suited for conical drums than belts.

The pulleys used with cords or ropes have a groove for holding the cord, as shown at A, B, C, Fig. 228. The sharp groove at B (angle of about 60°) causes the rope to wedge itself between the sides, thus producing greater friction and consequently a smaller tension. To prevent the cord from slipping, the sides of the groove are notched, as at CC; moreover grooved pulleys have the advantage of being suitable for both large and small cords. Stepped pulleys, too, are often used in rope gearing, especially in small lathes, Fig. 229 showing one, ACB, having three grooves. The diameter of cords, according to *Reuleaux*, should not be taken less than $4\sqrt{P}$ [$0.106\sqrt{P}$ in.] when P represents the force at the circumference, which is to be transmitted.

FIG. 228.

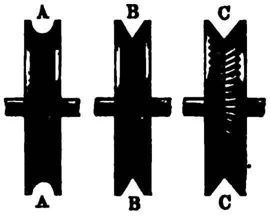


FIG. 229.

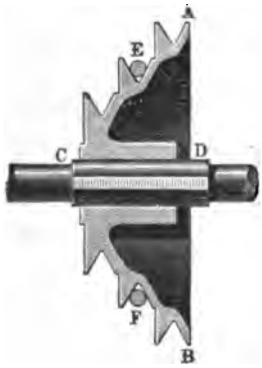
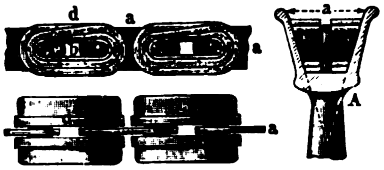
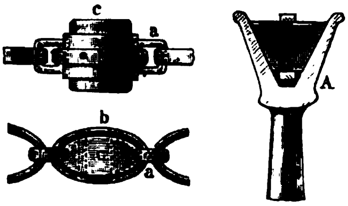


FIG. 230.



Recently, for the transmission of large forces, a sort of angular chain belt has been employed; its action, like that of ordinary belts, depending upon friction, and its chain-like structure serving simply to unite the separate portions. Such a belt, invented by *Clissold* and shown in Fig. 230, has been employed with advantage in steam pile-drivers. This belt consists of iron links a and b united by bolts c, thus forming a chain whose flat links are covered with leather strips c which bear against the sides of the V-shaped groove of the pulley A. In this form of belt the friction is increased in the same way as in *Minotto's* friction wheels (§ 52). The calculation of such a belt gearing is like that of the ordinary belt gearing, except that we must take here, as in friction wheels, for the coefficient of friction the value,

FIG. 231.



$$\frac{\varphi}{\sin\frac{\alpha}{2} + \varphi \cos\frac{\alpha}{2}}.$$

If we again assume $\alpha = 30°$ and $\varphi = 0.28$, we shall get for the coefficient of friction:

$$\frac{0.28}{0.259 + 0.28 \times 0.966} = 0.53.$$

Another singular belt given by *Angström*[*] is shown in Fig. 231. Here the chain consists of short and long links, *a* and *b*, into the latter of which wooden wedges *c* are driven, which fit the groove of the pulley *A*. To increase the friction as well as the durability, the wedges are so cut that their rubbing surfaces are cross-grained.

§ 66. **Spur Wheels.**—The transmission of motion between two shafts, by means of friction wheels and belt pulleys, is possible only so long as the resistance to be overcome does not exceed the friction which arises at the circumferences of the wheels, when one wheel ceases to obey the driving action of the other. To prevent this, the friction must be made sufficiently large, in one case by pressing the friction wheels more closely together and in the other by tightening the belt sufficiently. But such a pressure or tension involves a corresponding amount of friction of the shafts in their bearings, so that friction wheels cannot be employed to advantage where the force to be transmitted is considerable, as is generally the case in slowly running shafts. Neither can they be employed where it is necessary that the motions of two shafts shall have *exactly* the desired ratio at every instant, as in screw-cutting machines, clocks, or measuring instruments, for experience shows that, even with the greatest pressures, friction wheels will occasionally slip. These difficulties have led to the construction of *toothed wheels*, *i. e.*, wheels whose surfaces have such a form that the motion of one wheel compels the other to move under all circumstances, provided the strength of the material is not exceeded. Toothed wheels, as is well known, have their circumferences provided with certain projections or *teeth* and intermediate depressions or *tooth spaces*, so that the teeth of one wheel can enter the spaces of the other and thus transmit motion from one wheel to the other without being compelled to generate friction by pressing the wheels together. We will now examine the particular case in which the shafts are parallel, that is, the case of *spur* wheels. Fig. 232 represents the portions of the two wheels A and B which are in gear, the tooth C of the wheel A pressing against the tooth D of the wheel B and thus turning the latter. If P is the point of contact of the two teeth C and D, and lies upon the line of centres AB, it is easy to see that when the wheel A turns through the small angle α, the wheel B will turn through the angle β given by the proportion

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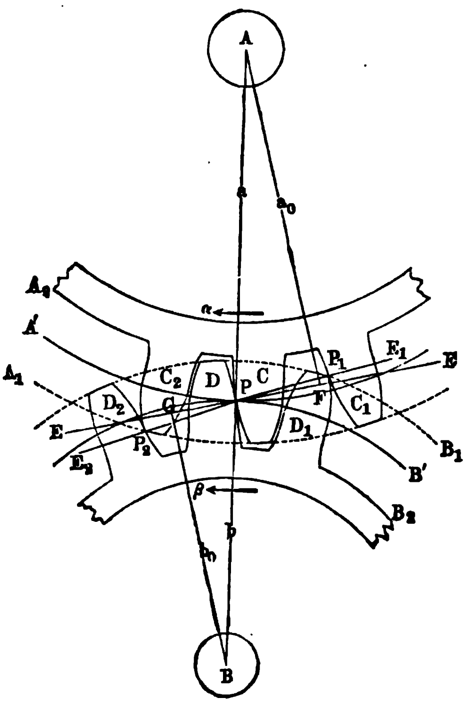
[*] Zeitschr. deutsch. Ingenieure, 1868, p. 706.

$$\alpha : \beta = b : a,$$

where a and b are

FIG. 232.



the distances of the point P from A and B respectively. This will be true whatever the direction in which the teeth act upon each other at P, for if this direction is represented by the straight line $E E$, and if

$$A F = a_0 \text{ and } B G = b_0$$

represent normals drawn from the axes to the line, the line $E E$ and the point P will shift through a distance $\eta = a_0 \alpha$ when the shaft A turns through the small angle α, and this shifting will correspond to a rotation β of the normal $B G$, for which $\eta = b_0 \beta$. Hence follows:

$$a_0 \alpha = b_0 \beta, \text{ or } \alpha : \beta = b_0 : a_0 = b : a.$$

At the instant when the point of contact P of two teeth lies upon the line of centres $A B$, the two wheels A and B turn exactly like two friction wheels whose radii are $A P = a$ and $B P = b$. If now, as is generally the case in practice, the condition is to be fulfilled that a uniform motion of one wheel A must produce a uniform motion of the other wheel B, evidently the velocity ratio $\frac{\alpha}{\beta}$ must have the same value every instant, *i. e.*, it must be equal to $\frac{b}{a}$ even when the point of contact of two teeth does not lie on the line of centres $A B$. When, therefore, the tooth C_1 touches the tooth D_1 at P_1, we must also for this position satisfy the condition:

$$\alpha : \beta = b : a.$$

We see from the drawing that *this condition can be satisfied only when the direction of pressure of the teeth at P_1 passes through the point P*, i. e., *when the normal $E_1 P_1$ to the toothed surfaces in contact at P_1 passes through the point P at which contact takes place in the line of centres*, for then only does the relation hold:

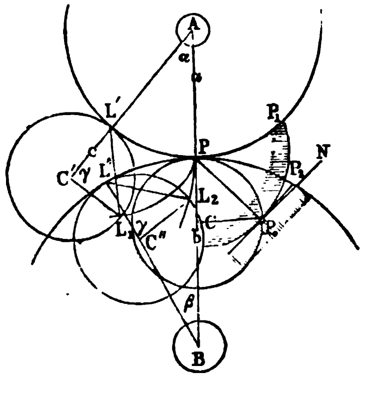
$$b : a = b'_0 : a'_0,$$

where a'_0 and b'_0 represent the perpendicular distances of the axes A and B from the direction $E_1 P_1$ of the pressure at P_1. Of course this holds when two teeth C_2 and D_2 are in contact at a point P_2 upon the other side of the line of centres; here also the normal $E_2 P_2$ to the toothed surfaces in contact at P_2 must pass through the point of contact P on the line of centres. We see from this that the forms of the teeth cannot be arbitrarily assumed, but must be so chosen as to satisfy the above condition.

If through the point P, at which two teeth come in contact on the line of centres, we pass two circles concentric to the axes and touching at P, we know from what has preceded that the circumferential velocities of these two circles of the wheels will be the same, and will be the same *for these circles only*, as is the case also with two friction wheels having the same radii. In a circle A_1 whose radius is greater than a, the circumferential velocity will be greater than that of the circle which touches it and is concentric to B, for the radius of the latter is smaller than b; in like manner in the circle A_2, whose radius is smaller than a, there must be a smaller circumferential velocity than in that circle of the wheel B with which the circle A_2 is in contact. Since the two circles A' and B' passing through P have equal velocities, they *roll* upon each other, *i. e.*, they move upon each other *without sliding;* on the other hand, on account of the difference of velocities which exists in any other two circles which come in contact, *sliding* will take place, the amount being greater the greater the difference of velocities, *i. e.*, the greater the distance of these circles from those passing through P. From this now follows that the distance of two adjacent teeth measured on the circumferences passing through P, must be *equal on each of the two wheels*, for the distance of two teeth on any other two circles which are in contact must be different on the two wheels. The distance of two teeth from centre to centre, or the distance between two homologous surfaces measured along the circumferences of the circles passing through P is called the *pitch of the teeth*, or simply *pitch*, consequently the oft-mentioned circles of equal velocity on which the pitch is laid off, are known as the *pitch circles*. From what has been said we know that these circles are not realized in actual constructions, but are ideal circles so situated between the *addendum circles* A_1 and B_1 and the *root circles* A_2 and B_2 of the wheels, that the ratio of their radii is inversely proportional to the angular velocities of the wheels. From §§ 40 and 41 it follows that the pitch circles of the wheels must be regarded as the *centroids* which represent the relative motion of the two wheels, the point of contact P being the *instantaneous centre*, and the straight line through P parallel to the axes, the *instantaneous axis*.

§ **67. Law Governing the Formation of Tooth Surfaces for Spur Wheels.**—With the help of what has been established in § 26, Introduction, concerning axoids, it will be easy to establish the conditions which the surfaces of the teeth of spur wheels must satisfy. Let α and β again represent the angular velocities of the parallel shafts A and B, Fig. 233, and let the point P be so assumed on the line of centres AB that its distances $PA = a$ and $PB = b$ will be inversely proportional to the angular velocities, and consequently $a\alpha = b\beta$, then we know from what has preceded that the cylindrical surfaces which pass through P and are concentric to the axes A and B, represent the axoids for the relative motion of A to B. We can also represent this motion by supposing the cylinder AP to be fixed, and the other axoid or cylinder BP *rolled about* the former with a velocity equal to $\alpha + \beta$ (see § 27, Introduction). If the wheels were constructed as smooth friction wheels, such a direct rolling would actually take place, for the cylindrical surfaces of the friction wheels may be regarded as the embodiment of the axoids or surfaces of instantaneous axes. But in toothed

FIG. 233.



wheels these pitch cylinders $A P$ and $B P$ are not actually constructed, consequently the wheels do not always touch at P, but generally elsewhere at P_0 along the line of contact of two teeth. These teeth are likewise to be regarded as cylindrical surfaces, whose elements are parallel to the axes A and B and whose normal sections are represented by the curves $P_0 P_1$ and $P_0 P_2$ which touch at P_0, that is, have a common tangent $N P_0$. The contact of the *tooth surfaces* is therefore a straight line L which is parallel to the axes A and B, and perpendicular at P_0 to the plane of the drawing, *i. e.*, perpendicular to the plane of the curves $P_0 P_1$ and $P_0 P_2$. If we again suppose the wheel A to be fixed and the wheel B rolled about A, the motion will no longer take place about the instantaneous axis P, but around the line of contact L at P_0, for this is now the line of support of the wheel B. We see from this that the instantaneous motion of the toothed wheels is in reality a slight rotation about the axis at P_0 while the condition to be fulfilled requires a slight rotation about the instantaneous axis at P. These two axes at P and P_0 are parallel, and since, according to § 4, Introduction, two equal rotations in the same direction about parallel axes differ only in a translation or shifting perpendicular to the plane of the axes, we may infer that the actually existing rotation about the axis P_0 can be replaced by the instantaneous rotation about the axis P prescribed by this problem, provided the tooth surfaces permit a simultaneous translation of the moved system in a direction perpendicular to the plane of the axes P and P_0. But two cylindrical surfaces which touch along a straight line b will evidently permit translation only in their plane of contact, consequently the plane of contact common to the tooth surfaces at P_0, which is projected at $P_0 N$, must be perpendicular to the plane of the axes P and P_0. In other words, ***wherever two teeth are in contact, the plane normal to the tooth surfaces at their line of contact must pass through the instantaneous axis at P***, for what has here been proved of an arbitrarily assumed point of contact P_0, holds for every other position of the point and of the line of contact L projected in it.

Having developed the fundamental condition which all tooth surfaces must satisfy, we can now deduce a general law for generating correct tooth surfaces, the application of which wil lgive us the various correct forms of teeth employed in practice.

In order that we may develop this law, let us imagine a third cylindrical axoid $C P$ parallel to the two cylindrical axoids $A P$ and $B P$, and let us for the present assume that the base $C P$ of the third axoid is also circular, although, as we shall show later, the law holds good when the cylinder has a non-circular base.

Let us suppose this auxiliary axoid $C P$ with a radius c to be brought into such a position that its element L will coincide with the instantaneous axis P, the auxiliary axoid C will then touch the primary axoids A and B in this line, L or P touching A on the outside and B on the inside, when, as assumed in the figure, the two wheels A and B gear externally, *i. e.*, lie upon *opposite* sides of the common tangent plane. When the wheels A and B gear internally, the third axoid C touches simultaneously the two axoids A and B on the inside or on the outside, *i. e.*, they lie upon the *same* side of the common tangent plane. If, now, the auxiliary axoid C rolls without sliding through an arc $P L'$ on the outside of the cylinder A, the line of contact L will pass from its position P to another L_1 determined by the equality of the arcs $P L'$ and $L_1 L'$. During this motion the line L generates a cylindrical surface parallel to the axis A, the base of the surface being the epicycloid $P L_1$. In the same way by rolling the cylinder C upon the inner surface of B, the motion of the straight line L generates a cylindrical surface parallel to B with a hypocycloid $P L_2$ for its base.

It may now be easily shown that these two cylindrical surfaces $P L_1$ and $P L_2$, when they are employed as the profiles of two teeth of the wheels A and B, will satisfy the general condition developed above for correct gearing. To do this, let us suppose the cylinder C to roll through the angle $\alpha = P A L'$ about the axoid A into the position C', then from what has preceded we know that this rolling of the cylinder C will be equivalent to two rotations, one α about A and another $\gamma = \frac{a\,\alpha}{c}$ about C. This angle γ is represented in the figure by $L' \, C' \, L_{11}$ and we also see from the figure that the plane passed through $L' \, L_1$ parallel to the axes, is perpendicular to the surface $P \, L_1$ at L_{11} for the element of this surface at L_1 can be regarded as a small element of a circular cylinder whose axis is at L'. If we now imagine the whole system, *i. e.*, the cylinder A as well as C, to have a rotation about the axis A equal to $-\alpha$, the cylinder C' will return to its original position C, the line of contact L' coinciding with the instantaneous axis P, and the whole motion of the cylinder C reducing to a rotation about its own axis equal to $\gamma = \frac{a\,\alpha}{c}$. Now, the straight line L_1 coincides with P_0 when $P\,C\,P_0 = \gamma$, and if we suppose the surface $P\,L_1$ to be rigidly connected with A, it will take the position $P_0 \, P_1$ when the system is turned back, so that the plane passed through $P \, P_0$, parallel to the axes, is again perpendicular to the surface $P_0 \, P_1$ at P_0.

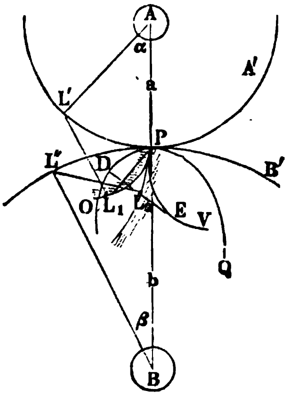
In the same way we can show that a rolling of the cylinder C on the inside of the cylindrical surface B from C to C'' through an angle $-\beta = P\,B\,L''$, corresponds to a two-fold rotation of the cylinder C, one $-\beta$ about B and another $\gamma' = + \frac{b\,\beta}{c}$ about its own axis. This angle γ' is represented in the figure by $L'' \, C'' \, L_2$, and the plane passed through $L'' \, L_2$ parallel to the axis is evidently normal to the surface $P \, L_2$ at L_2. Now, if the cylinder B, its surface $P \, L_2$ and the cylinder C be all turned about the axis B through the angle $+\beta$, C will pass from the position C'' to the original position C, and the whole motion of C will reduce to a rotation $\gamma' = \frac{b\,\beta}{c}$. As the required motion of the wheels A and B must satisfy the condition $a\,\alpha = b\,\beta$, it follows that $\gamma = \gamma'$, *i.e.*, in the two cases here considered, the auxiliary axoid C has turned through one and the same angle:

$$\gamma = \frac{a\,\alpha}{c} = \frac{b\,\beta}{c}$$

and in the same direction. From this follows that the two positions L_1 and L_2 of the generating line L finally coincide in the same line P_0, and that the plane passed through this line and the instantaneous axis P is normal to the two profiles P_0P_1 and $P_0 P_2$. The latter profiles must therefore touch in the straight line P_0, and any pressure exerted by one tooth surface on the other must pass through the instantaneous axis P and be perpendicular to it. This evidently satisfies the general condition to which all correct tooth surfaces are subject.

The property just found for surfaces generated by rolling the cylinder C on the axoids of the wheels, is by no means limited to the case in which the rolling cylinder C has a circular base, but holds good whatever curve be chosen as its base, provided the elements of this cylinder are parallel to the axes. For if we suppose any curve $O\ P\ Q$ Fig. 234, to be assumed as the base of the cylinder which rolls on the axoids A and B, the line of contact P will again generate two cylindrical surfaces whose cross sections are represented by the curves $P L_1$ and $P L_2$. When the cylinder C rolls about the axis A through an angle $P A L' = \alpha$, the line of contact L passes from P to L_1, and at the same time the point O of the curve $O\ P\ Q$ passes to L', the length of the arc $P\ O$ being equal to $P L'$.

FIG. 234.



When the rolling takes place on the inside of the axoid B, this same point O passes to a point L'', so that

$$arc\ P\ O = arc\ P\ L'',$$

and the line of contact L passes from its position P to L_2. Evidently here also the plane through $L'\ L_1$ parallel to the axes is normal to the surface $P L_1$ at L_1, and the plane through $L''\ L_2$ parallel to the axes is normal to the surface $P L_2$ at L_2. But it is also evident that the rotation of the system A and C around A, through the angle $L' A P = -\alpha$ and a rotation of the system B and C about B equal to $L'' B P = \beta = \frac{a\,\alpha}{b}$ will bring the rolling cylinder in both cases back into the same position, *i.e.*, the elements L_1 and L_2 of the generated surfaces $P L_1$ and $P L_2$ will come in contact at the same straight line P_0, and the normal plane common to these two surfaces at the line of contact will pass through the instantaneous axis P. To prove this we have only to show that in each of the two cases considered, the result of the rolling on the axoids A or B, and the subsequent backward rotation about the axes A or B, is a rotation of the rolling cylinder $O P Q$ through the same angle. This is easily done as follows: suppose normals to be drawn for all points of the curve $O P Q$, they will, as is well known, envelop a curve $P E V$, which is the *evolute* of $O P Q$ and contains the centres of curvature of the latter; we can therefore regard any element, for example D, of the surface $O P Q$, as an infinitely small portion of a cylinder whose axis is E and whose radius is $D E$. If we represent the angle at the centre (of curvature) subtended by this element by $\delta\gamma$ and the radius (of curvature) of the element by ϱ, then if we roll this element on the element $\delta\alpha$ of A or on $\delta\beta$ of B, the rolling cylinder will turn about its own axis E through the angles $\frac{a\,\delta\alpha}{\varrho}$ and $\frac{b\,\delta\beta}{\varrho}$ respectively. Since these two values are equal, it follows that for the rolling of every element of the cylinder $O P Q$ on the corresponding elements $\delta\alpha$ and $\delta\beta$, the rotations of $O P Q$ about its own axis will be equal. Consequently the sums

$$\int \frac{a\,\delta\alpha}{\varrho} \quad \text{and} \quad \int \frac{b\,\delta\beta}{\varrho}$$

must be equal, and these sums represent the total angles of rotation of the cylinder $O P Q$, which result from a rolling equal to $P D O$, and a subsequent backward rotation about A and B respectively.

The fundamental law governing the formation of tooth surfaces for spur wheels may therefore be expressed as follows:

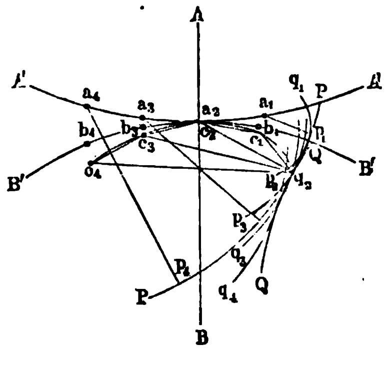
Two tooth surfaces which will work together correctly, i. e., *transmit a constant velocity ratio, can always be generated by an element of any cylinder parallel to the axes, which is rolled without sliding on the axoids A and B of the wheels.* As till more general law governing the formation of tooth surfaces, will be deduced hereafter.

It is hardly necessary to remark that while in *external* gearing the cylinder C simultaneously rolls on the outside of the axoid A and on the inside of B, or the reverse, in *internal* gearing the rolling on the axoids takes place simultaneously either on the inside or the outside of both axoids, so that with internal gearing the curves generated at the same time lie upon *one and the same side* of the line of contact of the pitch circles.

§ **68. General Form of Tooth.**—The general law just developed for spur wheels immediately leads us to the various forms of teeth used in practice, when we make the assumptions corresponding to the particular cases.

We can first suppose the form of a tooth on *one wheel* to be assumed *at pleasure,* for theoretically it is generally possible to find a form of tooth for the other wheel which will work correctly with the arbitrarily assumed form. If PP, Fig. 235, is a cylindrical tooth surface united and parallel to the axis A, we can always find a cylinder c_1, c_2, c_3, c_4 . . . whose rolling on a_1, a_2, a_3, a_4 . . . will generate the tooth surface PP, provided the normals from all points of PP meet the curve a_1, a_2, a_3, a_4 . . . For if we erect at any points p_1, p_2, p_3, p_4 . . . of the

FIG. 235.



profile PP of the tooth, and prolong them until they intersect the cylinder A at a_1, a_2, a_3, a_4 . . , and construct the triangle $c_2 p_2 c_1$ with the three sides

$$c_2 p_2 = a_2 p_2; \quad c_1 p_2 = a_1 p_1; \quad c_2 c_1 = a_2 a_1,$$

also make

$$c_2 c_3 = a_2 a_3 \quad \text{and} \quad c_3 p_2 = a_3 p_3,$$

$$c_3 c_4 = a_3 a_4 \quad \text{and} \quad c_4 p_2 = a_4 p_4, \text{ etc.,}$$

also

$$c_2 c_1 = a_2 a_1 \quad \text{and} \quad c_1 p_2 = a_1 p_1, \text{ etc.,}$$

we then have a polygon $p_2\ c_1\ c_2\ c_3\ c_4$ . . . , or when the distances are small enough, a curve which is the section of a cylinder parallel to the axes whose rolling on the cylinder $a_1\ a_2\ a_3$ . . ., will generate the given tooth surface PP. Hence if this cylinder, $p_2\ c_1\ c_2\ c_3$ . . . is rolled on the inside of the cylinder $b_1\ b_2\ b_3$ . . ., the element p_2 of the rolling cylinder will generate the tooth surface $Q\,Q$ of the other wheel B, which will work correctly with PP.

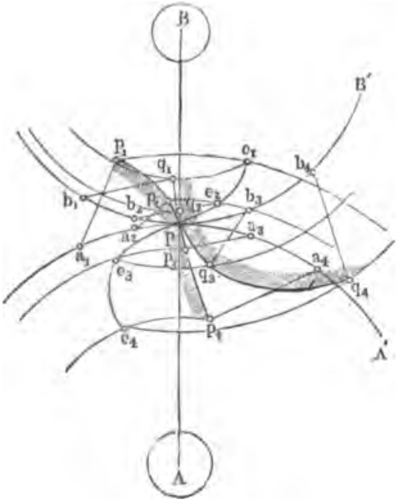
To draw $Q\,Q$, however, it is not necessary to specially construct the auxiliary cylinder $c_1\ c_2\ c_3$ . . . , for $Q\,Q$ may be more easily obtained by describing from the points b_1, b_2, b_3, b_4 . . . (which meet the points $a_1, a_2, a_3, a_4,$ . . .) as centres, and the corresponding normal distances, $a_1\,p_1, a_2\,p_2, a_3\,p_3, a_4\,p_4$ . . . as radii, the circular arcs $q_1\,q_2\,q_3\,q_4$ . . ., for the latter evidently envelop the required curve $Q\,Q$.

The curve $Q\,Q$ thus found, satisfies the kinematic conditions of the motion, but it does not follow from this that it is also practically useful, or even that it can be realized in construction, for it is possible for these curves to intersect each other, in which case they would of course not be applicable to actual teeth. The conditions which the tooth surfaces must satisfy in this respect and in respect to the durability of the teeth will be more fully discussed hereafter.

The method given by *Reuleaux* for finding a tooth surface which will work with another given profile, is shown in

Fig. 236. Let the curve $p_1 p_2 p_3 p_4$ be the given profile on the wheel A, and let the normals $a\,p$ again be drawn for various points of this curve, and find the points of intersection $a_1, a_2, a_3, a_4 \ldots$, of the normals with the pitch circle A' and the corresponding points $b_1, b_2, b_3, b_4 \ldots$, of the pitch circle B'. Now if any one of the points $a_1, a_2, a_3 \ldots$, coincides with the instantaneous centre, contact of the teeth will take place at the corresponding points $p_1, p_2, p_3 \ldots$, of the normals. Consequently, if, with A as a centre, circles be described through $p_1, p_2, p_3 \ldots$, and other circles be described with P as a centre, and the normals $a\,p$ as radii, the points of intersection of the arcs will form a curve $e_1 c_2 e_3 c_4$ which is the locus of the points of contact, and is called the *line of contact*. It is clear that the points $q_1, q_2, q_3 \ldots$, of the required second profile can now be found by describing, through these points c, circles whose centre is at B and then cutting these circles by arcs described from b_1, b_2, b_3, as centres with radii equal to the length of the normals already mentioned, so that

FIG. 236.



$$b_1 q_1 = a_1 p_1, \qquad b_2 q_2 = a_2 p_2 \ldots.$$

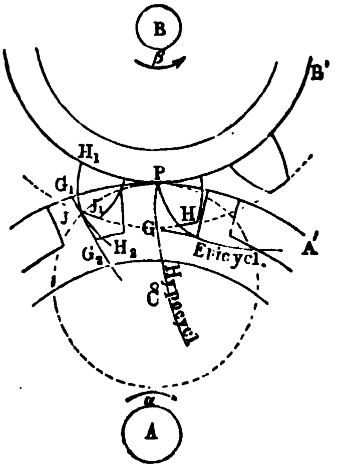
This construction gives us the required profile $q_1 q_2 \ldots$, and at the same time the line of contact $c_1 c_2 \ldots$, the latter being of course cut at right angles by the two profiles at the point of contact, which in the figure is assumed to be the instantaneous centre P. The line joining any point e of this curve with the instantaneous centre P, gives the direction of the pressure exerted by the two teeth upon each other at the instant in which they are in contact at *e*.

The method just given is seldom used in practice, and is of interest only because it shows that for any assumed form of tooth on one wheel we can find the proper form of tooth for the other wheel; this method is suitable principally for cases where for some reason the form of one tooth is prescribed.

§ **69. Cycloidal Profiles.**—In accordance with the law developed in § 67, it is customary in practice to give to the teeth of spur wheels surfaces which can be generated by an element of a cylinder rolled on the axoids of the pair of wheels. The rolling cylinders which are chosen always have *circular* cross sections, and the nature of the generated tooth surfaces can therefore be easily ascertained from the known properties of the roulettes which are generated by a point of a circle rolling on other circles.

Let A and B, Fig. 237, be two parallel axes, and let P be the point of contact of the pitch circles A' and B'. From the point C on the line of centres $A\ B$, describe a circle which will pass through P; then, if this circle is rolled on the outside of B' and on the inside of A', a point J on this circle will generate the *epicycloid* $H_1 J H_2$ while its circle is rolling on B', and the point J will generate the *hypocycloid* $G_1 J G_2$ while the rolling is taking place in A'. These two curves touch at J, and have there a common normal passing through P. Therefore, if we give to the teeth of the two wheels cylindrical surfaces whose elements are parallel to the axes and whose bases

FIG. 237.



are the two cycloids $G_1\,G_2$ and $H_1\,H_2$, they will have correct forms of teeth.

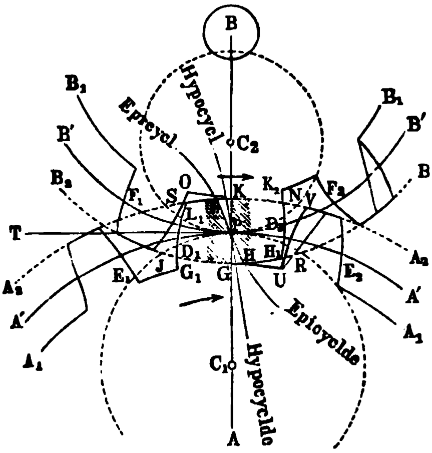
If we suppose the wheel A to drive, and to turn with uniform velocity α in the direction of the arrow, the tooth surface $G_1\,G_2$, which is fixed to this wheel, will act against the tooth surface $H_1\,H_2$ of the wheel B at J, and give to the latter a uniform rotation $\beta = -\alpha\frac{A\,P}{B\,P}$ in the direction of its arrow. During this rotation the point of contact J of the two cycloidal profiles will change its position and gradually reach the point of contact P of the two pitch circles, for when the two pitch circles have turned through equal arcs, $G_1\,P = H_1\,P$, the initial point G_1 of the hypocycloid $G_1\,G_2$ will coincide at P with the initial point H_1 of the epicycloid $H_1\,H_2$. At this instant the line of centres $A\,B$ will be the common tangent to the two curves at P. It is plain from the mode of generating the two cycloidal curves $G_1\,G_2$ and $H_1\,H_2$, that their point of contact while moving from J to P will remain on the circumference of C, for through every point J_1 of this circumference we can imagine two cycloids to be drawn which are identical with the first, and which touch at J_1, the distances of their initial points from P, measured along the pitch circles, being equal. The circle C between J and P is therefore the *line of contact* (or *locus of the points of contact of the curves*), while the pitch circles turn through the equal arcs $G_1\,P_1$ and $H_1\,P$ which lie *in front of the line of centres.* When the point of contact of the two profiles has moved to P in the line of centres, *i.e.*, when the profiles have taken the positions $P\,G$ and $P\,H$, and the uniform rotation is then continued, these profiles can no longer act on each other, and therefore the arrangement must be such that at the instant when two teeth are at $P\,G$ and $P\,H$ and are about to separate, two other tooth surfaces, having, say, the positions $G_1\,G_2$ and $H_1\,H_2$, will begin to act on each other. The point J, at which a new pair of teeth come in contact, is easily found at the intersection of the circle C with the *addendum circle*, *i. e.*, with the circle described from B as a centre with the radius $B\,H$ of the outermost point H of the tooth surface. If we suppose the pitch circles A' and B' to be provided with teeth whose distances apart, measured along the circumferences, are PG_1 and PH_1, and give to the teeth the cylindrical surfaces PG and PH, then a uniform rotation of the wheel A with a velocity α will evidently cause a similar rotation of the wheel B with a velocity β. With this arrangement the teeth will be in contact only *in front* of the line of centres $A\ B$, that is, only while the arcs H_1P and G_1P are passing the line of centres, and while the wheel A drives. But it is also evident that with the same tooth surfaces the wheels can rotate in a direction opposite to the arrows, provided the wheel B drives, the difference being that the tooth PH first comes in contact with PG at P, the point of contact of the profiles traveling from P to J along the circumference of C. The contact of the teeth then takes place *behind* the line of centres, *i. e.*, after the profiles have passed the point of contact P of the pitch circles. The arcs of the pitch circles which pass the line of centres while two teeth are in contact are called the *arcs of action*, and the angles at the centre which subtend these arcs are the *angles of action.*

§ 70. General Form of Cycloidal Teeth.

—The difference between the two wheels A and B, is that the axoid of B has projections with profiles like PH, while the other wheel A has depressions bounded by the profiles PG, the projections of B entering the depressions of A. It is evident that the wheel A could also be provided with projections outside of the pitch circle and the wheel B with corresponding depressions on the inside of the pitch circle, as in the following construction.

If on the two pitch circles $A'\,A'$ and $B'\,B'$, Fig. 238, we roll any circle C_1 on the outside of $B'\,B'$ and on the inside of $A'\,A'$, then from what has preceded we know that a point P of the rolling circle will give the epicycloid PH as the profile for the projections of the wheel B and the hypocyloid PG as the profile for the depressions of the wheel A. If in the same manner we roll any other circle C_2 on the outside of

FIG. 238.



$A'\,A'$ and on the inside of $B'\,B'$, we shall get the epicycloid PL as the profile for the projections of the wheel A and the hypocycloid PK for the corresponding depressions of B. Now if these curves are drawn through the points $F_1\,F_2$ . . ., as well as $E_1\,E_2$ . . ., of the pitch circles, so that the rectified arcs PE_1, PE_2 . . ., are equal to each other and to the rectified arcs PF_1, PF_2 . . ., and if the curves are limited by the circles A_1, A_2, B_1 and B_2, concentric to A and B, we shall get the profiles of all the tooth surfaces that work together. Were the direction in which the wheels revolve invariable, we could give to the backs of the teeth $L_1\,D_1\,G_1$ and $H_1\,D_2\,K_1$ any forms whatever, provided they did not prevent the following tooth surfaces from acting; for example, they could be made flat as shown in the teeth E_2 and F_2. In order that either of the wheels A and B may be employed as driver, whatever the direction of rotation, it is customary to give a form to the teeth of such character that the two sides of a tooth will be symmetrical with reference to the radius passing through the middle of the tooth, *i.e.*, the curves on the two sides will be alike but will be generated by opposite directions of rolling.

It is evident from what precedes that each of the teeth $L\,G\,G_1\,L_1$ and $H\,K\,K_1\,H_1$ can be regarded as composed of two parts, which are separated from each other by the surfaces of the axoids or pitch cylinders. The parts which project beyond the pitch circles (in external gearing), as $P\,L\,L_1\,D_1$ and $P\,H\,H_1\,D_2$, are called the *addenda;* the parts inside of the pitch circles, as $P\,G\,G_1\,D_1$ and $P\,K\,K_1\,D_2$, are called the *roots;* the tooth surface above the pitch circle, as PL, $D_1 L$ and PH, $D_2 H_1$, are called the *faces* of the teeth, and the parts below, as PG, $D_1 G_1$ and PK_1, $D_2 K_1$, the *flanks* of the teeth; the space between two teeth, as $L G U V$, is simply called the *space.* The innermost circles of the teeth, as $A_1 A_1$ and $B_1 B_1$, are called the *root circles,* the outermost circles, as $A_2 A_2$ and $B_2 B_2$, are called the *addendum circles.* The distance between the root circle and the addendum circle is called the *length* of the tooth, the dimension of the tooth in a direction parallel to the axes is called its *breadth.* By the thickness of a tooth is understood the dimension of the tooth which is measured along the pitch circles, for example, the arcs PD_1 and PD_2 respectively; in most cases it would be sufficiently accurate to substitute the chords of the arcs for the arcs themselves. By *pitch of the tooth,* or simply *the pitch,* we are to understand the arcs of the pitch circles included between two similarly situated tooth surfaces for example, the arcs PE_1, PF_1, PE_2, PF_2. . . . From what has been said we know that these arcs are equal in the two wheels that work together, there being on the wheels no other two circles in contact in which the distances of the teeth from each other on one wheel are equal to those on the other. This evidently depends on what has already been said of the axoids, namely, that in them, *i.e.*, in the pitch cylinders, the velocities of the two wheels are equal. Although the pitch circles are not evident to the eye in actually constructed wheels, they have nevertheless *decided significance* for the whole motion, and are *by no means* to be regarded as *something arbitrary.*

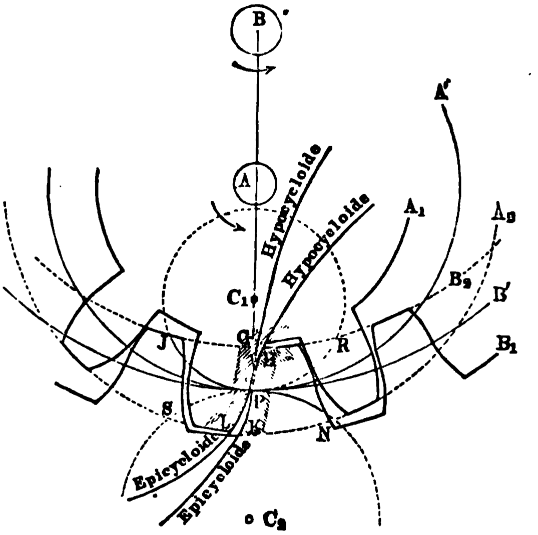
The generating circle C_1 is, according to the above, the line of contact or locus for the points of contact of two tooth profiles, consequently the point of intersection J of this circle with the addendum circle B_2 of the wheel B, will give the point at which the top of the tooth on B will be first touched by the flank of the tooth on the wheel A. After the wheels have turned so far that the two profiles touch on the line of centres, that is, at P, further motion will not be accompanied by cessation of action of these teeth, as was the case in Fig. 237, where the teeth of A had flanks, and those of *B* faces, only; for now the face *PL* will act on the flank *PK* and will cause the wheel *B* to turn till the tooth of *A* leaves the tooth of *B* at *N*. This point *N* will be found at the intersection of the second generating circle C_2 with the addendum circle A_2 of the wheel *A*, for it is evident that for the just described action of *PL* on *PK*, the generating circle C_2 will be the line of contact. When, therefore, the teeth of each wheel have faces as well as flanks, the teeth will be in action both *before* and *behind* the line of centres; for when *A* drives, as in the present case, the point of contact will travel from *J* through *P* to *N*, while when the direction of rotation is opposite and *B* is the driver, the point of contact will travel in the opposite direction on the generating circles, namely, from *N* through *P* to *J*. It is also clear that if this opposite rotation is effected by *A* as a driver, the symmetrical form of the teeth will cause the point of contact of the tooth profiles to travel from *R* along the circle C_1 to *P*, and from there along the circle C_2 to *S*, the point of contact having the opposite path *SPR* when the wheel *B* turns *A* in the direction of the arrow. When, therefore, the profiles of the teeth touch at some point of the double arcs *JPN* and *RPS* respectively, we can find the direction of the pressure between the teeth for the instant by drawing a line between this point and the point of contact *P* of the pitch circles.

We see from this that the direction of pressure is perpendicular to the line of centres for a single instant only, namely, when contact of the profiles takes place on the line of centres, and that in general the direction of pressure will deviate from the normal direction *TP*, the deviations extending to the directions *JP*, *PN* and *RP*, *PS* respectively. These deviations are evidently greater the farther the points *J*, *N*, *S* and *R* are from *P*, or the more the teeth project beyond the pitch circles. This shows that the construction in which the teeth have both faces and flanks is more advantageous than the construction shown in Fig. 237, where the teeth of one wheel are without faces, and those of the other without flanks, for in the latter case, the deviation of the direction of pressure from the normal *TP* is greater. This deviation causes friction on the shafts and, as will be shown later, increased friction between the teeth.

It follows from the foregoing remarks, that two teeth act on each other in such a way that the face of one tooth works with the flank of the other, and that faces can never work with faces, nor flanks with flanks, if the condition of uniform transmission of motion is to be satisfied.

That the forms of the teeth for *internal gearing*, Fig. 239, will differ from those found for external gearing, follows from what was said in § 67. Since here the axes of two wheels lie upon *the same* side of the common tangent plane, the describing cylinder must be rolled on the outside of both axoids or on the inside of both. in order to generate the tooth surfaces which work together. Hence the profile of the wheel A will have for its flank the hypocycloid generated by the describing circle C_1 on A' and for its face the epicycloid generated by the describing circle C_2 on A', as in the preceding case, but for the wheel B the profile of the tooth will have the *hypocycloid* described by C_1 in B' for its *face* and the *epicycloid* of C_2 on B' for its *flank*. The double arc JPN of the describing circles C_1 and C_2 is again the locus of the points of contact of the tooth curves, the point of contact traveling from J through P to N when the rotation is in the direction of the arrow, and A is the driver. When the direc-

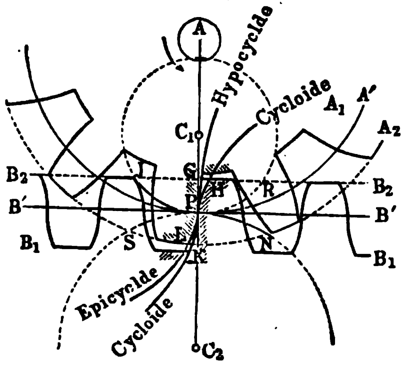
FIG. 239.



tion of rotation is the same and the larger wheel *B* is the driver, the contact of two profiles will take place along the line of contact *SPR*.

The case in which one of the wheels has an infinite radius and becomes a rack may be regarded as intermediate between the two cases of external and internal gearing. Here also the proper profiles for the teeth can be deduced from the general law, *PH* and *PK*, Fig. 240, representing the face and flank of a tooth of the rack, these two curves being ordinary cycloids generated by the describing circles C_1 and C_2, and the profiles of the teeth on the wheel *A* remaining the same as before. The straight line *B′ B′* is evidently to be regarded as the pitch circle of the rack, in the same manner the straight lines $B_1 B_1$ and $B_2 B_2$ take the place of the root circle and addendum circle; the line of contact will therefore be *JPN* or *SPR*, according as the wheel *A* or the rack *B* is the driver, the direction of motion being that represented by the arrow.

FIG. 240.



§ 71. **Teeth with Radial Flanks.**—The results thus far developed are independent of the size of the describing circles, consequently the cycloidal profiles obtained will form correct tooth surfaces however large the generating circles are taken. By making certain assumptions with reference to these describing circles, we shall get special forms of teeth of importance in practice, which we will now examine. Since a hypocycloid becomes a straight line and a diameter of its pitch circle when the diameter of the rolling circle is half that of the pitch circle, we have here a means of forming correct tooth surfaces which are partially *plane*.

For if we take the radius *A P*, Fig. 241, of the pitch circle of A as the diameter of the describing circle, we shall get for the flank of the tooth A, a flat surface which is projected into the radius PG; in like manner the flank of the tooth on B will have the radial direction PK, when its describing circle C_2 has a diameter equal to the radius $B\ P$ of the other pitch circle. The flanks of the teeth are, of course, the epicycloids generated by rolling C_1 on B' and C_2 on A'. This *radial flank gearing* was formerly much used for the wooden wheels of mills, etc., we therefore find in the corresponding text books the rule laid down that the flanks of teeth should be radial and their faces epicycloids generated by rolling on each pitch circle a describing circle, whose diameter is equal to the radius of the other pitch circle.

FIG. 241.

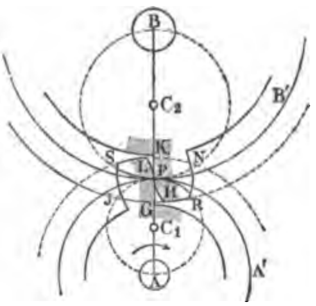


FIG. 242.

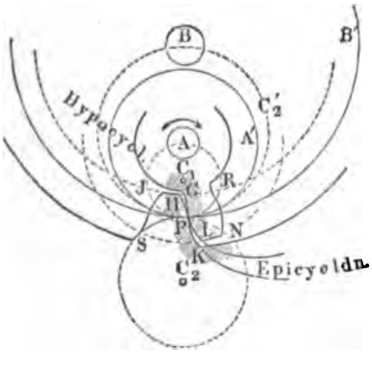
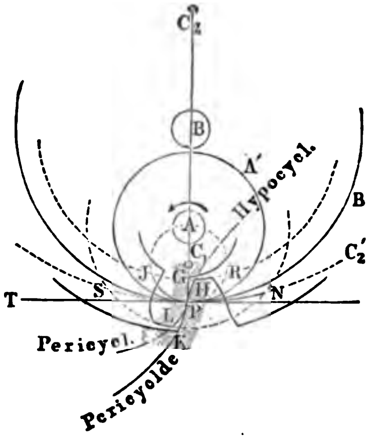


FIG. 243.



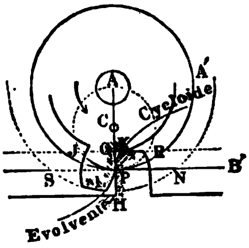
Also in internal gearing, Fig. 242, the smaller wheel A may be given a radial flank PG, in which case the face of the larger wheel will be the *hypocycloid* generated by rolling the circle C_1 in the pitch circle B', the diameter of the rolling circle being equal to the radius $A P$. The faces of A and the flanks of B may be made epicycloids generated by rolling any describing circle C_2 on the pitch circles. On the other hand, the faces of the larger wheel B can be made radial planes *only when the diameter of the smaller wheel A exceeds the radius of the larger wheel.* The reason for this is that the radial face of the larger wheel corresponds to a flank of the smaller wheel generated by rolling a circle C_2 of diameter $B P$, on the *inside* of the pitch circle A'. Now if the rolling circle C_2 is greater than the pitch circle A', the rolling will produce a *pericycloid*, *i.e.*, a curve which lies entirely *outside* of the pitch circle A' and consequently cannot be used for the flanks of A which lie on the *inside* of this pitch circle. Since the faces of the teeth of one wheel can only work with the flanks of the teeth of the other, it is evident that the general law for forming tooth curves by rolling holds good only *so long* as the two pitch circles at their point of contact lie on the *same* side of the rolling circle, but does *not* hold when the circumference of the rolling circle lies *between* the pitch circles, as is the case with C'_2, Fig. 242, whose circumference lies between A' and B'.

From this follows that in external gearing, Fig. 238, the two describing circles C_1 and C_2 can be arbitrarily chosen only within certain definite limits, which are determined by the condition that neither of these circles shall be larger than the pitch circle in which it is rolled; for example, C_1 cannot be taken greater than A', nor C_2 greater than B', in Fig. 238. On the other hand, it follows from this condition that with internal gearing, Fig. 243, instead of making the faces of the smaller wheel A and the flanks of the larger wheel B *epicycloids*, as was assumed before, we can give them the forms of the two *pericycloids* $P L$ and $P K$, which are generated by rolling on the inside of the pitch circles A' and B' a circle C'_2, whose diameter is greater than the larger pitch circle B', for in this case the elements of the pitch circles at the point of contact evidently lie on the same side of the rolling circle. If we make the flanks of the wheel A radial, and consequently the faces of the wheel B hypocycloids generated by rolling the circle C_1 of diameter $A\,P$, in the pitch circle B', evidently the line of contact will be the double arc $J\,P\,N$ or $S\,P\,R$, according as A or B is the driver.

If we suppose the circle C'_2 to gradually increase until it becomes infinite, the circumference C'_2 will finally become a straight line, namely, the common tangent $T\,P$ and the tooth profiles $P\,L$ and $P\,K$ will now be the *involutes* of the pitch circles. The point of contact of these curves will then travel along the straight line to P, during which time the direction of the pressure of the two teeth will be constantly normal to the line of centres $A\,B$, and therefore work very favorably, because there is no component of the pressure acting against the axes in the direction of the line of centres. It is clear that in this case the involutes are in a certain sense the limit for the epicycloids, Fig. 242, and the pericycloids, Fig. 243.

If we apply radial flank gearing to racks, Fig. 244, evidently the radial flanks $P\,G$ of the wheel A will work with the faces $K\,P$ of the rack, which are ordinary cycloids generated by the circle C of diameter $A\,P$. The rack itself may be regarded as a wheel of infinite diameter, the pitch circle being the straight line $P\,B'$, from which immediately follows that the radial flanks $P\,H$ of the rack are normal to $P\,B'$, and work with the involute faces $P\,L$ of the wheel A'. The line of contact is either $J\,P\,N$ or $S\,P\,R$, according as the wheel or the rack drives. The tangential position of the portion $P\,N$ of this line of contact shows that only one point P of the flank $P\,H$ of the rack comes in contact with the points of the face $P\,L$, so that the form of $P\,H$ is unimportant. Of course, with this construction, the wear of the teeth of the rack will be very great, because the pressure is constantly concentrated on the same point.

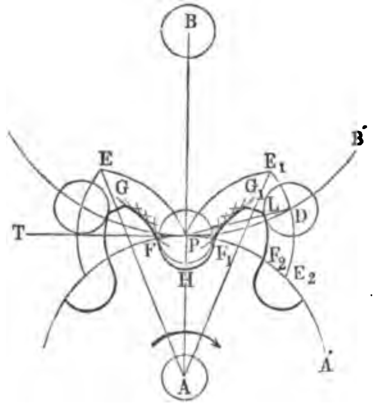
FIG. 244.



§ 72. Pin Gearing.

—If in two wheels A and B, Fig. 245, which gear externally, we give to the describing circle C_2 the greatest possible diameter, *i.e.*, make it equal to the pitch circle B', the curve described by one of its points P, will reduce to the point P itself, while the faces of the wheel A will evidently be epicycloids $P\,E$ or $P\,E_1$, generated by rolling C_2 or B' on A'. These epicycloids may therefore be employed when the wheel B has a tooth at P which is infinitely small. Sometimes in practice wheels are constructed with ***pins*** instead of teeth, *i.e.*, with cylindrical staves of circular cross section, as $P\,F$, such wheels being called *pin wheels* or ***trundles.*** The proper form of tooth for the wheel A, which will work with these pins, can be easily obtained from the epicycloid $P\,E$ by finding its equidistant curve $F\,G$, whose normal distance from $P\,E$ is everywhere equal to the radius $P\,F$ of the pin. The profile of the tooth is therefore the curve $F\,G$ or $F_1\,G_1$ which envelops the circular arcs described from the points of the epicycloid $P\,E$ or $P\,E_1$ as centres with a radius equal to that of the pin. Inside the pitch circle the wheel A must, of course, be provided with recesses $F\,H\,F_1$ sufficiently large to admit the pins, the profiles of the recesses being unimportant, as they do not come in contact with the pins. That the curve $F\,G$ thus obtained will give a correct tooth surface for the teeth of A may easily be shown. If the pin were infinitely small and therefore only a straight line, the epicycloid $P\,E$ would be the proper tooth curve, and for any position of the pin, say D, the straight line $P\,D$ would be normal to the epicycloid $E_1\,D\,E_2$ passing through D, consequently the straight line $P\,D$ is also perpendicular to the curve $L\,F_2$ parallel to the epicycloid; in other words, the fundamental condition for uniform transmission of motion is satisfied. The tooth $E\,P$ will commence to act on the pin P when the latter is on the line of centres,

FIG. 245.



and therefore the equidistant curve $F\,G$ will come in contact with the pin of radius $P\,F$ at the same instant and at a point F, which is found by laying off the radius of the pin $P\,F$ on the normal to the epicycloid at P, *i.e.*, on the tangent $P\,T$. In order that the motion may be continuous, it is now necessary that the preceding tooth $F_1\,G_1\,L\,F_2$ drive the pin D until $F\,G$ comes in contact with P at F. The height of the tooth of the wheel A necessary to accomplish this, $P\,D$ being the distance between the pins, may be found by taking the intersection L of the straight line $P\,D$ with the profile $L\,F_2$ as the outermost point of the tooth, the teeth being extended at least far enough to reach the circle described through L with A as a centre.

For internal pin gearing the forms of the teeth are found in exactly the same manner as for external gearing. If the pinion A, Fig. 246, receives the pin, the teeth of the wheel must be formed according to the curve $F\,G$, which is an equidistant to the hypocycloid obtained by rolling the pitch circle A' in B'; but when the pins are given to the larger wheel B, Fig. 247, the form of the tooth is determined by the equidistant $F\,G$ to the pericycloid obtained by rolling the pitch circle B' about A'.

FIG. 246.

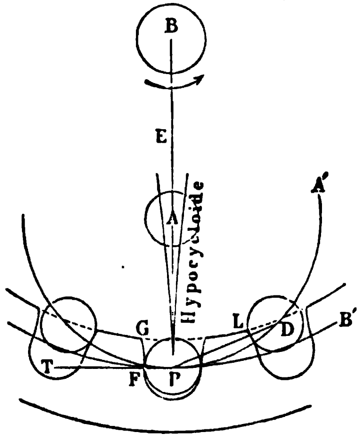
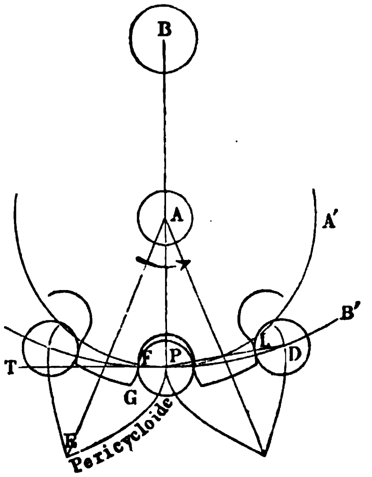


FIG. 247.



If a rack B gears with a trundle A, Fig. 248, the form of the tooth is determined by a curve $F\ G$ parallel to the cycloid $P\ E$ of the pitch circle A', while for the case represented by Fig. 249, where the pins are given to the rack, the teeth are formed according to a curve $F\ G$ which is parallel to the involute $P\ E$ of the pitch circle A'. In all the cases represented from Fig. 245 to Fig. 249, the intersection L of the straight line $P\ D$ with the profile of the curve gives the minimum height of the teeth. In practice it is customary to give the *teeth* to the *driver* and the *pins* to the *follower* or *driven wheel*, the reason for which will be given hereafter, when the friction of teeth is discussed. As may be seen from the figures, motion is transmitted principally *behind the line of centres* between P and L, and through a shorter distance before the line of centres between F and P, $P\ F$ being always smaller than $P\ L$.

FIG. 248.

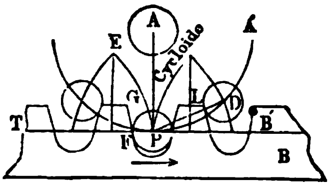
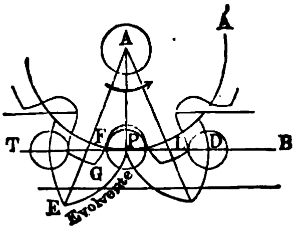
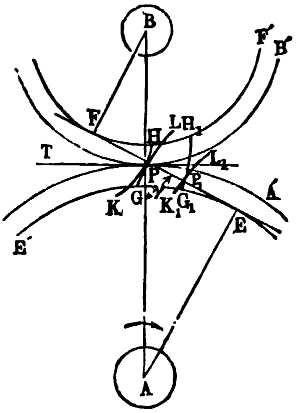


FIG. 249.



§ **73. Involute Gearing.**—If we give to the rolling circle which describes the profiles of the teeth an infinite diameter, it will become a straight line, namely, the common tangent $P\ T$, Fig. 250, whose rolling on the pitch circle will generate involutes of the latter. These involutes cannot be employed for tooth profiles, at least not for external gearing, because they lie wholly outside of the pitch circles, and can therefore be employed only for the faces of the teeth, which, of course, do not work together. That involutes of the pitch circles can be employed with internal gearing for the faces of the pinion and the flanks of the wheel, has already been shown in § 71.

FIG. 250.



But the involutes of circles may be employed for the profiles of teeth of wheels in external gearing, as the following discussion will show. If through the point of contact P of the pitch circles a straight line $E\,F$ be passed, making the acute angle $A\,P\,E = \gamma$ with the line of centres $A\,B$, and if from A and B as centres we describe two circles touching $E\,F$ at E' and F', the involutes to these two circles will likewise give correct tooth profiles. By rolling the straight line $E\,F$ on these circles, the point P will generate the involute $G\,P\,L$ to E' and the involute $H\,P\,K$ to F', both curves touching at the instantaneous centre P. It may now be easily shown that, for any rotation of the shafts A and B with the angular velocities α and β, these curves, which are fixed to their respective wheels, will touch each other at only one point of the fixed straight line $E\,F$, and that this straight line is normal to these curves at their point of contact and passes through the instantaneous centre P. For this purpose it is only necessary to call to mind the method employed in generating the involutes. Evidently the element of the involute arc $P\,G$ which contains the point P is generated by an instantaneous rotation of the straight line $E\,P$ about E, the rotation being left-handed (contrary to the hands of a watch), and equal to α. For the rotation we may substitute (Introduction, § 4) an equal left-handed rotation about A and a translation $\alpha \times A\,E$ perpendicular to $A\,E$, in the direction from P to E. If we now imagine the whole system, *i.e.*, the wheel A and the straight line $P\,E$, to receive an additional right-handed rotation α around A, the resultant motion of the straight line $P\,E$ will simply be a translation from P to, say P_1, which is expressed by $P\,P_1 = \alpha \times A\,E$. Similar considerations apply to the wheel B, for the element of the involute arc PK containing P is generated by a small right-handed rotation β of FP about F. For this rotation we may again substitute an equally large right-handed rotation about B and a translation $\beta \times BF$ in the direction P to E. If we again suppose the system to receive an additional left-handed rotation β about the axis B, the resultant motion of the straight line FP and of course of the point P will be simply this translation $\beta \times BF$. If now α and β have been chosen in accordance with the required motion, *i.e.*, $\alpha : \beta = BP : AP$, or if $\alpha \times AP = \beta \times BP$, we shall have the two translations $\alpha \times AE$ and $\beta \times BF$ equal, because

$$\frac{AE}{AP} = \frac{BF}{BP} = \sin \gamma.$$

From this follows that the point P of the straight line EF is shifted equal amounts in both cases along FE, namely, $PP_1 = \alpha \times AE = \beta \times BF$, when the wheels are turned through α and $-\beta$ respectively. Since the elements at P_1 of the involute arcs $G_1P_1L_1$ and $H_1P_1K_1$ have a common normal FE at P_1, which passes through the instantaneous centre P, it follows from the principle established in § 67 that involutes may be employed as correct profiles for teeth.

It is also plain from what has preceded that the oblique line FE is the locus of contact of the two profiles, and consequently that the force exerted by the teeth on each other has always the same direction FE. Supposing this force to be resolved into rectangular components, it is evident that the component in the direction of the line of centres AB will become greater the more acute the angle $APE = \gamma$, and as this component creates hurtful pressures in the bearings of the shafts A and B, the angle γ must not be taken too small, not less than 75°.

The construction of the teeth now readily follows. If in wheels with external gearing, Fig. 251, we pass through the point of contact P of the pitch circles the straight line EF so as to make the angle

$$\gamma = APE = 75^\circ \text{ to } 80^\circ,$$

and roll it on the two circles E and F (base circles) in contact with it, the resulting involutes to these circles will be the profiles for the tooth surfaces of the wheels. Each side of a tooth will therefore be composed of a single continuous curve, the methods employed in preceding articles giving special curves for the faces and flanks of the profiles. At the points of intersection of the addendum circles A_2 and B_2 with the lines of contact FE and $F_1 E_1$, we have the points S, R, N and J, at which contact of the teeth begins or ceases, so that when the arrow represents the direction of rotation, the point of contact of two teeth will travel from N to J when B is the driver, and from R to S when A is the driver.

FIG. 251.

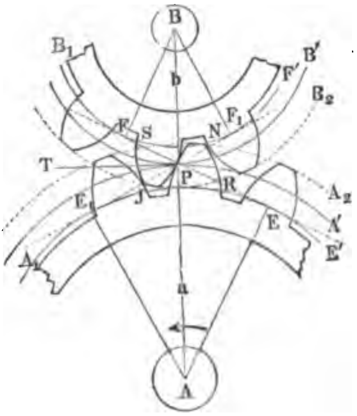
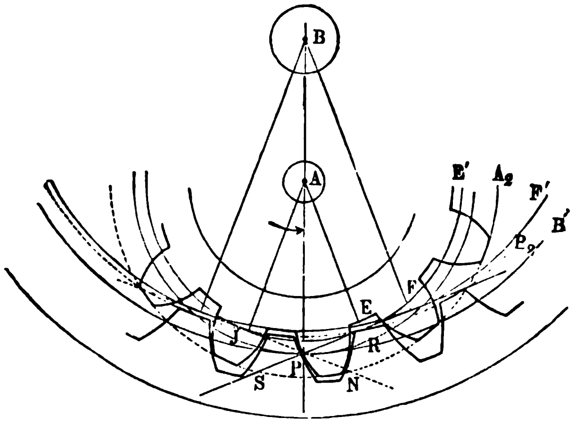


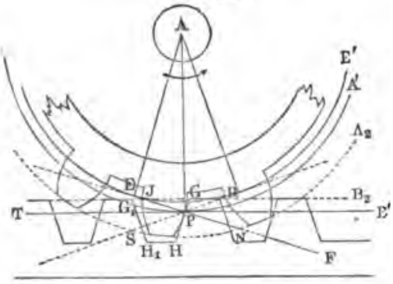
FIG. 252.



Evidently these remarks hold for wheels which gear internally, Fig. 252, and also for the rack and pinion, Fig. 253.

The rack may again be regarded as an infinitely large wheel, consequently that circle of this infinite wheel which is in contact with EF will become a straight line coinciding with EF. By rolling this line EF on itself, the point P will describe a straight line PH normal to EF at P, whence follows that the surfaces of the teeth of the rack will be planes GH and G_1H_1, which on each side of the tooth are inclined to the rack B' at an angle γ. As the point of contact of two teeth here travels from J to N or from R to S, we see that for this form of profile, the pressure of a tooth is gradually exerted over a greater surface, and is not concentrated on a single point, as was the case with the rack, Fig. 244, in which, according to § 71, the faces of the wheel were likewise involutes, but involutes to the pitch circles.

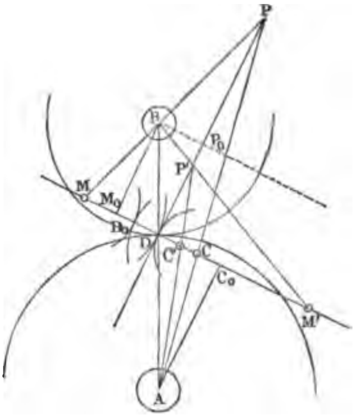
FIG. 253.



§ 74. **Approximations by Circular Arcs.**—As comparatively small portions only of the curves determined in the preceding articles are employed for the profiles of teeth, we may in practice substitute for these curves circular arcs which are close approximations to the exact curves, a suitable choice of centres and radii for the arcs giving sufficient accuracy for most cases. To do this we can construct the exact profile of a tooth by some one of the methods already given, and then find by trial a circular arc which will approximate as closely as possible to this profile. This method will always solve the problem, but suitable arcs may be found without first constructing an exact profile, as the following discussion will show.

Let A and B, Fig. 254, be the axes of two wheels, and D the instantaneous centre, and let us also assume that two teeth which work together have profiles which are circular arcs with centres at C and M respectively. The connecting line CM will evidently be the common normal to the tooth profiles at their point of contact, *i.e.*, the straight line CM is the direction of pressure and consequently should pass through the instantaneous centre D. When the wheels turn this straight line will not change its length, which is equal to the sum of the radii of the arcs, so long as the arcs remain in contact. According to § 7 of the Introduction, the motion of the line CM at each instant may be regarded as a slight rotation about the instantaneous centre, the latter lying at the intersection of the normals erected to the paths of the two points C and M. As these normals coincide with the radii AC and BM, the intersection P of the latter will give the instantaneous centre for the motion of the line CM.

FIG. 254.



The transmission of motion by means of circular profiles will be uniform during a slight motion, if the common normal CM to the profiles passes through the point of contact D of the pitch circles. This will only occur when the instantaneous radius PD is normal to the straight line CM, *i.e.*, when the instantaneous motion of the point D takes place in the direction CM.

This gives us the following construction. Through the point of contact D of the pitch circles, draw for the direction of pressure a straight line MC, making with the line of centres an angle ADC which is not too acute (say not less than 75°). At D erect the perpendicular DP, then unite any point P of this perpendicular with the wheel centres A and B. At the intersections C and M of these lines with the assumed direction of pressure, we shall have the centres of two arcs which pass through the same point of CM, and, for the instant, satisfy the required condition of uniform transmission.

These arcs may be struck through any point of CM; if we pass them through D, perfectly uniform transmission of motion will take place when the teeth touch on the line of centres, that is, at D; this also holds when the teeth are in contact at any point D_1 before or behind the line of centres through which the circular arcs have been passed. For all other positions of the teeth the transmission of motion is, of course, only approximately uniform.

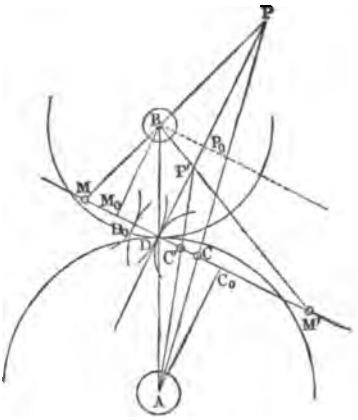
In the assumed case, the two centres C and M lie on opposite sides of the line of centres $A\,B$, and the two arcs may be said to have a convexo-convex contact. But if instead of the point P, we suppose some other point P' of the perpendicular DP to be chosen as the instantaneous centre, so that the instantaneous radii $P'A$ and $P'B$ shall cut the line of pressure CM on the same side of D, we shall get at C' and M' centres for the circular tooth profiles which are subject to a convexo-concave contact. Finally, by choosing the instantaneous centre on DP at P_0, where the instantaneous radius BP_0 is parallel to CM, the centre C will be infinitely distant, *i.e.*, the profile of the tooth of B will be perpendicular to CM, and the centre of the profile for the tooth on A will be at the intersection of $A\,P_0$ with CM.

If we assume the instantaneous centre P to be infinitely distant from D, the instantaneous radii to the wheel centres A and B will evidently be parallel to DP or perpendicular to CM, consequently C_0 and M_0 will be the centres for the two circular profiles. These points coincide with the centres of curvature of the two involutes at their point of contact, these involutes being drawn, as explained in the previous article, for the direction of the pressure CM, and for the instant at which they are in contact with the very point D or D_1 through which the arcs have been described. From the last remarks we can deduce a simple construction for finding the required centres C_0 and M_0 of the two arcs which are to be employed as approximate profiles for teeth, it being only necessary to project the wheel centres A and B on the assumed direction of pressure CM; the projections C_0 and M_0 are the required centres, and if we pass the arcs through the point of contact

D of the pitch circles, the teeth will transmit perfectly uniform motion at the instant in which they are in contact on the line of centres *A B*.

To determine the centres *C*, *M* and *C'*, *M'* respectively, let (Fig. 255) *a* and *b* be the radii of the pitch circles of the wheels *A* and *B*, and let *p* represent the distance of the instantaneous centre *P* from *D*, the required distances *C D* and *M D* for *convexo-convex* contact being represented by $\mathfrak{a}$ and $\mathfrak{b}$ respectively. We then have:

FIG. 255.



$$\mathfrak{a} = D\,C_0 - C\,C_0 = a \cos \gamma - C\,C_0,$$

or, since from

$$\mathfrak{a} : p = C\,C_0 : a \sin \gamma,$$

we get

$$C\,C_0 = \frac{\mathfrak{a} \times a \sin \gamma}{p},$$

we have

$$\mathfrak{a} = a \cos \gamma - \frac{\mathfrak{a} \times a \sin \gamma}{p},$$

and

$$\mathfrak{a} = \frac{p \times a \cos \gamma}{p + a \sin \gamma}.$$

We also have

$$\mathfrak{b} = D\,M_0 + M_0\,M = b \cos \gamma + M_0\,M,$$

or, since from

$$\mathfrak{b} \cdot p = M_0\,M : b \sin \gamma,$$

we get

$$M_0\,M = \frac{\mathfrak{b} \times b \sin \gamma}{p},$$

we have

$$\mathfrak{b} = b \cos \gamma + \frac{\mathfrak{b} \times b \sin \gamma}{p},$$

and

$$\mathfrak{b} = \frac{p \times b \cos \gamma}{p - b \sin \gamma}.$$

In exactly the same manner we find for the *convexo-concave* contact, when the distance DP' is represented by p', and the required distances DC' and DM' by $\mathfrak{a}'$ and $\mathfrak{b}'$ respectively:

$$DC' = \mathfrak{a}' = \frac{p' \times a \cos \gamma}{p' + a \sin \gamma},$$

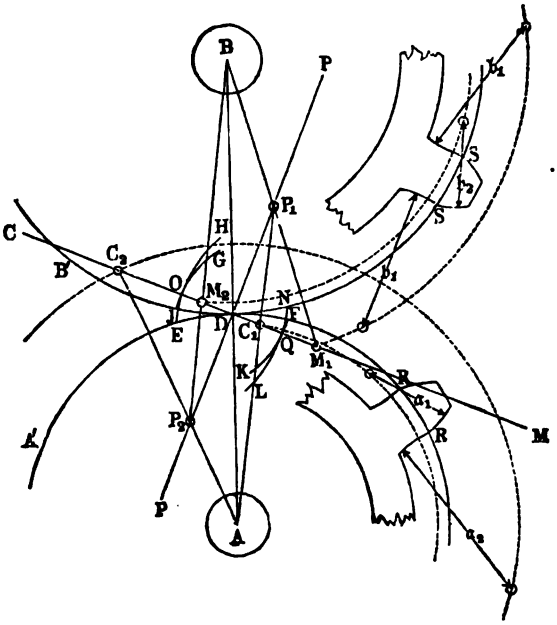
and

$$DM' = \mathfrak{b}' = \frac{p' \times b \cos \gamma}{b \sin \gamma - p'}.$$

These formulas may be employed to determine the centres of curvature when p and p' are known, instead of determining them by graphical construction as detailed above. For $p = b \sin \gamma$ we get $\mathfrak{b} = \infty$, and for $p = \infty$ we get as above $\mathfrak{a} = a \cos \gamma$, and $\mathfrak{b} = b \cos \gamma$.

In what has preceded it has been tacitly assumed that the tooth profile of each wheel could be replaced by a single circular arc. In this case the tooth profile, consisting of a single curve, presents a certain similarity to involute teeth, and perfectly uniform transmission of motion between the two wheels only takes place for a single instant, namely, when the two arcs touch on the line of pressure MC. But each tooth profile can also be composed of two circular arcs, one for the flank and the other for the face, the two arcs in a certain sense corresponding to the two different curves of cycloidal teeth. In this way two tooth surfaces which work together can be made to transmit perfectly uniform motion at two different instants, namely, when one pair and then the other pair of circular arcs which work together come in contact on the assumed line of pressure CM. This will, of course, give a closer agreement between the circular profiles and the exact ones, particularly if the two points of exact transmission just mentioned be assumed at equal distances, say half the pitch of a tooth, *before* and *behind* the line of centres. To accomplish this we can evidently make use of the following construction. Through the point of contact *D* of the two wheels, Fig. 256, draw the line of pressure *C M*, making with the line of centres *A B* an angle *A D M* of, say, 75°, and erect the normal *P P* to *C M*, then will the two lines drawn from *A* and *B* to

Fig. 256.



any point P_1 of this normal contain at C_1 and M_1 the centre for the face of *A* and for the flank of *B* respectively. If we choose a point *E* on the pitch circle *A'* at a distance from *D* equal to half the pitch, and pass through this point an arc *E G* with C_1 as a centre, it will give the face of a tooth on *A*, while an arc *J H* described from M_1 as a centre and touching *E G* at its intersection *O* with *C M*, will give the flank of a tooth on B. In the same way, by drawing the two lines $A\,P_2$ and $B\,P_2$ through another point P_2 of the straight line PP, we shall get, at their intersections C_2 and M_2 with CM, two centres for the arcs which form the flanks of A and the faces of B respectively. We accordingly pass through the point F of the pitch circle A', which is at a distance from D equal to, say, half the pitch, the arc FL with C_2 as centre for the profile of the flanks on A, and through the intersection Q of the arc with $C\,M$ we pass an arc $N\,K$ with M_2 as a centre, which latter arc will serve as profile for the faces of the wheel B. With the radii thus found we can now easily draw all the teeth, for the circles through C_1 and C_2 concentric with A are the loci of the centres of the two arcs forming the profile of the teeth on A, and the circles through M_1 and M_2 concentric with B are the corresponding loci for the teeth of B. The radii of these circles are shown in the figure, $\mathfrak{a}_1 = C_1\,O$, $\mathfrak{a}_2 = C_2\,Q$; $\mathfrak{b}_1 = M_1\,O$; and $\mathfrak{b}_2 = M_2\,Q$. These arcs, however, are not tangent to each other at the points R and S, where they meet on the pitch circle; it will therefore be well to slightly round the corners thus made.

REMARK. We can also assume for the radius of the arc the radius of curvature of the exact profile at the point in which the transmission of motion is to be perfectly uniform. If this procedure is to be applied to, say, cycloidal profiles, which are to be replaced by approximate arcs, the radii of the latter may be found by calculation as follows:

If a is the radius of a pitch circle and β the radius of the describing circle which is rolled on the outside or inside of the former, we can determine for the epicycloid and hypocycloid thus generated, the radius of curvature at a point whose angular distance from the point of contact of the rolling circle with the pitch circle is equal to φ, from the formula

$$r = 4\,\beta\,\frac{a \pm \beta}{a \pm 2\beta}\,sin\,\frac{1}{2}\,\varphi,^*$$

the positive signs being used for epicycloids and the negative signs for hypocycloids; the angle φ being measured at the centre of the rolling circle. If we again suppose the point at which the circular arc coincides with the correct profile to be at a distance from the instantaneous centre equal to half the pitch $\frac{t}{2}$, we shall have for this point $\beta\,\varphi = \frac{t}{2}$, and therefore $\frac{\varphi}{2} = \frac{t}{4\beta}$.

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* See *Zehme*, Elementare und analytische Behandlung der verschiedenen Cycloiden.

If, because the angle is small, we take $\sin \frac{1}{2}\,\varphi = \frac{1}{2}\,\varphi$, we shall have

$$r = \frac{a \pm \beta}{a \pm 2\beta}\,t.$$

From this formula we get the approximate radii $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{b}_1$, and $\mathfrak{b}_2$ for two wheels having the radii a and b, when the generating circles have the radii β_1 and β_2, namely:

$$\mathfrak{a}_1 = \frac{a + \beta_1}{a + 2\,\beta_1}\,t; \qquad \mathfrak{a}_2 = \frac{a - \beta_2}{a - 2\,\beta_2}\,t;$$

$$\mathfrak{b}_1 = \frac{b + \beta_2}{b + 2\,\beta_2}\,t; \qquad \mathfrak{b}_2 = \frac{b - \beta_1}{b - 2\beta_1}\,t.$$

For gearing with radial flanks $\beta_1 = \frac{1}{2}\,b$ and $\beta_2 = \frac{1}{2}\,a$; hence we have for the radii of the faces:

$$\mathfrak{a} = \frac{a + \frac{b}{2}}{a + b}\,t = \frac{n + \frac{1}{2}}{n + 1}\,t = \frac{2n + 1}{n + 1}\,\frac{t}{2},$$

and

$$\mathfrak{b} = \frac{b + \frac{a}{2}}{b + a}\,t = \frac{1 + \frac{n}{2}}{1 + n}\,t = \frac{2 + n}{n + 1}\,\frac{t}{2},$$

$n = \frac{a}{b}$ representing the velocity ratio.

§ 75. Interchangeable Wheels.—In what has preceded it has been tacitly assumed that a wheel A always works with a certain wheel B, and that the motions of the two always have the particular ratio $\alpha : \beta$ which is equal to the inverse ratio $b : a$ of the radii. Now, in practice it is often necessary that any two of a set of wheels $A, B, C, D \ldots$ shall work with each other, so that in each case the ratio of their velocities of rotation $\alpha, \beta, \gamma, \delta \ldots$ shall equal the inverse ratio of their radii $a, b, c, d \ldots$ For this purpose it is first necessary that the pitch, or distance apart of the teeth on the pitch circles, shall be the same for all the wheels of the set, moreover, it is necessary that the tooth profiles of any two wheels shall work together correctly, *i.e.*, satisfy the condition of uniform transmission of motion. Such wheels are called *interchangeable wheels.* The practical importance of this class of wheels will be evident when we remember that toothed wheels, especially the larger ones, are cast from accurately made patterns, and that the number of expensive patterns would become exceedingly large if each wheel could only work with one particular wheel, for the number is already large on account of the different diameters and different pitches employed. The principles according to which interchangeable wheels must be formed may be easily deduced. It is clear that if two wheels A and B formed by one of the above methods work together correctly, one of them, for example, A, *will not work correctly with a third wheel C different from B, if the size or radius of the wheel B has had any influence in determining the tooth profile of A.* Only for that case in which the tooth curves depend solely on the *dimensions of their own wheel* is it possible to construct interchangeable wheels.

From this follows that it is *impossible* to have interchangeable wheels in *radial flank gearing*, for the radial flanks of B require that the faces of A shall be epicycloids generated by a circle whose diameter is equal to the radius of B, consequently, this face will only work with a wheel of the diameter of B. Similar remarks apply of course to pin gearing, discussed in § 72.

On the other hand, it is easy to see that the general method employed for cycloidal gearing, § 70, in which the describing circles C_1 and C_2 are arbitrarily assumed, affords opportunity for forming interchangeable wheels, and that we shall obtain such wheels if the two describing circles C_1 and C_2, which are rolled on the inside and outside of the pitch circles, *are taken equal to each other and are the same for all the wheels,* for then only will the face of each wheel work correctly with the flank of any other wheel, because these two profiles have been generated by rolling the same circle on the inside and outside of the two pitch circles. This may be easily proved.

Let *A*, *B*, *D* be any three wheels, and let the faces of *A* be epicycloids generated by rolling any describing circle *C* on the pitch circle of *A*; the flanks of the wheel *B* must then be hypocycloids described by rolling this same circle *C* inside the pitch circle of *B*. But this form of profile for the flanks of *B* requires that the faces of *D* be epicycloids generated by rolling *C* on the pitch circle of *D*, and this form of face for *D* requires, in order that *D* may work with *A*, that the flanks of *A* shall be hypocycloids generated by rolling this same circle *C* inside the pitch circle of *A*. This applies to every wheel of the set.

It is also evident that similar remarks apply to internal gears and to racks, consequently, to form a set of interchangeable wheels, it is only necessary to properly choose the rolling circle. This choice is principally determined by the fact that when flanks are radial, the teeth are weak near the base, and this evil is still more marked when the describing circle has a diameter larger than the radius of the pitch circle in which it is rolled. It is well, therefore, to make the diameter of the describing circle *at most equal to the radius of the smallest wheel in the set*, or, better still, to make it somewhat smaller. According to *Reuleaux* the radius of the describing circle is to be assumed equal to 0.875 t, which gives for the number of teeth in the smallest wheel in the set

$$z_0 = \frac{2\pi(2 \times 0.875\, t)}{t} = 11.$$

[In the United States and in England, 12 is the number of teeth usually chosen for the smallest wheel of the set, the radius of the describing circle being 0.955 t.]

Involute gearing, § 73, also permits the formation of interchangeable wheels,* for in this case the profile of a tooth on the wheel *A* is completely determined when we know the radius *A E*, Fig. 257, of the base circle. The latter is determined from $a \sin \gamma$, a being the radius of the pitch circle and γ the angle *A P E* made by the line of action *P E* with the radius $A\,P$ drawn to the intersection of $P\,E$ with the pitch circle. In order that the wheel A may work correctly with any other, it is only necessary that the line of action passing through the point of contact be the same for each of the wheels, *i.e.*, that the angle $A\,P\,E$ have the same value γ for all the wheels of the set. From this follows that the radii of the base circles, which are smaller than the radii of the pitch circles in the ratio $sin\,\gamma : 1$, are to each other exactly as the radii of the pitch circles, and also that with interchangeable wheels, the pitch measured on the base circles must be the same in all wheels of the set. Like remarks hold for internal gears and for racks.

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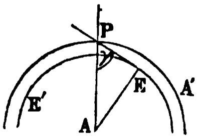
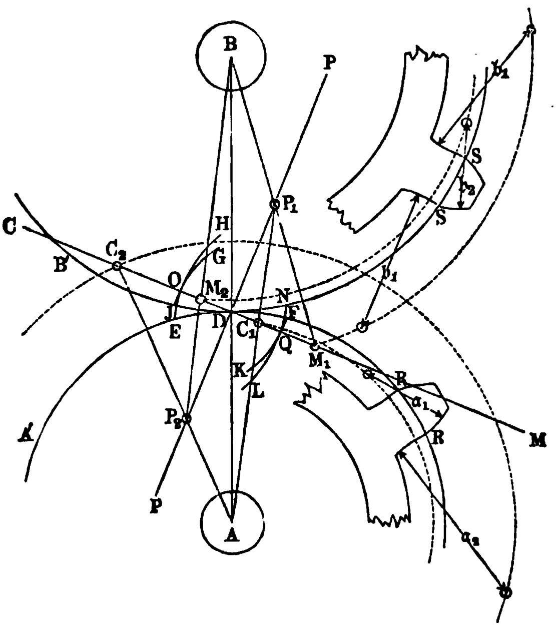
* See article by *Büttner*, Zeitschr. deutsch Ing., 1871.

FIG. 257.



Interchangeable wheels may also be constructed according to the method of approximate arcs, described in § 74. For this, it is only necessary to make the values $D\,P_1 = p_1$ and $D\,P_2 = p_2$, Fig. 258, equal to each other and equal to p for all the wheels of the set, and make the angle $A\,D\,M$ also the same (75°) for all the wheels. For we see from the figure that under these suppositions the diameter $B\,D$ of the wheel B is without any influence on the position of the centres C_1 and C_2 of the arcs of the wheel A, this diameter only influencing the distances from D of the centres M_1 and M_2 of its own arcs. In this case, therefore, it is only a question of assuming suitable values for the angle $A\,D\,M$ and the distance $D\,P_1 = D\,P_2 = p$. For the angle $A\,D\,M$ it will be well to take a value about 75°, because with a smaller value that component of the pressure of the wheels which falls into the line of centres $A\,B$ will cause an unnecessarily large amount of friction on the shafts. The value for the distance $p = D\,P_1 = D\,P_2$ of the instantaneous centre P_1 from the point of contact D of the pitch circles, is determined from the following considerations: If, with this distance p the teeth be constructed for a wheel whose radius is $r = \dfrac{p}{sin\,\gamma}$, evidently the ray drawn from the centre of this wheel to the instantaneous centre P_1 will be parallel to the direction $C\,M$ of the force, and the required centre M_1 for the arc JH forming the flanks of the teeth will lie at an infinite distance, *i.e.*, the flanks will be radial. If the radius r of the wheel were taken still smaller, the centre M_1 would pass to the other side of D, and the arcs thus formed would cause the flanks of the wheel B to undercut, thus greatly weakening the teeth. Consequently, p should be so chosen that the flanks of the smallest wheel of the set will at most be radial. If, therefore, r represents this smallest radius, p may be determined from the above relation $r = \frac{p}{\sin \gamma}$, *i.e.*, $p = r \sin \gamma$. Now we may readily calculate from the formulas given in § 74, for any wheel A whose radius is a, the distances $\mathfrak{a}_1$, $\mathfrak{a}_2$ of the centres of the arcs C_1 and C_2 from the point of contact D of the pitch circles. Since we have here a convexo-concave contact, we shall evidently find for the wheel whose radius is a, the centre for the convex curvature or face

FIG. 258.



$$D C_1 = \mathfrak{a}_1 = \frac{p \times a \cos\gamma}{p + a \sin\gamma} = \frac{r \sin\gamma \times a \cos\gamma}{r \sin\gamma + a \sin\gamma} = \frac{r \times a \cos\gamma}{a + r},$$

and for the centre C_2 of concave curvature or flank:

$$D C_2 = \mathfrak{a}_2 = \frac{p \times a \cos\gamma}{a \sin\gamma - p} = \frac{r \sin\gamma \times a \cos\gamma}{a \sin\gamma - r \sin\gamma} = \frac{r \times a \cos\gamma}{a - r}.$$

If z represents the number of teeth in the wheel A, and z_0 the number in the smallest wheel of radius r, we shall have, when t represents the pitch,

$$2 \pi a = z t, \text{ and } 2 \pi r = z_0 t,$$

hence

$$D C_1 = \mathfrak{a}_1 = \frac{z_0 z \cos\gamma}{2 \pi (z + z_0)} t,$$

and

$$D C_2 = \mathfrak{a}_2 = \frac{z_0 z \cos\gamma}{2 \pi (z - z_0)} t.$$

By means of these formulas we can calculate a table of values for $\mathfrak{a}_1$ and $\mathfrak{a}_2$ corresponding to different pitches and numbers of teeth. This has already been done for $z_0 = 12$ and $\gamma = 75^\circ$ by Willis,* to whom this method of approximation by circular arcs is due. To render the construction easier, *Willis* has also designed a small card-board or brass instrument called the *Odontograph*, which in its essential features resembles a composing stick whose legs make an angle of 75° with each other. On each side of the vertex along one edge the instrument is graduated into equal divisions. The graduations on one side are for the values of $\mathfrak{a}_1$ and on the other for $\mathfrak{a}_2$, which are taken from the table accompanying the instrument. However, in designing teeth, one can easily do without this apparatus.

REMARK. The patterns for a set of interchangeable wheels whose pitch is 30 millimeters [1.18 in.] are to be given circular tooth profiles ; how large must we assume the distance $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of the centres for the faces and flanks for a wheel having 40 teeth, supposing the angle made by the direction of the force with the line of centres to be $\gamma = 75^\circ$, the smallest wheel of the set to have 12 teeth and radial flanks?

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* Principles of Mechanism.

We have here for a wheel with 40 teeth:

$$\mathfrak{a}_1 = \frac{12 \times 40 \times \cos 75^\circ \times 30 \text{ millim.}}{2 \times 3.14 (40 + 12)} = 0.382 \times 30 = 11.5 \text{ millimeters } [0.45 \text{ in.}],$$

and

$$\mathfrak{a}_2 = \frac{12 \times 40 \times \cos 75^\circ \times 30 \text{ millim.}}{2 \times 3.14 (40 - 12)} = 0.707 \times 30 = 21.2 \text{ millimeters } [0.83 \text{ in.}].$$

When the pitch is t we have for a wheel of 40 teeth:

$$\mathfrak{a}_1 = 0.382\, t, \qquad \text{and} \qquad \mathfrak{a}_2 = 0.707\, t.$$

Under these suppositions the required distance of the instantaneous centres becomes

$$p = r \sin 75^\circ = \frac{12\, t}{2 \times 3.14}\, 0.966 = 1.84\, t = 55.2 \text{ millimeters } [2.17 \text{ in.}].$$

In like manner the calculation can be made for every other number of teeth. The mode of constructing the teeth with the found values $\mathfrak{a}_1$ and $\mathfrak{a}_2$, is given in § 74.

§ 76. Number of Teeth.—Since the pitch, *i.e.*, the distance measured on the pitch circles between two similarly situated tooth surfaces, is exactly the same for both pitch circles, it follows that the numbers of teeth z_1 and z_2 on the two wheels A and B are to each other as the circumferences, and consequently as the radii $\mathfrak{a}_1$ and $\mathfrak{a}_2$, of these pitch circles. We always have $z\, t = 2\, \pi\, r$, and hence

$$t = 6.28 \frac{r_1}{z_1} = 6.28 \frac{r_2}{z_2}.$$

Therefore we can also employ for the velocity ratio $n = \frac{r_1}{r_2}$ of two wheels, the ratio of the numbers $n = \frac{z_1}{z_2}$.

It follows from this method of expressing the velocity ratio of two toothed wheels, that we must always express the ratio by *whole* numbers, which is a limitation to which friction wheels and belt pulleys are not subject.

From the relation $z\,t = 2\,\pi\,r$, we can find any one of the three magnitudes, pitch, number of teeth and radius of pitch circle, when the two others are given. When these values have to be frequently determined, it is customary to make use of a table which gives the values $\frac{z}{2\,\pi} = 0.1592\,z$ for the ordinarily occurring numbers z (say up to 300). In some establishments the pitch, instead of being expressed directly in terms of the unit of length, is expressed in terms of the quotient $\frac{t}{\pi}$, and may then be designated by (t), so that the pitch will equal 3.14 (t) millimeters, inches, lines, etc., the radius r being then directly expressed by $z\,(t)$ in ordinary units of length (millimeters, inches, etc.). [In the United States the so-called *diametral pitch* is much used, it is equal to the *number of teeth per inch of diameter*, *i.e.*, $= \frac{d}{z} = \frac{\pi}{t}$, the diameter of the pitch circle being easily found from $z \times$ diametral pitch.]

In laying off the pitch t on the pitch circle we can of course only take the chord s of the arc of the pitch in the dividers, hence, if we do not prefer to divide the whole pitch circle into z equal parts, we must take between the points of the dividers a length

$$s = 2\,r\,\sin\frac{\tau^\circ}{2}\,,$$

τ° representing the *pitch angle* or angle subtending the pitch t, *i.e.*,

$$\tau^\circ = \frac{360^\circ}{z} = \frac{180^\circ}{\pi\,r}\,t = 57^\circ.296\,\frac{t}{r}.$$

If we substitute the approximate value

$$\sin\tau^\circ = \frac{\tau}{2} - \frac{1}{6}\left(\frac{\tau}{2}\right)^3 = \frac{\tau}{2}\left(1 - \frac{\tau^2}{24}\right) = \frac{t}{2\,r}\left(1 - \frac{t^2}{24\,r^2}\right),$$

we shall get for the chord

$$s = t\left(1 - \frac{t^2}{24\,r^2}\right) = 2\,\pi\frac{r}{z}\left(1 - \frac{4\,\pi^2}{24\,z^2}\right) = \frac{6.283\,r}{z}\left(1 - \frac{1.645}{z^2}\right).$$

It will always be better, however, to divide the pitch circle into z equal parts, as the calculated values of s are only approximate, and when frequently laid off on the pitch circle the small errors in s will gradually sum up so as to make considerable of a mistake at the last tooth.

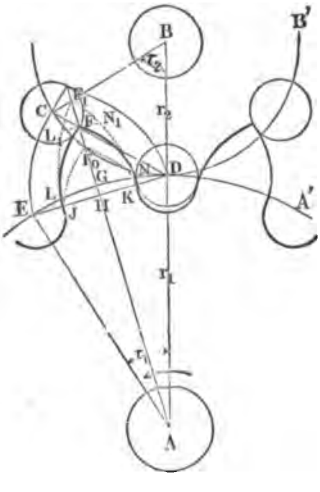
The magnitude of the pitch being determined with reference to the strength of the teeth, as will be shown hereafter, we may determine from it and the given radii of the wheels, the number of teeth.

Although in general the radius of the pitch circle may be chosen at pleasure, wheels being constructed of many different sizes, the *mutual relation* of the radii or the number of teeth of a pair of wheels is nevertheless dependent on certain conditions which are partly due to practical considerations of construction, but principally to considerations relating to the *possibility of transmitting motion.* The latter particularly holds with reference to the minimum number of teeth which will permit continuous transmission of motion. As regards the ratio of a pair of wheels, it may be said that when the wheels are unequal, each tooth of the smaller wheel will come into action oftener than a tooth of the larger wheel, the relative frequency of contact being expressed by the ratio of the larger wheel to the smaller. Therefore to diminish the rapid wear of the teeth of the smaller wheel, it is quite customary in practice not to make the ratio of the two wheels greater than 4 or 5 when the velocities are large. On the other hand, it is customary in windlasses and other slowly moving wheels driven by hand, to make the ratio of the two radii equal to 8 and even more. In the case of the rack and pinion, the ratio is infinite, but we must remember that the motion of the rack is reciprocating, and that the ratio of the *circumference of the pinion* to the *whole stroke of the rack* is what governs the frequency of contact of the teeth.

The reason underlying the practical rule to so choose the number of wheels that they will not have a common divisor, is that each tooth may gradually come in contact with every tooth of the other wheel, and thus prevent in some degree unequal wear. On this account, in the smaller wheels or pinions the numbers of teeth are very often prime numbers (11, 13, 17, 19, etc.), while the larger wheels receive numbers of teeth divisible by the number of arms, in accordance with constructive considerations. It should be mentioned, however, that it is not important to always rigidly follow this rule, and that, as with interchangeable wheels, it is not even possible to do so for all cases.

That the *possibility* of continuous transmission of motion is dependent on a certain *minimum number* of teeth for each wheel, is evident from the following considerations: in each case the action of a tooth must last at least while its wheel is rotating through its pitch angle. Therefore, when the number of teeth z is too small, the pitch angle $t^\circ = \frac{360}{z}$ will become so large that with the dimensions given to the teeth, the tooth cannot act through the whole of so great an angle of rotation. This minimum number z_0 of the teeth of a wheel depends partly on the form of the teeth, but principally on the number of teeth of the wheel which works with it, and consequently on the relative size of the latter. We will now establish methods for determining this minimum number.

FIG. 259.



Let us first assume that the wheel A, Fig. 259, has z_1 teeth, and that it works with another wheel B having z_2 pins D of diameter δ_2, then, according to § 72, the tooth must be profiled by a curve FLJ which is an equidistant to the epicycloid *CE* generated by rolling the pitch circle *B'* on *A'*, the distance between the two being $\frac{\delta}{2}$.

A pin does not come into action until its centre comes to *D* on the line of centres, and if the motion is to be continuous, the preceding pin *C* must not be out of contact with the driving tooth *FL* of the wheel *A*, until the centre of the following pin has reached *D* on the line of centres. This instant is represented by the figure, and it follows from the former investigation that the point of contact *F* of the tooth and pin must lie in the direction of pressure *DC*.

Now if t again represents the pitch, and

$$\tau_1 = \frac{t}{r_1} = \frac{2\pi}{z_1},$$

also

$$\tau_2 = \frac{t}{r_2} = \frac{2\pi}{z_2}$$

the pitch angles, we shall have for the chords:

$$s_1 = DE = 2r_1 \sin\frac{\tau_1}{2}.$$

and approximately,

$$s_1 = r_1\tau_1 - r_1\frac{\tau_1^3}{24} = t\left(1 - \frac{\tau_1^2}{24}\right),$$

when

$$\sin\frac{\tau}{2} = \frac{\tau}{2} - \frac{\tau^3}{2\,.\,3\,.\,8},$$

the higher powers being neglected when τ is small.

In like manner we get

$$s_2 = DC = t\left(1 - \frac{\tau_2^2}{24}\right).$$

If we now assume that the thickness δ_1 of the teeth, *i.e.*, *LN* or *JK* approximately, and the diameter δ_2 of the pins, shall each equal half the pitch,

$$\delta_1 = \delta_2 = \frac{t}{2},$$

we shall get:

$$\delta_1 = 2JH = 2(DJ - DF \cos CDE)$$

$$= 2\left[s_1 - \frac{\delta_1}{2} - \left(s_2 - \frac{\delta_2}{2}\right) \cos \frac{\tau_1 + \tau_2}{2}\right],$$

for the angle

$$CDE = \frac{\tau_1 + \tau_2}{2}.$$

Substituting the values above given for s_1 and s_2, also

$$\cos \frac{\tau_1 + \tau_2}{2} = 1 - \frac{(\tau_1 + \tau_2)^2}{8},$$

we get after reduction:

$$\delta_1 = \frac{t}{2} = 2t\left(\frac{3}{4} - \frac{\tau_1^2}{24}\right) - 2t\left(\frac{3}{4} - \frac{\tau_2^2}{24}\right)\left(1 - \frac{(\tau_1 + \tau_2)^2}{8}\right)$$

$$= \frac{t}{4}\left(-\frac{\tau_1^2}{3} + \frac{\tau_2^2}{3} + \frac{3}{4}(\tau_1' + \tau_2)^2\right),$$

Substituting,

$$\tau_1 = \frac{2\pi}{z_1}, \quad \text{and} \quad \tau_2 = \frac{2\pi}{z_2},$$

we get

$$\delta_1 = \frac{t}{2} = \frac{\pi^2 t}{12}\left(\frac{5}{z_1^2} + \frac{18}{z_1 z_2} + \frac{13}{z_2^2}\right).$$

Therefore, if δ_1 is to be equal to $\frac{t}{2}$, the following condition must be satisfied:

$$\frac{5}{z_1^2} + \frac{18}{z_1 z_2} + \frac{13}{z_2^2} = \frac{12}{2\pi^2} = 0.608.$$

For $z_1 = z_2$ we get:

$$\frac{36}{z_1^2} = 0.608,$$

or

$$z_1 = z_2 = \sqrt{\frac{36}{0.608}} = 7.7, \text{ i.e., at least } 8.$$

We see from this that when the pitch circles are equal, the *minimum* number of teeth and of pins must be taken equal to 8 in order to fulfil the condition that $\delta_1 = \delta_2 = \frac{t}{2}$. If the number of teeth or pins were smaller, the tooth LFN would be too short, for its two curves LF and NF would intersect at F_0 instead of F, and the tooth could not then act on the pin during a whole pitch.

On the other hand, a larger value for z_1 or z_2 than 8, would cause the two tooth curves to intersect at F_1 outside the point F, and the action of the tooth GF_1 on the pin would take place during an arc which is greater than the pitch t, consequently, with a greater number of teeth or pins transmission of motion will be assured. In this case the teeth may be shortened to the point F, so that they will no longer be pointed but truncated by the surface L_1N_1. It is therefore clear that the above equation of condition will give the *minimum number* of pins or teeth, and hence *greater* values for z_1 and z_2 may be assumed without hesitation.

Although for equal wheels the minimum number of teeth or pins is found equal to 8, this value does not remain constant when the velocity ratio changes, but, as may be seen from the equation of condition, varies so that an increase of z_1 causes the minimum value of z_2 to be less than 8, and inversely an increase of z_2 makes the minimum for z_1 less than 8. To ascertain the extreme limits for z_2 and z_1 we have only to make z_1 and z_2 respectively equal to infinity, in which case the wheel in question becomes a rack having teeth or pins. For $z_1 = \infty$, *i.e.*, for a rack, we get:

$$z_2 = \sqrt{\frac{13}{0.608}} = 4.6,$$

so that 5 is the *smallest number of pins* that can be given to a trundle working with a rack. On the other hand, if we put $z_2 = \infty$, *i.e.*, give the pins to the rack, we shall have for the *minimum number of teeth* in the toothed pinion which works with the rack,

$$z_1 = \sqrt{\frac{5}{0.608}} = 2.9, \text{ or about } 3.$$

Hence while a trundle needs at least 5 pins to work with a toothed rack, a toothed pinion need only have 3 teeth in order to work correctly with a rack having pins. The following table gives the minimum number of pins which will work with 3–8 toothed pinions and the number of teeth for trundles having 5–8 pins, the values having been determined from the above equation of condition.

No. of teeth z_1	3	4	5	6	7	8	10	14	48 (∞)
No. of pins z_2	117 ∞	18	12	10	9	8	7	6	5

We see from this that a 3-toothed pinion will work correctly, not only with a rack but with a pin-wheel having 117 pins, and that a trundle with 5 pins can also work with a wheel having 48 teeth.

When the two wheels have internal contact, the wheel A being the external one, the above investigation remains the same, with the single exception that the angle CDE of the chords is now found equal to $\frac{\tau_2 - \tau_1}{2}$, the equation of condition becoming

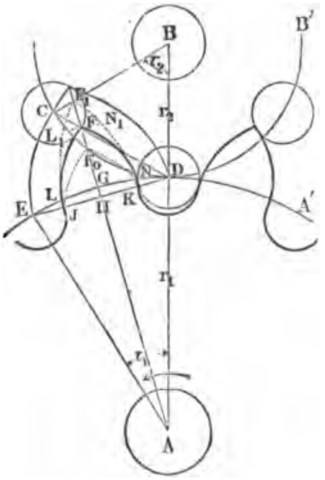
$$\frac{5}{z_1^2} - \frac{18}{z_1 z_2} + \frac{13}{z_2^2} = 0.608,$$

where z_1 again represents the number of teeth and z_2 the number of pins. If we assume the *toothed wheel* for the *outside* wheel and the trundle for the inside pinion, we shall get $z_1 = 21$ for $z_2 = 4$, from the equation of condition. This number give the ***maximum*** number of teeth in the wheel, for if z_1 were taken greater we should have from

$$\frac{5}{z_1^2} - \frac{18}{z_1 z_2} + \frac{13}{z_2^2},$$

a value which is greater then 0.608, *i.e.*, the dimension JK, Fig. 260, would be greater than $\frac{t}{2}$. It is also clear that with internal gearing the arc of contact *diminishes* when the diameter of the outside wheel increases.

FIG. 260.



In like manner we get $z_1 = 7$ for $z_2 = 3$, and $z_1 = 3$ for $z_2 = 2$, *i.e.*, a trundle with three pins could only work with an outside wheel (with internal teeth) having less than 7 teeth, and a trundle with two pins could only be driven by an outside wheel having 3 teeth, provided such an arrangement were possible. If we finally take z_2 so large that $\frac{13}{z_2^2} \underset{<}{=} 0.608$ (the least number which will give this being $z_2 = 5$, and hence $z_1 = 1.4$), an outside wheel with two teeth would suffice theoretically to drive a trundle with 5 pins, provided such an arrangement were practically possible.

At any rate we recognize that in this case, where $\frac{13}{z_2^2} < 0.608$ and therefore z_2 at least equal to 5, we can take any number whatever for z_1, for then the expression

$$\frac{5}{z_1^2} - \frac{18}{5 z_1} + \frac{13}{5^2}$$

will always be less than 0.608. So that we may conclude that a trundle with at least 5 pins can be in internal contact with an outside wheel of any number of teeth, also that a trundle which works with a toothed rack, must have at least 5 pins.

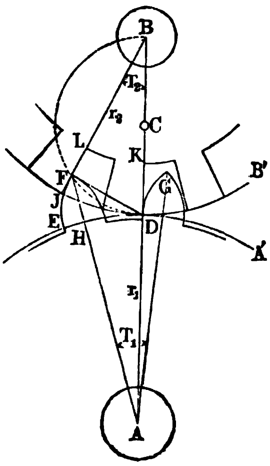
If we give the pins (z_2) to the larger wheel and the teeth (z_1) to the inside pinion, the investigation remains the same. We then find as above that, for the number of teeth z_1 which makes $\frac{5}{z_1^2} \overset{=}{<} 0.608$ (the least whole number for z_1 which will give this being 3, from which follows $z_2 = 2$), any number for z_2 greater than 2 will make the expression

$$\frac{5}{z_1^2} - \frac{18}{z_1 z_2} + \frac{13}{z_2^2},$$

less than 0.608. A three-toothed pinion may therefore work on the inside of a pin-wheel having any number of teeth and consequently may also work with a rack which is provided with pins. On the other hand if we make $z_1 = 2$, we shall get $z_2 = 13$ for the greatest number of pins in the larger wheel.

In a manner similar to that employed for pin gearing, we may determine the minimum number of teeth for other kinds of profiles. In Fig. 261, let the wheel B have radial flanks DK, the faces EF of the wheel A will therefore be epicycloids generated by rolling a describing circle C of diameter $DB = r_2$ on the pitch circle A'. Let us also assume that A has only faces and B only flanks, then the point F of the tooth EF cannot cease to be in contact with the flank JL till the following face DG is in contact with the following flank DK at D on the line of centres. If we let $s_1, s_2, \tau_1, \tau_2, \delta_1$ and δ_2 represent the same quantities as before, we shall have, since $FDE = \tau_2 + \frac{\tau_1}{2}$, for the limiting case, in which the tooth is pointed, the thickness:

FIG. 261.



$$\delta_1 = \frac{t}{2} = 2\,s_1 - 2\,DF \times cos\left(\frac{\tau_1}{2} + \tau_2\right)$$

$$= 4\,r_1\,sin\,\frac{\tau_1}{2} - 2\,r_2\,sin\,\tau_2\,cos\,\frac{\tau_1 + 2\,\tau_2}{2};$$

or as before

$$\delta_1 = \frac{t}{2} = 2\,r_1\,\tau_1 - \frac{r_1\,\tau_1^3}{12} - \left(2\,r_2\,\tau_2 - \frac{1}{3}\,r_2\,\tau_2^3\right)\left[1 - \frac{1}{2}\left(\frac{\tau_1 + 2\,\tau_2}{2}\right)^2\right]$$

$$= t\left[-\frac{\tau_1^2}{12} + \frac{\tau_2^2}{3} + \left(\frac{\tau_1 + 2\,\tau_2}{2}\right)^2\right] = t\left(\frac{\tau_1^2}{6} + \tau_1\,\tau_2 + \frac{4\,\tau_2^2}{3}\right).$$

If we again substitute $\tau = \frac{2\,\pi}{z}$ we shall get for the equation of condition:

$$\frac{2\,\pi^2}{3}\left(\frac{1}{z_1^2} + \frac{6}{z_1\,z_2} + \frac{8}{z_2^2}\right) \leqq \frac{1}{2},$$

or

$$\frac{1}{z_2^2} + \frac{6}{z_1\,z_2} + \frac{8}{z_2^2} \leqq 0.076.$$

For $z_1 = z_2$ we get

$$z = \sqrt{\frac{15}{0.076}} = 14.$$

With this form of tooth, therefore, equal wheels must each have at least 14 teeth.

The minimum number of teeth in a pinion which works with a rack having straight flanks, may be obtained by substituting $z_2 = \infty$, with the result, that

$$z_1 = \sqrt{\frac{1}{0.076}} = 3.6, \text{ in round numbers} = 4,$$

while the number of teeth in a pinion with radial flanks which works with a rack having faces will at least be

$$z_2 = \sqrt{\frac{8}{0.076}} = 10.25, \text{ or about } 11.$$

We may here, as was done for pin gearing, calculate a table from which the least number of teeth in a wheel can be obtained when the other wheel has a given number of teeth, the condition being observed, however, that the wheel with radial flanks does not receive less than 11 teeth, and that with epicycloidal faces not less than 4 teeth.

With internal gearing the only difference in the calculation arises from the angle

$$FDE = \tau_2 - \frac{\tau_1}{2},$$

and therefore we have for the condition of continuous transmission

$$\frac{1}{z_1^2} - \frac{6}{z_1 z_2} + \frac{8}{z_2^2} \overset{=}{<} 0.076.$$

This equation of condition again gives the greatest number of teeth in the larger wheel when the number of teeth in the smaller wheel does not exceed

$$z = \sqrt{\frac{8}{0.076}} = 10.25 = \text{about } 11,$$

or

$$z = \sqrt{\frac{1}{0.076}} = 3.6 = \text{about } 4,$$

according as the smaller and inner wheel has radial flanks or curved faces.

§ 77. **Addendum of Tooth.**—The investigations of the preceding article were confined to the case in which one wheel has only faces while the other has only flanks, the action of the two teeth taking place *behind* the line of centres when the faces are given to the driver. That pins are really nothing more than flanks is evident from the fact that they are not acted upon by the teeth of the other wheel till their centres have passed the instantaneous centre.

But generally each of the two wheels receives flanks as well as faces, consequently the action of the teeth on each other begins at a certain distance *in front of* the line of centres, and ceases at a certain distance *behind* this line. If we regard these distances as arcs measured on the pitch circles and represent the arc in front of the line of centres by e_1 and that behind the line by e_2, it is evident that their sum must at least be equal to the pitch if there is to be continuous motion; in other words, we must have the total arc of action $e_1 + e_2 \overset{=}{>} t$. The arc e_1 is usually called the *arc of approach* and e_2 the *arc of recess.*

It is also evident that the size of these arcs depends principally on the size of the addenda, *i.e.*, on the extent to which the heads of the teeth project beyond their pitch circles. It is also clear that each of these addenda can be smaller than in the cases discussed in the preceding articles, where the action was confined to one side of the line of centres along an arc e, which had to be at least equal to the pitch. The reason why these teeth which act on both sides of the line of centres are preferred in practice, is principally because the resistance due to the friction of the teeth increases rapidly with the length of the addenda (see § 79). The determination of the minimum number of teeth (maximum with internal gearing) is effected in a manner similar to that employed in the preceding articles for the one-sided action. In the present instance, the limiting case for the minimum number of teeth will be that in which both wheels have symmetrical heads terminating in sharp points and having the required thicknesses $\left(\delta_1 = \delta_2 = \text{say}\,\frac{t}{2}\right)$, the addenda being sufficient to make $e_1 + e_2 = t$.

The general analytical investigation of this case would, however, lead to such extended and complex calculations that its practical value would be very small, and as the required results can always be easily obtained by graphical methods, it seems preferable to employ the latter, and also more natural, because we must necessarily make use of graphical methods in designing the wheels and the tooth profiles.

Let A and B, Fig. 262, be two wheels with the cycloidal type of tooth, whose profile is generated by rolling the two describing circles C_1 and C_2 in and on the pitch circles (§ 70). These circles C_1 and C_2 are at the same time the lines of contact or loci for the points of contact of the profiles. Let e_1 be the arc of approach and e_2 the arc of recess, then if we lay off from P

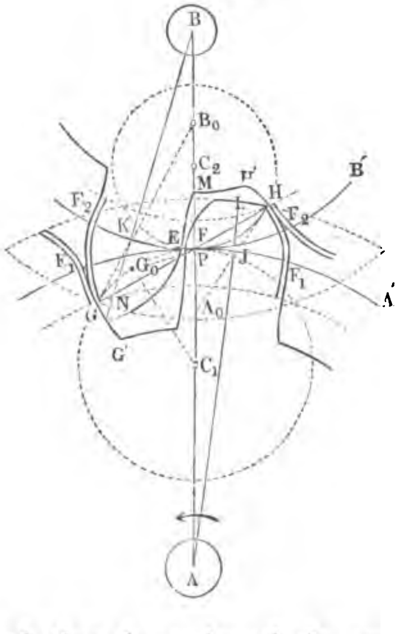
$$e_1 = \widehat{PE_1} = \widehat{PE_2} = \widehat{PG}$$

along the circles A B and C_1, we shall have at G the point at which the teeth begin to act. The profiles in contact at G pass through E_1 and E_2, the latter points coming into contact at the instantaneous centre P. In like manner if we make

$$e_2 = \widehat{PF_1} = \widehat{PF_2} = \widehat{PH},$$

we shall have at H the point at which action ceases, the profiles in contact at H passing through the points F_1 and F_2, and these latter points subsequently coming into contact at the instantaneous centre P. Therefore, if we pass through G a circle having the centre B, the latter will cut the faces of the wheel so as to give them the proper addendum, and the same may be said of the circle passed through H concentric to A, with respect to the faces of the wheel A.

FIG. 262.



Now, in order to ascertain whether the numbers of the teeth are sufficiently large, in other words, whether the ratios of the radii of the wheels to the pitch are sufficiently large, we must lay off from E_2 and F_1 the required thicknesses of the teeth on the corresponding pitch circles, so that $\delta_1 = F_1 F$ and $\delta_2 = E_2 E$, and then draw through F and E the profiles of the backs of the teeth, symmetrically to the fronts $F_1 H$ and $E_2 G$. These curves for the front and back of each tooth evidently must not cut each other inside of the addendum circles passing through H and G respectively, otherwise the teeth would not have the length required by the assumed arcs of action e_1 and e_2. It the two curves of a tooth intersect outside of its addendum circle, the teeth will be truncated, as in the figure. If the points of intersection were on the addendum circles, the teeth would terminate in sharp points, and this case, when $e_1 + e_2 = t$, corresponds to the above mentioned limit below which the radii cannot pass.

If we suppose the thickness $\delta_1 = F F_1$ of the tooth on A to be bisected at J, the figure shows that the radius drawn from A to the end H of the arc of contact will approach nearer to the point J the smaller the radius $A\ P$, and we can easily imagine a position A_0 of the centre A, for which the radius drawn from A_0 to H will pass exactly through the centre J of the tooth. This position A_0 evidently corresponds to the above mentioned lower limit of the radius of A, for which the profile becomes pointed at H. Similar remarks apply to the position of B and its smallest permissible radius $P B_0$.

If, instead of the cycloidal type of tooth, we employ the involute form, Fig. 263, the foregoing remarks remain the same in all essential points. Here the line of contact is given by the direction of pressure $G P H$, making the angle $A\ P\ G = \gamma$ with the line of centres. Let G again represent the beginning and H the end of the line of contact, then it is easy to see that the arc of approach is

$$e_1 = \overset{\frown}{P E_1} = \overset{\frown}{P E_1} = \frac{P G}{\sin \gamma}$$

and similarly for the arc of recess,

$$e_2 = \widehat{PF_1} = \widehat{PF_2} = \frac{PH}{\sin\gamma},$$

for the length PH is equal to the rectified arc of the base circle $O\,O'$, which is subtended by the angle at the centre:

$$PAF_1 = \frac{e_2}{AP}$$

the radius of the base circle being $AO = AP\sin\gamma$; hence we have

$$PH = AP.\sin\gamma\frac{e_2}{AP} = e_2\sin\gamma.$$

The arrangement here chosen is such that the profile of the tooth in the wheel B terminates in a point G, consequently its radius BD cannot be taken smaller, if the condition is to be satisfied that the arc of approach shall equal

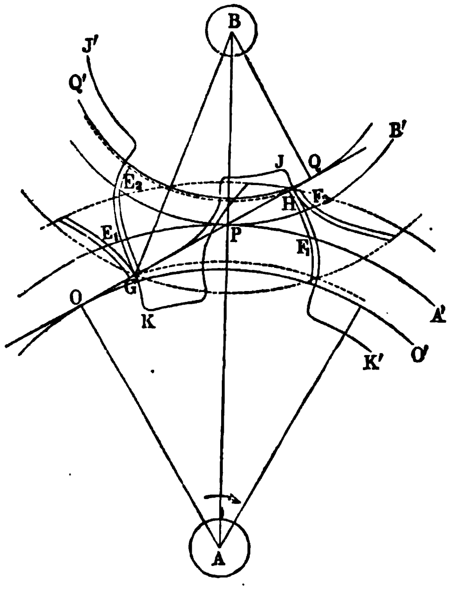
$$e_1 = E_1P = E_2P.$$

The thicknesses of the teeth are here also chosen somewhat smaller than the half pitch, so as to allow of a certain play between the teeth, concerning which more will be said hereafter.

After what has been said it is unnecessary to discuss the case of internal gearing.

Although we have always assumed in preceding investigations that the total arc of action should be at least equal to the pitch t, it by no means follows that this arc cannot be greater. Sufficient reason for not making this arc equal to the pitch is found in the fact that if the arc were equal to the pitch, the slightest irregularity in the construction of the teeth, or in the position of the axes, would cause the transmission of motion to become irregular or to cease entirely. But there is another important reason for assuming a larger arc of action, namely, that more than one pair of teeth will be in action at the same time, and consequently the pressure on each tooth and the wear caused by it will be less. For wheels whose centres approach and recede, as is the case with the gears for two rolls, the arc of action and therefore the addenda, must be made so large that with the greatest distance of the rolls the arc of action e will still exceed the pitch. In practice the arc of contact varies from 1.1 t to about 3.5 t, and it is evident from what has been said that the arc e can be given the larger values only when the number of teeth is sufficiently great, and that for small pinions, such as are employed in windlasses, counters, etc., where the number of teeth in the pinions is sometimes as low as 4, or even 3, the arc of action must always have a value which but slightly exceeds the pitch. What has been said explains why it is customary in practice to keep the number of teeth in quickly running wheels above certain numbers, say above 20, while for slowly running gears much smaller numbers are permitted. With the larger numbers of teeth we may attain a smoother motion on account of the greater arc of action. On the other hand, the work of friction increases with the arc of action, so that for this reason alone it would not be well to employ a very large arc of action.

FIG. 263.



In what has preceded, no assumptions have been made concerning the ratio of the arcs e_1 and e_2 of approach and recess; we are therefore still free to decide upon the ratio of the two. As these arcs e_1 and e_2 depend upon the addendum employed for the driver and follower respectively, it is clear that a large value for the arc of approach e_1 will require a large addendum k_2 for the driven wheel B, and that a large arc of recess e_2 will require a large addendum k_1 for the teeth of the driving wheel A. Moreover as the work of friction increases with the addenda, *i. e.*, as the projections beyond the theoretical axoids increase, it would, at first glance, seem justifiable either to assume the arcs e_1 and e_2 equal, or to assume k_2 and k_1 equal.

That these two assumptions are not equivalent, may be readily seen from Fig. 263, equal addenda k_1 and k_2 giving equal arcs e_1 and e_2 only when the size of the wheels as well as the form of the teeth is the same. In practice, one or the other of the above assumptions is usually made. According to *Redtenbacher*, the arc of approach and the arc of recess are each to be made equal to the pitch, *i.e.*, $e_1 = e_2 = t$. On the other hand, *Willis* gives the rule in common use among English builders of machinery (and among American builders also), according to which the addenda of both wheels are the same, being equal to $0.3\,t$, a rule which is also accepted by *Reuleaux*. The radial depths f_1 and f_2 of the teeth may be found from the addenda k_1 and k_2, for the space between the flanks of the wheel A must have sufficient depth f_1 to easily receive the head k_2 of the other wheel, and like remarks apply to the flank f_2 of B and the head or addendum k_1 of A. In order to make allowance for dust, etc., and imperfect construction, it is customary to make $f_1 = f_2 = 0.4\,t$, so that between the tops of the teeth of one wheel and the bottom of the spaces of the other, there will be a *bottom clearance* equal to $0.1\,t$. The assumption of *equal addenda* for wheels that work together is closely connected with the methods hitherto commonly employed in constructing toothed wheels, and is the only one which can be made for interchangeable wheels (§ 75), as may be easily seen from the following considerations. Let two wheels A and B of a set, have respectively the addenda k_1 and k_2, and let us suppose that the spaces between the flanks are provided with corresponding depths f_1 and f_2, so that if σ represents the bottom clearance, we shall have $f_1 = k_2 + \sigma$, and $f_2 = k_1 + \sigma$. Now, if the third wheel C, of the set, having the addendum k_3, is to work with A, and if we assume the bottom clearance to be constant, we shall have $k_3 = f_1 - \sigma$, and $f_3 = k_1 + \sigma$. Since the wheel C must also be able to work with B, we must also have $k_3 = f_2 - \sigma$, and $f_3 = k_2 + \sigma$, from which follows $k_1 = k_2 = k_3$ and $f_1 = f_2 = f_3$. As the like obtains for every other wheel of the set, it is evident that the assumption of a constant bottom clearance involves equal addenda k and equal depths f of the flanks for all the wheels of the set.

But, as has already been remarked, the arcs of approach and recess e_1 and e_2 are not equal, when the addenda are equal and the wheels unequal. It depends wholly on the form of the tooth, *i.e.*, on the line of contact, which of these two arcs is the smaller when the larger wheel drives. It is evident from simple geometrical relations, that with involute teeth whose line of contact is the straight line GH, Fig. 264, the arc of approach e_1 is *smaller* than the arc of recess e_2 when the *wheel A* drives. The reverse is of course true when the *pinion* (smaller wheel) drives, for then the arcs e_1 and e_2 exchange their characters, the arc of approach becoming the arc of recess and *vice versa*. *Willis* has shown that just the opposite is true of teeth with radial flanks, Fig. 265, where C_1 and C_2 are circles having the radii r_1 and r_2 as diameters

FIG. 264.

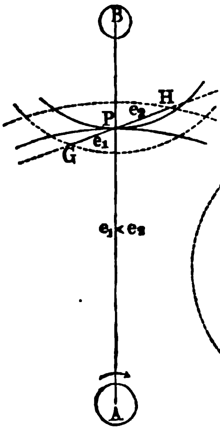


FIG. 265.

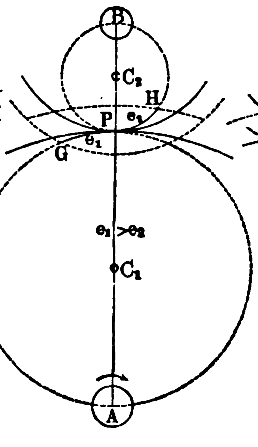
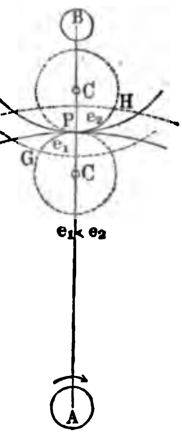


FIG. 266.



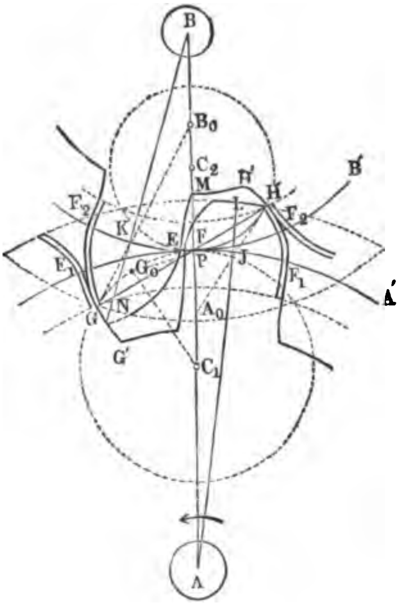
and represent the lines of contact; here calculation shows e_1 or GP to be greater than e_2 or PH, when the larger wheel drives. On the other hand, with interchangeable wheels having cycloidal teeth, which are generated by the smaller circle C, Fig. 266, the arc of approach $GP = e_1$ is again smaller than e_2, when the larger wheel drives.

This investigation of the relative magnitude of the arcs of approach and recess is not without practical value, because the durability or rapid wear of the teeth is closely connected with it. For it seems to be a fact established by experience that the friction of the tooth surfaces (see § 79), which takes place *in front of* the line of centres is of a much more vibratory character and more destructive to the material of the teeth, than the friction which occurs *behind* the line of centres. On this account teeth with radial flanks are not particularly suitable for places where the larger wheel drives. The preference still shown by clock makers and formerly by millwrights for pin gearing is largely due to this circumstance, for in this form of gearing but a slight action takes place in the front of the line of centres, the arc of approach being equal only to the radius of the pins, and therefore not even equal to $\frac{t}{4}$. This is also the reason why clock makers give to toothed wheels different addenda, so that the addendum of the driving wheel is about 1.5 times as great as that of the driven one, when the driver, as is generally the case, is the larger wheel.

Recently, wheel-moulding machines have been extensively used in foundries, and to a certain degree, have made the builders of machinery independent of costly wheel patterns, and by giving to the designer greater freedom in choosing addenda, have enabled him to give greater weight to considerations of smooth running than was heretofore possible, on account of the desire to make use of existing patterns.

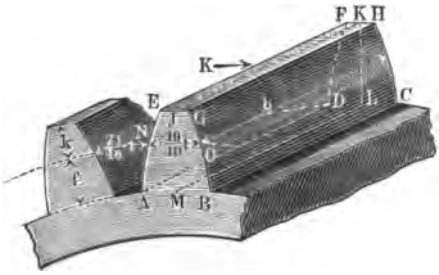
As regards the spaces between the teeth, we must make their depth, as has already been mentioned, so great that the shortest distance between the bottom of the space and the top of the tooth which enters it shall give a clearance of about 0.1 *t*, when the teeth are cast, and half this clearance (0.05 *t*) when the teeth are formed by cutters. We need only to draw that portion of the flank which actually comes in contact with the faces of the other wheel, as this portion of the correct profile will suffice to transmit uniform motion. This contact portion of the flank never extends to the bottom of the space, being evidently limited by the circle which is passed through the end of the line of contact concentric to the axis of its wheel. Thus in Fig. 267, only a portion $E_1 G$ of the flank of the wheel *A* and a portion $F_2 H$ of the flank of the wheel *B* actually come in contact with the opposite faces, consequently the remaining portions of the flanks *G G'* and *H H'* may be given any profile which will allow sufficient space for the heads of the teeth of the other wheel. These latter portions of the flanks are often made radial, the teeth being strengthened by rounding the corners *G'* and *H'* where they join the rim of the wheel. It may here be remarked that with involute teeth, the involutes do not usually extend to the root circles, as the latter are often smaller than the base circles, and consequently the portions of the flanks inside the base circles may be assumed at pleasure, provided they are chosen so that they will not come in contact with the teeth of the other wheel. In Fig. 263 and Fig. 267, the portions of the profile actually coming in contact are designated by double lines.

FIG. 267.



That the pitch circles of two wheels *do not lie half way between the addendum and root circles*, even when both wheels have the same addendum k, is evident from the fact that there is a bottom clearance. In order to ascertain the pitch circles of two wheels that work together, it is only necessary to divide the distance between their axes in the ratio of the two numbers representing the teeth of the two wheels.

FIG. 268



§ **78. Dimensions of Tooth.**—The teeth of a wheel must receive such dimensions as will enable them to resist without injury the pressure to be transmitted. The pressure K acting on the tooth $A\,G\,F\,C$, Fig. 268, tends to fracture the tooth along the surface $A\,B\,C\,D$ where it joins the rim of the wheel, for the bending moment is greatest for this surface, being equal to $K\,l$, where l is the length $M\,J$ of the tooth. The pressure K is nearly tangential, differing but little from the circumferential force P, and may therefore be taken equal to the latter. If we now assume the pressure P to be uniformly distributed over the whole breadth $E\,F = b$ of the tooth, and suppose it to act at the outermost edge $E\,F$ in a direction perpendicular to the radial plane $J\,K\,L\,M$, we shall be making the most unfavorable supposition possible with reference to the strength, for the pressure P is generally more or less inclined and nearer the root of the tooth. Let h represent the thickness $A\,B$ at the root; we shall then have the formula for strength the same as for a beam fixed at one end and loaded at the other (see Vol. I., § 235), namely:

$$P\,l = \frac{W}{e}\,k = \frac{b\,h^2}{6}\,k,$$

where k is the greatest permissible stress on the unit of surface.

The thickness h of the tooth at the root will generally, if we except the teeth with radial flanks, be somewhat greater than the thickness $\delta = NO$ at the pitch circle, and as this latter dimension is usually taken somewhat smaller than the half pitch, being about $\frac{19}{40}t$, we may assume for the purposes of calculation, the thickness at the root $h = \frac{1}{2}t$. When the thickness at the pitch circle is $\frac{19}{40}t$, the *side clearance* or *backlash* is equal to $\frac{1}{20}t$, thus making ample allowance for imperfections of workmanship and for the presence of oil, dust, etc., between the teeth.

The breadth AD of the teeth, measured in an axial direction, is generally taken proportional to the pitch, as a rule between $2t$ and $3t$, the smaller breadth $2t$ being given to slowly moving wheels driven by hand, as windlasses, etc., and the greater breadth $2\frac{1}{2}t$ to $3t$, and sometimes even more, to rapidly running wheels. The whole length l of the tooth is also taken proportional to the pitch, as has already been mentioned, then

$$l = k + f = 0.7\,t \text{ to } 0.75\,t.$$

Assuming

$$l = 0.75\,t,\ b = 2\,t, \text{ and } h = 0.5\,t,$$

we get from the above formula:

$$P \times 0.75\,t = \frac{2\,t \times 0.25\,t^2}{6}\,k,$$

the value

$$t = 3\sqrt{\frac{P}{k}}.$$

If we assume the greatest permissible (working) stress for cast-iron teeth to be $k = 3$ kilograms [about 4,300 lbs. per square inch], we shall have,

$$t = 1.73\sqrt{P}, \qquad [t = 0.046\sqrt{P}].$$

For wooden teeth, made of the best ash, we may take $k = 1$ kilogram per mm^2 [1,422 lbs. per sq. in.], and for these teeth get

$$t = 3\sqrt{P} \qquad [t = 0.08\sqrt{P}],$$

while in the small wrought-iron wheels used in windlasses the pitch is

$$t = 3\sqrt{\frac{P}{6}} = 1.22\sqrt{P} \qquad \left[t = 3\sqrt{\frac{P}{8,500}} = 0.0324\sqrt{P}\right].$$

The value P in the preceding formulas must be understood to mean the *whole* of the pressure transmitted from one wheel to another, for although the pressure is distributed on several teeth simultaneously when the arc of action is greater than the pitch, nevertheless, a slight inaccuracy in spacing the teeth would, with an unyielding material like cast-iron, suffice to bring the whole pressure on a single tooth.

It is only with elastic wooden teeth that we can assume that the pressure will be borne simultaneously by two teeth and can consequently take the pitch smaller, and even with these teeth the arc of action must at least be equal to $2t$, for only then do we have two pairs of teeth *continually* in contact. As the relation between the moment Pr of the force transmitted and N horse powers transmitted when the wheel makes n revolutions, is

$$Pr = 716,200\frac{N}{n} \qquad \left[Pr = 63,025\frac{N}{n}\right],$$

r being the radius of the pitch circle. From this we find the pitch:

$$t = 1,470\sqrt{\frac{N}{nr}} \qquad \left[t = 11.5\sqrt{\frac{N}{nr}}\right] \text{ for cast iron ;}$$

$$t = 1,050\sqrt{\frac{N}{nr}} \qquad \left[t = 8.1\sqrt{\frac{N}{nr}}\right] \text{ for wrought iron ;}$$

$$t = 2,540\sqrt{\frac{N}{nr}} \qquad \left[t = 20\sqrt{\frac{N}{nr}}\right] \text{ for wood.}$$

The foregoing determinations of the thickness, and consequently of the pitch, of the tooth, were made simply with reference to its strength, no account being taken of the velocity of the circumference and the wear resulting therefrom. This is always permissible when the circumferential velocity is small, as is the case, for example, with the wheels of windlasses, presses, etc., worked by hand. With greater speeds, like those to which the wheels of shafting are generally subject, we must take account of the wear and of the effects of shocks which inevitably arise in such cases. To diminish the wear, it is therefore well to increase the breadth b of the teeth, so as to reduce the pressure per unit of surface as much as possible, while the effects of shocks may be taken into account by diminishing the permissible stress k as the velocity v increases. *Reuleaux* accordingly assumes for cast-iron wheels:

$$b = 0.01\, t \sqrt{Pn} = 8.5\, t \sqrt{\frac{N}{r}},$$

$$\left[b = 0.0067\, t \sqrt{Pn} = 1.7\, t \sqrt{\frac{N}{r}}\right],$$

and

$$k = \frac{constant}{\sqrt{v}},$$

when v is the velocity of the circumference of the pitch circles in millimeters [in feet], and n the number of revolutions of the *pinion* (or *smaller* wheel), the wear being greater on the teeth of the pinion because of the more frequent contact. From this we may deduce the expression:

$$t = 2.1 \sqrt[4]{Pr} = 60 \sqrt[4]{\frac{N}{n}},$$

$$\left[t = 0{,}154 \sqrt[4]{Pr} = 2.44 \sqrt[4]{\frac{N}{n}}\right],$$

which corresponds to:

$$k = \frac{120}{\sqrt{v}} \qquad \left[k = \frac{10{,}000}{\sqrt{v}}\right].$$

For wooden teeth, which by the way are mostly used for swiftly running wheels, we have:

$$b = 0.0125\, t \sqrt{Pn} = 10.6\, t \sqrt{\frac{N}{r}}$$

$$\left[b = 0.0084\, t \sqrt{Pn} = 2.12\, t \sqrt{\frac{N}{r}}\right],$$

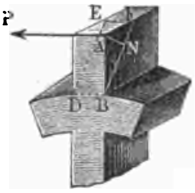
and

$$t = 3.23 \sqrt[4]{Pr} = 92.4 \sqrt[4]{\frac{N}{n}}$$

$$\left[t = 0.233 \sqrt[4]{Pr} = 3.7 \sqrt[4]{\frac{N}{n}}\right].$$

When the axes of the wheels are not exactly parallel, or some foreign substance accidentally comes between the teeth, the latter may be more unfavorably acted upon than was assumed for the preceding formulas. For example, if the pressure P is concentrated principally at the edge A of the tooth, Fig. 269, it will tend to fracture the latter along the oblique edge $DBFE$. If, for this case, we substitute in

FIG. 269.



$$Pl = \frac{bh^2}{6} k,$$

for l the normal distance

$$AN = AB \times \sin ABN = l \sin \varphi = 0.75\, t \sin \varphi,$$

for b the length

$$BF = \frac{l}{\cos\varphi} = \frac{0.75\,t}{\cos\varphi},$$

and for h, the thickness of the tooth at the pitch circle,

$$\delta = \frac{19}{40}t = 0.475\,t,$$

we can get the maximum working stress k_1 at the dangerous section from

$$P \times 0.75\,t\sin\varphi = \frac{0.75\,t \times 0.475^2\,t^2}{6\cos\varphi}k_1,$$

equal to

$$k_1 = \frac{6}{0.226}\sin\varphi\cos\varphi\frac{P}{t^2} = 13.3\sin 2\varphi\frac{P}{t^2}.$$

This becomes a maximum for $\varphi = ABN = 45°$, being equal to:

$$k_1 = 13.3\frac{P}{t^2}.$$

If we substitute in this formula the value

$$\frac{P}{t^2} = \frac{k}{9},$$

found from the formula

$$t = 3\sqrt{\frac{P}{k}},$$

we shall have:

$$k_1 = 1.48\,k.$$

In this case, therefore, the material will be subject to a stress 1.5 times as great as that first assumed, *i.e.*, the stress in cast iron may amount to 4.5 kilograms [6,600 lbs. per sq. in.]. As this value is still sufficiently far from the limit of elasticity, it is unnecessary to increase the thickness of the tooth in order to take account of this exceptional case. Moreover, as the frictional resistances increase with the pitch, it is not advisable to assume the latter larger than is absolutely necessary. Still another reason for keeping the pitch as small as possible, is that the number of teeth in each of the wheels will then be larger, and the wheels will run more uniformly and smoothly because of their closer approximation to the ideal condition of friction wheels.

EXAMPLES: 1). What must be the pitch of a cast-iron wheel 1.5 meters [59.06 ins.] in diameter, to drive the drum of a windlass when the drum has a radius of 180 millimeters [7.09 ins.], and the pull on the chain is 3,000 kilograms [6,614 lbs.]?

Here

$$P = \frac{3{,}000 \times 180}{750} = 720 \text{ kilograms [1,587 lbs.].}$$

The required pitch is therefore

$$t = 173 \sqrt{720} = 46.4 \text{ millimeters [1.83 ins.],}$$

or in round numbers, 50 millimeters [1.97 ins.]. The breadth of the teeth should be $b = 2\,t = 100$ millimeters [3.94 ins.], the length $l = 0.75\,t = 37.5$ millimeters [1.48 ins.], the thickness $\delta = \frac{19}{40}\,t = 23.7$ millimeters [0.95 in.], the width of the space at the pitch circle 26.3 millimeters [1.05 ins.], and the side clearance or backlash 2.6 millimeters [0.1 in.]. The circumference of the pitch circle is $1{,}500 \times 3.14 = 4{,}712$ millimeters [185.51 ins.], consequently the number of teeth is $\frac{4{,}712}{50} = 94.24$. If there are six arms, the number of teeth may be taken equal to 96, and then the circumference of the pitch circle will be $96 \times 50 = 4{,}800$ millimeters [188.98 ins.], which corresponds to a diameter of $\frac{4{,}800}{3.14} = 1.528$ meters [60.16 ins.]. If we make the addendum $0.3\,t = 15$ millimeters [0.59 in.], and the depth of the flanks $0.4\,t = 20$ millimeters [0.79 in.], the outside diameter of the wheel will be

$$1{,}528 + 2 \times 15 = 1{,}558 \text{ millimeters [61.34 ins.],}$$

and the diameter of the root circle

$$1{,}528 - 2 \times 20 = 1{,}488 \text{ millimeters [58.58 ins.].}$$

2). Suppose a toothed wheel 5 meters [196.85 ins.] in diameter to be employed as fly wheel for a steam engine of 40 horse powers making 30 revolutions per minute, and suppose the wheel to gear with a pinion 1.5 meters [59.05 ins.], in diameter, and provided with wooden teeth. What dimensions should be given to the latter?

According to *Reuleaux* the required pitch may be found from

$$t = 92.4 \sqrt[4]{\frac{40}{30}} = 98.9 \text{ millimeters } [3.89 \text{ ins.}],$$

or in round numbers 100 millimeters [39.37 ins.], which gives for the breadth:

$$b = 10.6\, t \sqrt{\frac{N}{r}} = 10.6 \sqrt{\frac{40}{750}} \times t = 2.44\, t = 244 \text{ millimeters } [9.6 \text{ ins.}].$$

The other dimensions of the tooth, the number of teeth, and the exact diameter of the pitch circle, may be determined as above.

§ **79. Friction of Teeth.**—When two wheels with parallel shafts touch along a straight line which coincides with the instantaneous axis, there will be no sliding friction at the place of contact, because the motion is then a purely rolling one. In friction wheels and those which have direct transmission, as belt pulleys, contact takes place continually at the instantaneous axis, so that in these cases there will be no sliding friction at the circumferences if the motion is perfectly transmitted, *i.e.*, without slipping. In toothed wheels, however, contact between two teeth takes place at the instantaneous axis P, Fig. 270, only for one instant of their action, for the point of contact travels from one end to the other of the line of contact $G\,P\,H$. It is clear that this causes certain frictional resistances, for example, if the contact takes place along a line through J parallel to the axes, there will occur, instead of the small rotation about the instantaneous axis P, a rotation equal to the latter about the straight line through J, accompanied by a sliding of the two tooth surfaces at J upon each other in a direction perpendicular to PJ. The amount of this sliding, and consequently the distance through which the friction is overcome, according to § 4, Introduction, is equal to $\lambda\,\delta\,\omega$ when λ represents the distance PJ and $\delta\,\omega$ the angle of rotation about the instantaneous axis. Now, if we let K be the pressure between the tooth surfaces acting in the direction PJ, and φ the coefficient of friction, the work of friction for the instant under consideration will be $\varphi K \lambda \delta \omega$. Moreover, if $l_1 = GP$ and $l_2 = PH$ represent the distances of the beginning and end of the line of contact GH from the instantaneous axis P, we shall have for the total work of friction occurring between two teeth:

$$A = \int_0^{l_1} \varphi K \lambda \delta \omega + \int_0^{l_2} \varphi K \lambda \delta \omega = A_1 + A_2.$$

Here the first expression

$$A_1 = \int_0^{l_1} \varphi K \lambda \delta \omega$$

represents the friction which occurs during the *approach* GP to the line of centres, and the second

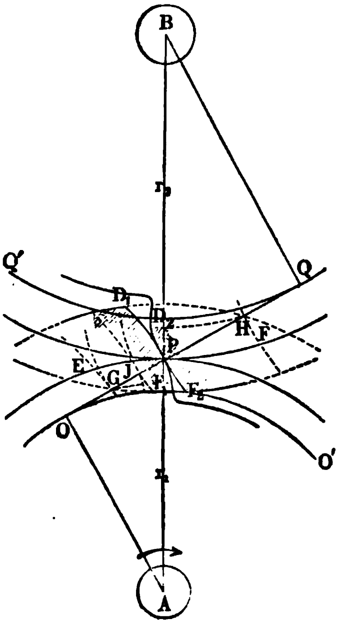
$$A_2 = \int_0^{l_2} \varphi K \lambda \delta \omega,$$

the friction during the *recess* PH from the line of centres.

It is easy to see that these two frictions act in opposite directions. For instance, when contact begins at G, the point F_1 of the driving tooth $D_1 F_1$ acts on the point F_2 of the driven wheel $D_2 F_2$, and then the tooth surface $F_1 P$ slides along the tooth surface $F_2 P$ in the direction of F_2 to P, till the two surfaces come in contact at the point P on the line of centres. The whole sliding during this period is evidently equal to the difference $PF_2 - PF_1$ in the lengths of the curves. In like manner, during the second period of contact, the tooth surface PD_1 will slide on PD_2 in the direction of D_2 to P, and therefore opposite to the preceding, this sliding being equal to the difference $PD_1 - PD_2$ of the curves. These opposite directions of the relative motions are characterized as follows: during the approach to the line of centres the tooth surface of each wheel slides *toward the axis of the other wheel*, and during the recess, *away from this axis*.

Experience shows that the friction occurring during the approach to the line of centres is much more injurious than the friction during the recess, and, as already mentioned, it is on this account desirable to make the addendum of the driven wheel smaller than that of the driver. This also explains why in pin gearing the teeth are given to the driver and the pins to the follower, for the pins must be principally regarded as flanks, since only a small portion of the pin, *i.e.*, the portion lying between the pitch circle and its tangent, acts as face, thus making an arc of approach which is less than one-quarter of the pitch (see § 77). There is, however, an objection to having no faces whatever, for then the line of action during recess, *i.e.*, the quantity l_2, must be made larger, thus again increasing the path along which friction is to be overcome. We must also remark that, as teeth cannot be constructed with perfect accuracy, there will be deviations from the ideal motion represented by the axoids, which deviations will be greater, the greater the distance from the instantaneous axis at which the teeth come in contact. Therefore, in spite of the hurtful action during approach, it is customary to give faces to the driven wheel also, and, as already mentioned (§ 77), economy in the use of patterns and cutters requires that the faces of both wheels shall have the same addendum.

FIG. 270.



We see from the expressions

$$A_1 = \int_0^h \varphi K \lambda \, \delta \omega$$

and

$$A_2 = \int_0^{l_2} \varphi K \lambda \, \delta \omega$$

for the work of friction before and behind the line of centres, that these losses of work depend on the form of the tooth surfaces, for λ varies with the form of the tooth; but, as will hereafter be shown, this difference is insignificant for the tooth profiles ordinarily employed. Let us first assume involute profiles, Fig. 270, the direction of pressure $O\,Q$ making the angle $O\,P\,A = \gamma$ with the line of centres, and let the arc of approach be represented by $e_1 = \widehat{E\,P}$ and the arc of recess by $e_2 = \widehat{P\,F}$. From what has preceded we know that on the line of contact

$$G\,P = e_1 \sin\gamma, \quad \text{and} \quad P\,H = e_2 \sin\gamma,$$

these lengths being equal to those arcs of the base circles O' and Q' which subtend the angles of approach and recess respectively. The radii of the base circles are

$$A\,O = r_1 \sin\gamma, \quad \text{and} \quad B\,Q = r_2 \sin\gamma.$$

It is also clear that for any position of the teeth for which the point of contact is at some point J between G and P, the distance from P is

$$\lambda = PJ = r_1 \sin\gamma \times \alpha,$$

where α is the angle through which the wheel A must turn while the point J is moving to P. If the wheel A were turned through $\delta\,\alpha$, the wheel B would turn through

$$\delta\,\beta = -\frac{r_1}{r_2}\delta\,\alpha\,,$$

and if B were held stationary the wheel A would have a rotation about the instantaneous axis P equal to

$$\delta\,\omega = -\,\delta\,\beta + \delta\,\alpha = \frac{r_1 + r_2}{r_2}\delta\,\alpha.$$

As this rotation does not actually take place around the instantaneous centre at P, but around the line of contact at J, the above mentioned sliding

$$\lambda\,\delta\,\omega = \lambda\frac{r_1 + r_2}{r_2}\,\delta\,\alpha$$

must take place, and the elementary quantity of work

$$\varphi K\lambda\frac{r_1 + r_2}{r_2}\,\delta\,\alpha$$

be performed. Consequently, if the angle of approach $\frac{e_1}{r_1}$ of the wheel A be represented by ε_1, the work of friction before the line of centres must be placed equal to:

$$A_1 = \int_0^{\varepsilon_1} \varphi\,K\lambda\,\frac{r_1 + r_2}{r_2}\,\delta\,\alpha = \varphi\,\frac{r_1 + r_2}{r_2}\,r_1\,sin\,\gamma\int_0^{\varepsilon_1} K\alpha\,\delta\,\alpha.$$

As the direction of the pressure K is invariable with involute teeth, we have K constant, and $K\,sin\,\gamma = P$ when P represents the circumferential force at the pitch circle; hence follows:

$$A = \varphi\,K\frac{r_1 + r_2}{r_2}\,r_1\int_0^{\varepsilon_1}\alpha\,\delta\,\alpha = \varphi\,K\frac{r_1 + r_2}{r_2}\,\frac{r_1\,\varepsilon_1^2}{2}.$$

In like manner we find the work of friction during recess to be

$$A_2 = \varphi\,K\frac{r_1 + r_2}{r_2}\,\frac{r_1\,\varepsilon_2^2}{2},$$

the whole work of friction is therefore:

$$A = A_1 + A_2 = \varphi\,K\frac{r_1 + r_2}{r_2}\,r_1\,\frac{\varepsilon_1^2 + \varepsilon_2^2}{2}.$$

This work is performed while the wheel rotates through the angle $\varepsilon_1 + \varepsilon_2$ about its own axis, the circumference describing the path $r_1\,(\varepsilon_1 + \varepsilon_2)$. If F represents the average force of friction reduced to the pitch circle, we shall have from $A = F r_1 (\varepsilon_1 + \varepsilon_2)$:

$$F = \frac{A}{r_1 (\varepsilon_1 + \varepsilon_2)} = \varphi K \frac{r_1 + r_2}{r_2} \frac{\varepsilon_1^2 + \varepsilon_2^2}{2 (\varepsilon_1 + \varepsilon_2)}$$

$$= \varphi K \frac{r_1 + r_2}{r_2} \frac{\varepsilon_1^2 + \varepsilon_2^2}{2 \varepsilon}.$$

It is now easy to see that this value becomes a minimum for a given *total* angle of action $\varepsilon = \varepsilon_1 + \varepsilon_2$, when $\varepsilon_1 = \varepsilon_2 = \frac{1}{2} \varepsilon$, that is, when the angle of approach is equal to the angle of recess, for if we write:

$$\frac{\varepsilon_1^2 + \varepsilon_2^2}{2 \varepsilon} = \frac{\varepsilon_1^2 + (\varepsilon - \varepsilon_1)^2}{2 \varepsilon},$$

and differentiate with respect to ε_1 we shall have:

$$2 \varepsilon_1 - 2 (\varepsilon - \varepsilon_1) = 0, \quad \text{or} \quad \varepsilon_1 = \frac{1}{2} \varepsilon = \varepsilon_2.$$

For this case the friction reduced to the pitch circle becomes:

$$F = \varphi K \frac{r_1 + r_2}{r_2} \frac{\varepsilon}{4}.$$

The angle ε must at least be equal to the angle of the pitch $\tau_1 = \frac{t}{r_1}$ if there is to be continuous transmission of motion, but, as was more fully explained in § 77, it is customary to take ε larger. If we make the arc of action $e = \nu t$, where ν is a number greater than 1, we shall have:

$$\varepsilon = \frac{e}{r_1} = \frac{\nu t}{r_1},$$

substituting this value, we get

$$F = \varphi K \frac{r_1 + r_2}{r_2} \frac{\nu t}{4 r_1} = \varphi K \left(\frac{1}{2 r_1} + \frac{1}{2 r_2}\right) \frac{\nu t}{2}.$$

If we also substitute

$$\frac{t}{2 r_1} = \frac{\pi}{z_1}, \quad \text{and} \quad \frac{t}{2 r_2} = \frac{\pi}{z_2},$$

we get

$$F = \varphi \pi K \left(\frac{1}{z_1} + \frac{1}{z_2}\right) \frac{\nu}{2}.$$

For the purposes of calculation $\nu = 2$ is usually chosen, that is, it is assumed that action begins at the distance t in front of the line of centres, and that it ceases at the same distance behind the line; when this is done the above expression becomes:

$$F = \varphi \pi K \left(\frac{1}{z_1} + \frac{1}{z_2}\right) = \frac{1}{3} \frac{P}{\sin \gamma} \left(\frac{1}{z_1} + \frac{1}{z_2}\right),$$

where the coefficient of friction between cast-iron teeth is assumed to be $\varphi = 0.11$, and therefore $\varphi \pi = \frac{1}{3}$.

It is evident from the foregoing investigation that for internal gearing we should have

$$F = \varphi \pi \frac{P}{\sin \gamma} \left(\frac{1}{z_1} - \frac{1}{z_2}\right) \frac{\nu}{2},$$

for then the motion of the inner wheel, when the outer wheel is held stationary, is expressed by

$$\delta \omega = \frac{r_2 - r_1}{r_2} \delta \alpha;$$

for the rack we have only to take r_2 or z_2 infinite, and we have:

$$F = \varphi \pi \frac{P}{\sin \gamma} \frac{1}{z_1} \frac{\nu}{2}.$$

It is evident from the development of these formulas that they apply only when the arc of approach and that of recess are each equal to $\frac{\gamma}{2} t$. It has been tacitly assumed in this investigation that the whole force transmitted, $K = \frac{P}{\sin \gamma}$, is borne by a single tooth, which evidently occurs only when $\nu = 1$, *i.e.*, when the arc of action is just equal to the pitch. But as e is generally taken greater than the pitch, more than a single pair will be in contact simultaneously. If we take, for example, $\nu = 2$, the pressure P will always be distributed on two teeth, and consequently there will be friction on two teeth simultaneously, but each tooth will transmit but one half the whole pressure, namely,

$$\frac{K}{2} = \frac{P}{2 \sin \gamma}.$$

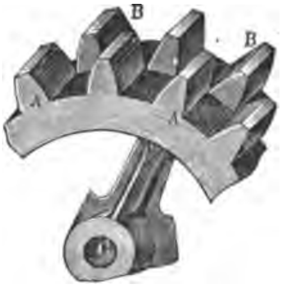
It follows from this, that the number of pairs of teeth in contact at the same instant has no direct influence on the amount of friction, except in so far as it influences the size of the arc of action νt. Moreover this is generally true, even when ν is not represented by a whole number, but by a mixed number, for example, 1.6. In this case the pressure is alternately borne by a single pair of teeth over a distance $0.4\, t$, and by two pairs, each sustaining half the pressure, through a distance $0.6\, t$, etc.

But the formulas developed above show that the friction is smaller the greater the number of teeth z_1 and z_2 of the wheels. The explanation of this is that with a great number of teeth, *i.e.*, with smaller pitch, the angle of action may also be made smaller and the friction thus diminished. From this point of view, therefore, it is desirable to make the pitch as small as the strength of the teeth, § 78, will allow. As this strength prescribes a certain minimum pitch, recourse is sometimes had to stepped gearing, partly to diminish the friction, but principally to produce as smooth a motion as possible for certain machines (planers for example). If we imagine an ordinary toothed wheel with z_1 teeth and pitch t to be divided into two wheels by a plane through the middle perpendicular to its axis, and then shift these two parts relatively to each other by exactly half the pitch, if the parts are now united we shall have a wheel with two steps, Fig. 271, and it is plain that such a wheel will work with its corresponding partner exactly as if it had $2\ z_1$ teeth, *i.e.*, as if it had a pitch $\frac{t}{2}$. . Now if it is required that the arc of action be ν times the pitch, we shall have, in a wheel with two steps and z_1 teeth, this arc equal to $\nu\frac{t}{2}$, *i.e.*, it will be the same as for a wheel of the same size having $2\,z_1$ teeth, and consequently the friction between this wheel and its two-stepped partner with z_2 teeth, will be

$$F = \pi\,\varphi\,\frac{P}{\sin\gamma}\left(\frac{1}{2\,z_1} + \frac{1}{2\,z_2}\right)\frac{\nu}{2} = \frac{1}{2}F,$$

and therefore only half as large as for ordinary wheels of the same number of teeth. But this, of course, presupposes that in wheels with two steps the length of the teeth is diminished sufficiently to make the arc of action equal to $\nu\frac{t}{2}$. For this, only about ½ of the addendum employed for ordinary wheels is necessary. If we retain the addendum of the ordinary wheel for the two-stepped one, there will be no diminution of the friction of the teeth, as this depends upon the angle of action ε, which is invariable when the addendum is invariable. By retaining this addendum, moreover, we lose the other advantage attainable by stepped wheels, namely, the smoother motion. For if we remember that the action of two teeth will deviate the more from the ideal motion represented by the axoids the greater the distance

Fig. 271.



from the instantaneous axis at which the teeth come in contact, we shall see that these irregularities can be diminished by stepped wheels only when the length of the teeth is diminished, *i.e.*, when the projections beyond the axoids are reduced. Hence when $e = v\frac{t}{3}$, we have for the friction of the three-stepped wheel, Fig. 272, only

$$F_1 = \pi\varphi\frac{P}{\sin\gamma}\left(\frac{1}{3z_1}+\frac{1}{3z_2}\right)\frac{v}{2} = \frac{1}{3}F.$$

If we suppose a stepped wheel to be composed of an infinite number of small wheels, the teeth will take the oblique position relatively to the axis shown in Fig. 273. These wheels, first mentioned by *White*,* reduce the friction, provided the teeth are very short; on the other hand their obliquity causes lateral pressures in the direction of the axes, which produce new frictional resistances at the collars or ends of the journals. Attempts have been made to neutralize these lateral pressures by combining two of these wheels, the teeth being of equal but opposite inclination, as in Fig. 274; but these and other oblique teeth have been little used in practice, principally because the pressure on the teeth is always confined to a very small surface, strictly speaking, to a point, causing rapid wear. Wheels having teeth which are oblique in only one direction, are used in machines requiring little

FIG. 272.

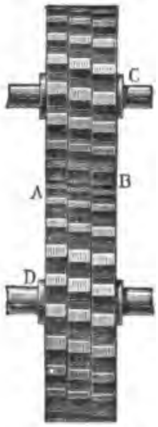


FIG. 273.

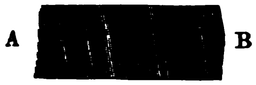
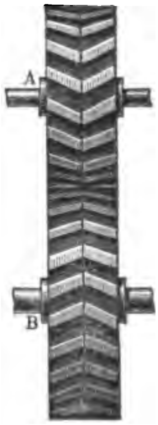


FIG. 274.



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* *White*, Century of Inventions, 1822.

force, but as smooth a motion as possible, for example, in watchmakers' lathes. Wheels whose teeth are inclined in either direction, Fig. 274, have also been proposed for railways having steep grades.

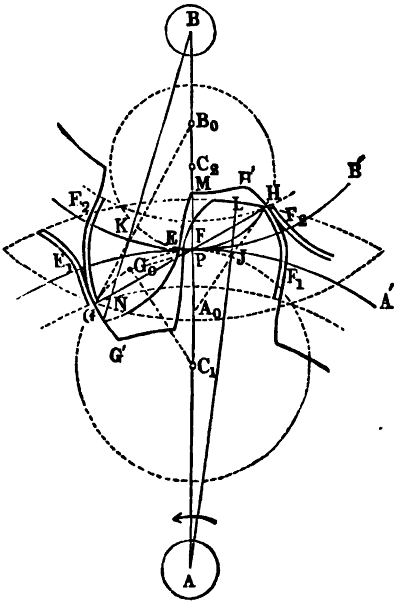
It is easy to see that the *form* of oblique teeth must be like that of screw surfaces of large pitch, on which account these wheels are often called screw wheels, although there is no *screw action*, properly speaking.

REMARK. The method of determining the friction of involute profiles may be followed for other forms of profiles. Let us take for example the cycloidal profiles generated by rolling the two circles C_1 and C_2, Fig. 275, with radii ρ_1 and ρ_2, on the pitch circles A' and B', and let us again take $G P = e_1$ and $P H = e_2$ for the arcs of contact, and consequently $E_1 P = E_2 P = e_1$ and $P F_1 = P F_2 = e_2$ for the arcs of action on the pitch circles. Here the direction of the pressure is variable, making an angle of 90° with $A B$ when the contact is at P, and having $G P$ and $P H$ for its direction when the contact is at the extreme points G and H respectively. If γ represents the average angle made by the direction of pressure with the line of centres, we shall have

$$K = \frac{P}{sin\,\gamma},$$

the angle γ differing but little from a right angle.

FIG. 275.



The distance λ of any point G_0 from the instantaneous centre P is given by

$$P\,G_0 = 2\,\rho_1\,sin\,\tfrac{1}{2}\,G_0\,C_1\,P = 2\,\rho_1\,sin\,\frac{r_1\,\alpha}{2\,\rho_1},$$

for

$$G_0\,C_1\,P = \frac{r_1}{\rho_1}\,\alpha,$$

α again representing the angle through which the wheel A turns when the point of contact travels from G_0 to P. Since for external gearing

$$\delta\,\omega = \frac{r_1 + r_2}{r_2}\,\delta\,\alpha$$

$\left(\frac{r_2 - r_1}{r_2}\,\delta\,\alpha \text{ for internal gearing}\right)$ we shall again have:

$$A_1 = \int_0^{\varepsilon_1} \varphi K \lambda\,\delta\,\omega = \varphi K \int_0^{\varepsilon_1} 2\,\rho_1\,\sin\frac{r_1\,\alpha}{2\,\rho_1}\,\frac{r_1 + r_2}{r_2}\,\delta\,\alpha$$

$$= \varphi K\,4\,\rho_1^2\,\frac{r_1 +{}_2 r}{r_1\,r_2}\left(\cos 0 - \cos\frac{r_1\,\varepsilon_1}{2\,\rho_1}\right).$$

If we here substitute the approximate value

$$\cos x = 1 - \frac{x^2}{2}$$

we shall have:

$$A_1 = \varphi K\,4\,\rho_1^2\,\frac{r_1 + r_2}{r_1\,r_2}\,\frac{r_1^2\,\varepsilon_1^2}{8\,\rho_1^2} = \varphi K\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\frac{r_1^2\,\varepsilon_1^2}{2}.$$

As the path along which this work was done is $r_1\,\varepsilon_1$ we shall have for the friction, reduced to the pitch circle,

$$F = \frac{A_1}{r_1\,\varepsilon_1} = \varphi K\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\frac{r_1\,\varepsilon_1}{2},$$

or if we again take

$$r_1\,\varepsilon_1 = e_1 = e_2 = \frac{e}{2} = \frac{\nu t}{2} \quad \text{and} \quad \frac{t}{2\,r} = \frac{\pi}{z}$$

we shall have

$$F = \pi\,\varphi K\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\frac{\nu}{2} = \pi\,\varphi\,\frac{P}{\sin\gamma}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\frac{\nu}{2}.$$

The expression just found

$$F = \pi\,\varphi\,\frac{P}{\sin\gamma}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\frac{\nu}{2}$$

may therefore be regarded as a general one, and if we place $\sin\gamma = 1$, we shall have

$$F = \pi\,\varphi P\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\frac{\nu}{2}.$$

It is evident, moreover, that the friction of involute teeth will be somewhat *greater* than that of the cycloidal teeth, the angle γ being smaller for the former than for the latter.

§ 80. General Remarks Concerning the Forms of Teeth.—With *rapidly running shafts* it is important that the forms of teeth be as exact and correct as possible, for an inaccurate or incorrect form is accompanied by a variable velocity ratio causing shocks, these being greater the greater the living force stored in the shafts. With slowly running shafts, for example the shafts of hoisting gear, the disadvantages resulting from incorrect forms of teeth are insignificant, and wheels may be employed whose profiles do not satisfy the above conditions of uniform transmission of motion, whenever practical reasons, such as the use of existing models, etc., render it desirable. But when the wheels run rapidly, transmitting large forces and setting large masses in rotation, such reasons should have no influence, unless the profiles or patterns on hand fully satisfy the conditions of uniform transmission of motion. That incorrect forms of teeth have decided disadvantages for high speed transmissions may easily be seen. Let us suppose a shaft A, which turns uniformly with the angular velocity α, to drive the shaft B with the angular velocity β, the two toothed wheels employed for this purpose having such radii a and b that $a\,\alpha = b\,\beta$. Now, if the forms of the teeth are incorrect, the directions of the pressure will not always pass through the instantaneous centre, but will cut the line of centres at distances from the axis A alternately greater (a_1) or smaller (a_2) than a; as a consequence the wheel B will have a variable angular velocity greater than β in the first case, *i.e.*, when $a_1 > a$, and smaller in the second case when $a_2 < a$. The result of this will be that the whole mass connected with the shaft B must at one time be subject to an acceleration caused by a pressure between the teeth which is greater than the average resistance due to the work of the shaft B, and at another time the shaft B, by virtue of the living force stored in it, will get in advance of A, overcoming some of the useful resistances by a part of its stored-up energy. At this instant the tooth of the wheel B will not be pressed upon by the tooth of the wheel A, but as it advances faster than A the back of the tooth on B will press against the back of the tooth on A. This is accompanied by a certain shock due to the side-clearance (or play between the teeth) which must be traversed by the tooth on B before its back can come in contact with the back of the tooth on A. Immediately after this takes place the next pair of teeth come in contact, the tooth on B being again overtaken by that on A, and there is a slight shock between the front surfaces of the teeth. As these occurrences are repeated $n z$ times a minute, n being the number of revolutions and z the number of teeth of one of the wheels, we have a sufficient explanation of the clattering which takes place when incorrect profiles are used with rapidly running wheels. That these continually repeated vibrations to which the material is subjected will ultimately diminish its strength, is a well known fact. The evil mentioned will be greater the greater the circumferential speed of the wheels, the greater the masses moved and the greater the side-clearance; the latter may be suitably reduced by careful construction, *i.e.*, accurate spacing of the teeth. The evils mentioned are more marked when, as is sometimes done in rolling mills, heavy fly-wheels are placed on a shaft parallel to the steam engine shaft for the purpose of attaining greater velocities, the fly-wheel shaft being driven by toothed wheels. In such a case it is almost impossible to prevent the wheels from becoming loose on the shaft, and their journals from heating.

We now see why wooden teeth are preferable for rapidly running wheels, the elastic property of wood enabling it to absorb the living forces. It is customary to give wooden teeth to only one of the two wheels which work together, usually to the driver. If we give the wooden teeth to the larger wheel they will last longer, because they are less often in contact, but when they do ultimately wear out, the cost of renewal will be correspondingly greater and the delay often considerable. It therefore seems better, in spite of the often given opposite rule, to give the wooden teeth to the smaller wheel, and it will be easy to have a spare wheel like the latter on hand.

Radial flanks are particularly suitable for wheels with wooden teeth, not only because the surfaces can be more easily produced, but also because the nearly parallel surfaces are cut with the grain. When the teeth curve rapidly, as with involute profiles, some of the fibres must necessarily be cut across the grain along the flank of the tooth. As a consequence, such teeth are rapidly worn out by the iron teeth in contact with them, especially when the hurtful action in front of the line of centres is considerable, the iron tooth then acting like a tool against the grain. But radial flanks are not desirable for iron wheels, particularly small ones having few teeth, because the rapid convergence of the flanks greatly diminishes the section of a tooth near its root; if, now, the pitch be increased to give sufficient strength, the length of the tooth will also be increased and will cause greater friction and irregularity of motion. Radial flanks are occasionally employed in very small pinions of wrought iron for hoisting gear, where it is desirable to use pinions with a small number of teeth (as low as 3), for radial flanks alone allow of so small a number of teeth, the material employed in this case giving sufficient strength. Pin gearing is not to be recommended except for wrought-iron wheels, in which case it gives very strong teeth; wooden pin gearing, such as was formerly employed in mill work, is seldom constructed at the present day. In clocks and windlasses, however, pins are still much used.

Teeth formed according to involutes are excellent as regards strength, for their form approximates most closely to that of a beam of uniform strength; consequently when involutes are employed, we can have a smaller pitch with the same transmission of force, and thus obtain the advantages of smaller friction and smoother running accompanying a greater number of teeth. But involute profiles cannot be so readily employed with very small numbers of teeth (windlasses, etc.), for their minimum number is greater than in cycloidal teeth. Moreover, when the two wheels which work together are very unequal, the teeth of the smaller wheel become so pointed that there is danger of the upper edges fracturing; it is therefore customary to choose some other form of tooth when the ratio of the wheels exceeds 5. An objection sometimes urged against involute teeth is, that the oblique direction of their line of pressures causes considerable pressure on the shafts, but this disadvantage is much less than it seems. If we suppose the inclination of the line of pressure to the line of centres to be 75°, which is the ordinary amount, we shall have for the pressure against the axes

$$\frac{P}{\sin 75^\circ} = \text{about } 1.04\, P,$$

P being the resistance at the circumference. In cycloidal teeth also, the line of pressure is inclined to the line of centres, forming a variable angle with the latter, which is equal to a right angle for one instant only, namely when the teeth touch at the instantaneous axis. Now although the average inclination of the line of pressure to the tangent of the pitch circles is somewhat less for cycloidal teeth than for involute, the variation of this direction of pressure, and the consequent variation of the pressure itself, is disadvantageous with respect to the wear of the teeth and the smooth running of the wheels. These disadvantages are usually greater than that arising from the slight increase of the pressure against the axes in the case of involute wheels.

One of the principal advantages of involute profiles is that they remain correct profiles when they are slightly moved toward or from each other. A slight separation of the shafts does often take place in consequence of wear in the bearings, and no teeth but involute teeth will then work together correctly. On the other hand, the possibility of bringing the shafts closer together permits the side clearance to be diminished as much as possible when the wheels are set up. That such a shifting of the axes may take place without interfering with the correct working of the profiles may be inferred from the fact that the base circles of the profiles always have a common tangent which cuts the line of centres at a point whose distances from the axes are proportional to the radii of the base circles. One effect of bringing the shafts closer together will be to diminish the pitch circles, or circles of equal velocity, without changing their ratio, and another effect will be that the line of pressures will assume a somewhat greater inclination to the line of centres ; the arcs of action are thereby slightly increased because the addenda of the teeth are increased. The opposite takes place when the shafts are separated. This property renders involute teeth very suitable where, as in rolling mills, the distance of the axes varies. All these reasons show involute teeth to be among the most perfect.

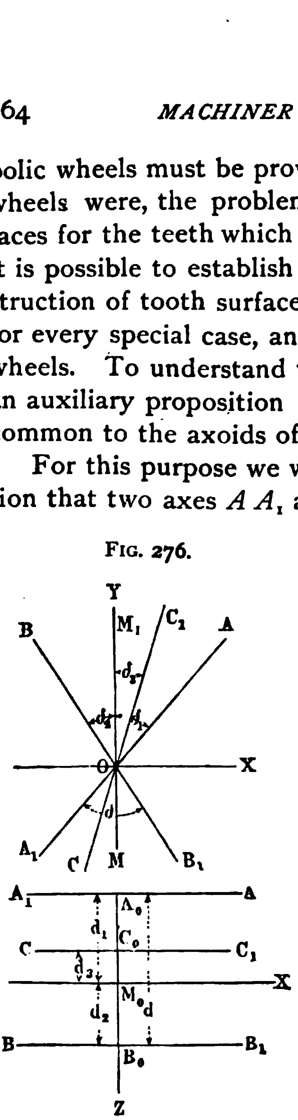
There is no especial advantage to be gained by employing circular profiles for the teeth, as they are only approximately correct forms, transmitting motion correctly at but one or two lines of contact, and their simpler form renders the construction of the wheels no easier; at most they facilitate the preparation of general drawings, where it is understood that the teeth are not to be constructed exactly as drawn. A particular disadvantage of a profile composed of two arcs, constructed as in § 74, is that these arcs are not tangent to each other at the pitch circles, but form a blunt edge which soon wears off, unless it is rounded off before the wheels are used.

Finally, we may remark that of late excellent wheel moulding machines are much employed in foundries, rendering the constructor of gearing less dependent on the pattern room, and consequently greatly diminishing the importance of interchangeable wheels. As, in the production of wheels by moulding machines, the form of tooth has little influence on the cost of construction, there is greater freedom of design, the formerly rigid rules concerning the addendum of teeth, etc., having less controlling influence.

§ 81. Wheels for Non-parallel Shafts.—The case in which the axes of wheels are not parallel has been already investigated in §§ 45 and 46. It was there found that when the axes intersect, the axoids are two conical surfaces whose common vertex is at the intersection of the axes, and that when the axes cross without intersecting the axoids are represented by two hyperboloids of revolution. Conic and hyperbolic wheels must be provided with teeth very much as spur wheels were, the problem here, as there, being to find surfaces for the teeth which will transmit uniform motion. Now it is possible to establish a perfectly general law for the construction of tooth surfaces, from which we may deduce rules for every special case, and therefore also for the case of spur wheels. To understand this law, we must first demonstrate an auxiliary proposition enunciating a remarkable property common to the axoids of every pair of wheels.

For this purpose we will again make the general supposition that two axes $A A_1$ and $B B_1$, Fig 276, cross without intersecting, the shortest distance between them being $A_0 B_0 = d$, and their angle of inclination equal to δ. Let α and β again represent the angles of rotation of the axes and the unit of time, then we shall get, as in § 46, for the corresponding instantaneous axis of the relative motion, a straight line $M M_1$ lying in a plane which is parallel to the pair of parallel planes passing through $A A_1$ and $B B_1$, and whose distances from the axes are given by the relation

FIG. 276.



$$\frac{d_1}{d_2} = \frac{\tan \delta_1}{\tan \delta_2},$$

δ_1 and δ_2 again representing the inclinations of the axes $A A_1$ and $B B_1$ to the instantaneous axes, for which we have

$$\frac{\sin \delta_1}{\sin \delta_2} = \frac{\beta}{\alpha}.$$

If we imagine this instantaneous axis $M M_1$ to be rotated once about $A A_1$ and once about $B B_1$, its inclination and distance remaining the same, we know from the former discussion that two hyperboloids of revolution will be generated which touch at $M M_1$ and are to be regarded as the axoids or surfaces of instantaneous axes for the present case. We may remark that these two axoids will be wholly determined when the mutual position of the axes, that is d and δ, and the velocity ratio $\frac{\alpha}{\beta}$ are given, and we may speak of these hyperboloids as the *axoids associated* for the velocity ratio $\frac{\alpha}{\beta}$. Now let us suppose with one of these axoids, say the one having the axis $A A_1$, a third axoid to be associated for any other velocity ratio, say $\frac{\alpha}{\gamma}$, and let us imagine it in such a position that contact between the first and third will take place along the same straight line $M M_1$ on which contact takes place between the axoids of $A A_1$ and $B B_1$ first considered. To obtain such a third axoid it is only necessary to assume, on the common normal $A_0 B_0$, at a distance $d_3 = C_0 M_0$ from M_0, an axis $C C_1$ which lies in a plane parallel to the axes $A A_1$ and $B B_1$, and forms with $M M_1$ an angle δ_3 for which

$$\frac{sin\ \delta_1}{sin\ \delta_3} = \frac{\gamma}{\alpha},$$

d_3 being at the same time limited by the condition:

$$\frac{d_1}{d_3} = \frac{tan\ \delta_1}{tan\ \delta_3}.$$

This third hyperboloid is then evidently of such a character that when it rotates through an angle γ, it will cause the hyperboloid A to turn through the angle α, if both hyperboloids are allowed to act upon each other as two friction wheels, in other words, the two hyperboloids are the correct axoids for the two axes $A A_1$ and $C C_1$, and correspond to a velocity ratio $\alpha : \gamma$.

If we now consider this hyperboloid of the axis $C C_1$ in its relation to the one belonging to the axis $B B_1$, we shall readily find that these two surfaces are the correct axoids for the two axes $B B_1$ and $C C_1$, and correspond to a velocity ratio $\beta : \gamma$. This is immediately seen when we divide

$$\frac{\sin \delta_1}{\sin \delta_3} = \frac{\gamma}{\alpha} \quad \text{by} \quad \frac{\sin \delta_1}{\sin \delta_2} = \frac{\beta}{\alpha}$$

and

$$\frac{d_1}{d_3} = \frac{\tan \delta_1}{\tan \delta_3} \quad \text{by} \quad \frac{d_1}{d_2} = \frac{\tan \delta_1}{\tan \delta_2},$$

thus getting

$$\frac{\sin \delta_2}{\sin \delta_3} = \frac{\gamma}{\beta} \quad \text{and} \quad \frac{d_2}{d_3} = \frac{\tan \delta_2}{\tan \delta_3},$$

which equations contain the conditions which must be satisfied by the axoids of the two axes $B B_1$ and $C C_1$ if they are to have the velocity ratio $\beta : \gamma$. From this may be deduced the following property of the axoids of wheels:

When two axes A and B have their position and their velocity ratio $\alpha : \beta$ given, and a third axoid C is so associated with the axoid of A as to represent the velocity ratio $\alpha : \gamma$, this third axoid C will likewise be so associated with the axoid belonging to B as to represent the velocity ratio $\beta : \gamma$.

To render this law clear we may imagine the three hyperboloids A, B and C to be constructed of very thin sheet metal and to be pressed together at the common line of contact $M M_1$ after the manner of friction wheels, so that when one of them turns, the others will turn correspondingly. Now if the third axoid C turns uniformly through the angle γ, it will cause a uniform rotation of the axoid A through the angle α, and simultaneously a uniform rotation of the axoid B through the angle β, the rotations α and β being exactly equal to the rotations which would be assumed if the axoids A and B acted directly upon each other as friction wheels. In the illustration given, the axoids A and B were turned by the help of a third axoid, which we can therefore call the *auxiliary axoid.* This auxiliary axoid may be chosen in an infinite number of ways, but is by no means perfectly arbitrary, being restricted by the perfectly definite conditions

$$\frac{\sin\delta_1}{\sin\delta_3} = \frac{\gamma}{\alpha} \quad \text{and} \quad \frac{d_1}{d_3} = \frac{\tan\delta_1}{\tan\delta_3},$$

underlying the case of two axes which cross but do not intersect. If therefore we imagine all the auxiliary axoids possible for the pair of axoids A and B, the axes C of all these axoids will lie upon a ruled surface whose character may be determined as follows.

If we suppose the instantaneous axis MM_1 of the axoids of two shafts A and B which cross but do not intersect, to be the axis of Y, and the shortest distance $A_0 B_0$ between the axes to be the axis of Z, of a rectangular system of co-ordinates, we shall have for any auxiliary axoid whose axis is at the distance $M_0 C_0 = z$ from the instantaneous axis $M M_1$, the equation

$$\frac{z}{d_1} = \frac{\tan\delta_3}{\tan\delta_1},$$

where d_1 is the radius of the gorge circle of the axoid A and δ_3 and δ_1 the inclinations of the instantaneous axis to the axes C and A.

If we substitute in this $\tan\delta_3 = \frac{x}{y}$, we shall have for the equation of the surface containing *the axes of rotation of all the possible auxiliary axoids*

$$\frac{x}{y} = \frac{\tan\delta_1}{d_1} z, \quad \text{or} \quad x = kyz,$$

k being the constant representing the pair of principal axoids A and B, *i.e.*,

$$k = \frac{\tan\delta_1}{d_1} = \frac{\tan\delta_2}{d_2}.$$

The ruled surface represented by this equation is an hyperbolic paraboloid. A plane parallel to the co-ordinate plane XY at a distance $z = c$ from the latter, will cut this ruled surface in the axis of an auxiliary axoid whose gorge-circle radius is c, while all planes parallel to the co-ordinate plane XZ will cut the ruled surface in straight lines which intersect the instantaneous axes, the equation of straight lines being $x = k c_1 z$, when the distance of the cutting plane is $y = c_1$. Finally, a plane parallel to the co-ordinate plane YZ at a distance $x = c_2$ from the latter, will intersect the ruled surface in an equilateral hyperbola represented by $yz = \frac{c_2}{k}$. We may call this ruled surface the *locus of the axes of rotation* of the auxiliary axoids belonging to the two primary axoids.

It is clear that when the axes are parallel (spur wheels) or when they intersect (bevel wheels) this surface will become a plane, namely, the plane of the axes. In the first case, every line of this plane parallel to the axes may be employed as the axis of the auxiliary axoid, while in the latter case, every straight line of the plane which passes through the point of intersection of the axes may be regarded as an axis for the auxiliary axoid.

§ 82. **General Law Governing the Formation of Tooth Surfaces.**—Let us suppose any third or auxiliary axoid C to be associated with the axoids A and B of any two wheels, and that the axoid C has rigidly attached to it any straight or curved line L. When all three axoids are given their proper rotations, the line L will generate two surfaces relatively to the two axoids A and B. It may help to render the matter clearer, to suppose the generating line L to be a cutting edge or a thin wire, and the axes A and B to be surrounded by plastic masses, into which the sharp edge or wire cuts while it is making its supposed motion, leaving traces in the form of the two surfaces F_1 and F_2.

At every instant these two surfaces have the generating line L in common, and, in order that they may be employed as tooth surfaces, it is necessary that they *touch* along this line, and also that they permit the relative motion peculiar to the axoids A and B. We know from the property of auxiliary axoids discussed in the preceding article, that the latter condition is always satisfied, but we may specially convince ourselves of this by the following calculation. Suppose the auxiliary axoid C to be brought to rest by giving to the whole system an additional rotation $-\gamma$ about the axis CC_1, and the two rotations to which the axoid A is then subjected (α about A and $-\gamma$ about C) to be replaced by a helical motion around the instantaneous axis MM_1, the rotating part ω_a of this motion being determined according to § 46, from

$$\alpha : \omega_a = \sin\delta_3 : \sin(\delta_1 + \delta_3),$$

or

$$\omega_a = \alpha\frac{\sin(\delta_1 + \delta_3)}{\sin\delta},$$

and the shifting portion of the motion from

$$s_a = (d_1 + d_2)\,\alpha\sin\delta_1.$$

In like manner the two rotations of the axoid B (β around B and $-\gamma$ around C) are equivalent to a helical motion around this same axis MM_1, the rotating and shifting portions of the motion being respectively

$$\omega_b = \beta\frac{\sin(\delta_2 - \delta_3)}{\sin\delta_3}$$

and

$$s_b = -(d_2 - d_3)\,\beta\sin\delta_2.$$

From this we find that the relative motion of the axoids A and B is likewise a helical motion around the instantaneous axis MM_1. The rotative component of this helical motion is:

$$\omega = \omega_a - \omega_b = \alpha\frac{\sin(\delta_1 + \delta_3)}{\sin\delta_3} - \beta\frac{\sin(\delta_2 - \delta_3)}{\sin\delta_3}$$

$$= \alpha\sin\delta_1\frac{\sin(\delta_1 + \delta_3)}{\sin\delta_1\sin\delta_3} - \beta\sin\delta_2\frac{\sin(\delta_2 - \delta_3)}{\sin\delta_2\sin\delta_3},$$

or since $\alpha \sin \delta_1 = \beta \sin \delta_2$:

$$\omega = \alpha \sin \delta_1 (\cot \delta_1 + \cot \delta_2) = \alpha \frac{\sin (\delta_1 + \delta_2)}{\sin \delta_2}.$$

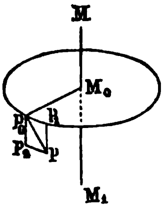
For the shifting component of the helical motion we have

$$s = s_a - s_b = (d_1 + d_2)\, \alpha \sin \delta_1 = d\, \alpha \sin \delta_1 = d \beta \sin \delta_2.$$

Now this relative rotation and translation to which the axoids A and B are subjected during the generation of the surfaces above mentioned, are exactly equal to the relative helical motion of the axoids A and B when the latter have the required motion. As the corresponding elements of the surfaces F_1 and F_2 are generated by motions exactly like those required of the axes A and B, it is proved that these surfaces will answer for correct tooth surfaces, provided they always *touch* along their common line L.

In order to ascertain when this is the case, we must note that when the auxiliary axoid twists as above described about the instantaneous axis $M M_1$, any point p_0 of the generating line, Fig. 277, will generate at every instant, elementary paths $p_0 p$ which are perpendicular to the instantaneous radius $p_0 M_0$ let fall from p_0 on the instantaneous axis. Now let some line T be the tangent to the generating line L at the point p_0 and let T_1 and T_2 be tangents to the paths $p_0 p$ described by the point p_0 at any instant, in consequence of the twisting motions; the two tangent planes $T p_0 T_1$ and $T p_0 T_2$ of the above mentioned surfaces F_1 and F_2 must coincide if these surfaces are to touch at p_0, *i.e.*, if they are to touch along the generatrix L. This condition is always satisfied in spur and bevel wheels, for in these the two tangents T_1 and T_2 to the paths coincide. On the other hand, in the hyperbolic axoids of axes which cross but do not intersect, the tangents T_1 and T_2 to the screw-like paths, deviate from each other, consequently contact at p_0 is

FIG. 277.



only possible when the tangent T to the generatrix is perpendicular to the instantaneous radius $p_0 M_0$ (as is always the case with the two tangents T_1 and T_2 to the paths), *i.e.*, contact is possible when the generatrix is a *straight line parallel to the instantaneous axis.*

We may, accordingly, express the general law governing the formation of tooth surfaces as follows:

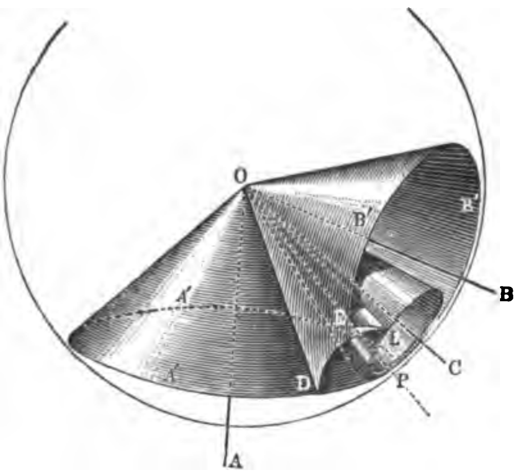
Correct tooth surfaces for any two wheels having parallel or intersecting axes are surfaces which may be generated, relatively to the axes, by any arbitrary line fixed to the auxiliary axoid, all three axoids having the rotations belonging to them. With axes which cross but do not intersect, however, the generatrix is restricted by the condition that it must be a straight line which is always parallel to the instantaneous axis. How the last mentioned case leads to a peculiar (*spiraloidal*) kind of toothed gearing, will be explained in § 86.

According to the above, with axes which cross but do not intersect, the twisting motion of any arbitrarily chosen axoid will produce only approximately correct surfaces, contact taking place only in points, namely those in which the tangent to the generatrix is normal to the instantaneous radius.

§ **83. Bevel Wheels.**—With the help of this perfectly general law for the formation of correct tooth surfaces, the law given in § 67 for spur wheels being only a particular case, we may now easily determine the tooth surfaces for conical wheels. The axoids of the wheels as well as the auxiliary axoid now become conical surfaces whose common vertex is at the intersection of the axes. For simplicity we will also here, as with spur wheels, assume a *straight line* on the surface of the auxiliary cone and consequently an element of the latter, as the generatrix of the tooth surfaces.

We may now find the rule for determining the tooth surfaces in a manner similar to that employed for spur wheels in § 70. Let OA and OB, Fig. 278, be the axes of two bevel wheels whose axoids touch along the straight line OP, and let OC be the axis of a third cone, which acts as auxiliary axoid and touches the cones A and B in this same instantaneous axis. Any element, for example $O\,L$, of this auxiliary cone will generate two curved surfaces relatively to the axoids A and B, provided all three cones have the rotation assumed in the preceding article. The generated surfaces are likewise conical surfaces, for the generating line $O\,L$ continually passes through the same fixed point O. It is evident from preceding discussions that surfaces thus formed can also be considered as generated by the rolling of the cone C on the outside of the cone A and on the inside of the cone B. In doing this the generatrix $O\,L$ will cut the surface of a sphere, having O as a centre and $O\,L$ as a radius, along two curves $L\,D$ and $L\,E$ called *spherical cycloids;** the curve $L\,D$ generated by external rolling being a spherical epicycloid, and $L\,E$ generated by internal rolling a spherical hypocycloid. These spherical curves can be regarded as the base lines of the conical surfaces which bound the teeth.

Fig. 278.



If through any point L of the line of contact or instantaneous axis we pass two planes perpendicular to the axes, they will cut the axoids in two circles $P\,A'A'$ and $P\,B'B'$ which roll on each other and have equal velocity. These circles evidently play a part in bevel wheels similar to that of pitch circles in spur wheels, and are also called pitch circles, but to avoid ambiguity, it is necessary to state at what distance from the intersection O the circles are which are called pitch circles. This is why we speak of outer, inner, or average pitch circles, and also of the outer, inner, and average pitch of bevel wheels, according as the data refer to the outer, inner, or average circumferences of the wheels. It is clear that with respect to these pitch circles we may, as in spur wheels, speak of *faces* and *flanks*, and that the relations determined for these in spur wheels, for example, that faces can only work with flanks, also holds for bevel wheels. Therefore in order that each wheel may have faces as well as flanks, it is here also necessary that there be two rollings for each wheel, the epicycloids being given to the faces and the hypocycloids to the flanks. Again, it is easily shown how far what was said concerning internal gearing in spur wheels applies to bevel wheels, and we may moreover refer to § 45 for an explanation of what is meant by internal gearing in bevel wheels. It is also easily shown how pins may be employed instead of teeth for bevel wheels; in such a case, of course, the pins cannot be of cylindrical form, but must be cones whose vertices are all at the intersection of the axes.

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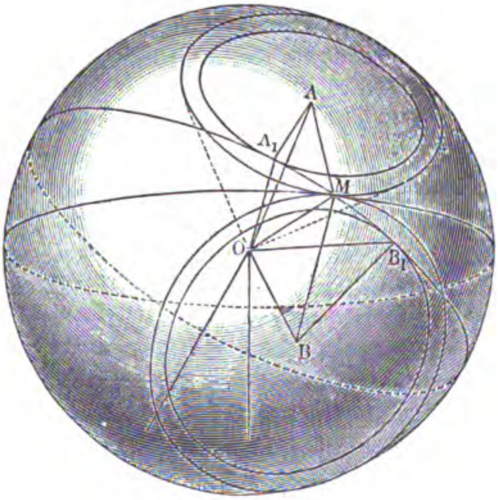
* See Verhandlgn. d. Ver. z. Beförderg. d. Gewerbfleisses, 1876.

As regards the limiting case between external and internal gearing represented in spur wheels by the rack, we have for bevel wheels the face wheel, for which the half angle at the vertex is 90°, this angle being acute in external and obtuse in internal gearing. In spur wheels when the describing cylinder became a plane, we obtained a particular form of tooth known as the involute tooth; in like manner, we obtain for bevel wheels a corresponding special form of tooth by rolling the face wheel, as auxiliary axoid, on both the axoids. Thus, if we let the half angle δ_3 at the vertex of the rolling cone C grow larger and larger till it is equal to a right angle, the conical surface will be gradually transformed into a plane which touches the two axoids at the instantaneous axis OP. When this plane is rolled on the two cones A and B, a straight line of this plane passing through O will generate conical surfaces whose intersections with the sphere described about O are known in geometry as ***spherical involutes.*** It is also evident that here, as in spur wheels, these spherical involute cones cannot be employed as tooth surfaces in external gearing, as they are only suitable for the faces of teeth, and it is well known that faces cannot work together. At most these surfaces can be used for internal gearing in bevel wheels, as in the corresponding case of spur wheels. But we may also employ involute profiles for bevel wheels by employing them as in spur wheels, namely, by rolling the face wheel C not directly on the axoids A and B, but on certain base cones A_1 and B_1, Fig. 279, obtained as follows: Through the line of contact OM of the two cones whose axes are OA and OB, pass a plane making an angle φ = about 75° with the plane of the axes, then suppose two cones to be described concentric to the axes OA and OB and tangent to this plane along the elements OA_1 and OB_1; these two cones may be regarded as the base cones on which the tangent plane or face wheel OA_1MB_1 must be rolled in order to obtain tooth surfaces which will work together. In this way a radius, say OM, of the face wheel will generate two conical surfaces whose intersections with the sphere passed through M and about O will be the spherical involutes mentioned, and it is easy to show, as in spur wheels, that these surfaces will give correct forms for the teeth. For let the half angles at the vertex of the cones AOA_1 and BOB_1 be represented by δ_1' and δ_2' respectively, we shall have from the corresponding spherical triangles $OAMA_1$ and $OBMB_1$:

Fig. 279.



$$\sin \delta_1' : \sin \delta_1 = \sin \varphi : \sin 90°$$

and $\sin \delta_2' : \sin \delta_2 = \sin \varphi : \sin 90°$,

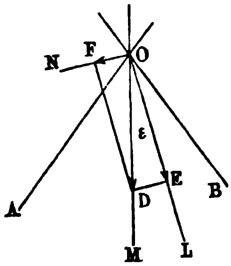
hence

$$\frac{\sin \delta_1'}{\sin \delta_2'} = \frac{\sin \delta_1}{\sin \delta_2} = \frac{\beta}{\alpha}.$$

When the two cones A and B move with their angular velocities α and β, the *base cones* A_1 and B_1 will turn with angular velocities whose ratio is the same as $\alpha : \beta$ and this, as in § 73 for spur wheels, proves the correctness of the assumption above made.

§ **84. Tooth Friction in Bevel Wheels.**—It is evident from the foregoing presentation that the successive positions of the generating line are identical with the successive positions of the line of contact of the teeth, and as the generating line lies on the surface of the auxiliary axoid, it is clear that this latter surface must be the *locus of contact*. This view of the matter also enables us to judge of the kind and amount of the friction occurring between the teeth. There is evidently no friction between the teeth when they touch along the instantaneous axis OM, for the relative motion is then simply a rolling about this line of contact. But if we suppose the wheel A, whose average radius is r_1, to have turned out of this position through a certain small angle α, the wheel B will likewise have turned through an angle $\beta = \frac{r_1 \alpha}{r_2}$ and contact will now exist along a certain straight line OL, Fig. 280, making the angle ε with the instantaneous axis OM. With the small angles here considered it will be sufficiently accurate to assume $r_1 \alpha = r_2 \beta = l \varepsilon$ when l represents the length of a conical element $OM = OL$ measured up to the average pitch circle. If we now suppose the wheel A to turn out of this position through $\delta \alpha$, the wheel B will turn through the angle $\delta \beta = \frac{r_1}{r_2} \delta \alpha$ and

FIG. 280.



these rotations will correspond to a relative motion which is equivalent to a rotation about the instantaneous axis $O M$, *i.e.*, to

$$\delta \omega = \sqrt{\delta \alpha^2 + \delta \beta^2 + 2 \delta \alpha . \delta \beta . cos \, \delta}$$

$$= \delta \alpha \sqrt{1 + \left(\frac{r_1}{r_2}\right)^2 + 2 \frac{r_1}{r_2} cos \, \delta},$$

where δ is the angle included between the axes.

Now, since the teeth at the instant under consideration, do not touch along the instantaneous axis $O M$, but along a line $O L$ making the angle ε with $O M$, we may resolve the motion $\delta \omega$ about the instantaneous axis into two rotations, one $\delta \omega . cos \, \varepsilon$ about the line of contact $O L$ as an axis, and another $\delta \omega . sin \, \varepsilon$ about an axis $O N$ perpendicular to $O L$ and lying in the plane $M O L$. It may be easily shown that the last mentioned axis is perpendicular to those elements of the surfaces of the teeth which are in contact at the instant, and consequently that the rotation component $\delta \omega . sin \, \varepsilon$ will cause the tooth surfaces to exert a sort of *drilling* action on each other which generates the friction, differing in this respect from spur wheels, where the friction is produced simply by sliding. Now, if P represents the normal pressure between the teeth and φP the friction, and if we suppose the pressure P to be concentrated at the middle point of contact, the work consumed by friction while the wheel A rotates through $\delta \alpha$ will be $\varphi P . l \delta \omega . sin \, \varepsilon$, and as ε is only a small angle, a sufficiently close approximation will be

$$\varphi P l \varepsilon \delta \omega = \varphi P r_1 \alpha \delta \omega$$

$$= \varphi P r_1 \sqrt{1 + \left(\frac{r_1}{r_2}\right)^2 + 2 \frac{r_1}{r_2} cos \, \delta} \; . \, \alpha \delta \alpha.$$

If we now assume that two teeth are in contact in front as well as behind the line of centres, and that the arc of action is equal to the pitch t, the angle of contact will be $\tau_1 = \frac{t}{r_1}$ for the wheel A, and $\tau_2 = \frac{t}{r_2}$ for the wheel B, and the total work of friction during such an angle of rotation will evidently be

$$F = \varphi P r_1 \sqrt{1 + \left(\frac{r_1}{r_2}\right)^2 + 2\frac{r_1}{r_2}\cos\delta} \int_0^{\frac{t}{r_1}} \alpha\, \delta\alpha$$

$$= \varphi P \frac{t^2}{2 r_1} \sqrt{1 + \left(\frac{r_1}{r_2}\right)^2 + 2\frac{r_1}{r_2}\cos\delta}\,.$$

If we here substitute for r_1 and r_2 the numbers z_1 and z_2, we shall have, since $r = \frac{tz}{2\pi}$, the expression

$$F = \varphi P \frac{t}{2}\frac{2\pi}{z_1} \sqrt{1 + \left(\frac{z_1}{z_2}\right)^2 + 2\frac{z_1}{z_2}\cos\delta}$$

$$= \varphi P \pi t \sqrt{\frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{2}{z_1 z_2}\cos\delta}\,.$$

This formula applies equally to contact in front and behind the line of centers when P represents the pressure of two teeth on each other; the formula therefore also expresses the work consumed by friction during rotation through the pitch arc when P represents the force transmitted by the wheels, as was more fully explained in § 79, with respect to spur wheels. It is also clear that the work of friction will be greater or smaller according as the pitch angle $\frac{t}{r_1}$ is greater or smaller. For axes at right angles, we have $\cos\delta = 0$, and

$$F = \varphi P \pi t \sqrt{\frac{1}{z_1^2} + \frac{1}{z_2^2}}\,.$$

For parallel axes and external contact $\cos\delta = \cos 0 = +1$ and

$$F = \varphi P \pi t \left(\frac{1}{z_1} + \frac{1}{z_2}\right),$$

while for internal contact we have *cos* 180° = − 1 and

$$F = \varphi P \pi t \left(\frac{1}{z_1} - \frac{1}{z_2} \right),$$

which agrees with § 79.

§ 85. Approximate Forms of Teeth for Bevel Wheels.—The exact forms of teeth for bevel wheels resulting from the general law (§ 82) present great practical difficulties when it is sought to realize them by the method of templets hitherto employed for patterns, for it is evident from what has preceded that the templets for exact teeth must have the forms of spherical cycloids or spherical involutes respectively. This would necessitate designing the templets on corresponding spherical surfaces, which would be difficult, as the spherical surface cannot be developed. This is the reason why *Tredgold* (see Tredgold's Edition of Buchanan's Essay on Millwork) proposed a simpler but approximate method of constructing the teeth. This method is given in most of the text-books on machine construction, and has also been generally introduced in practice, for the deviations from the correct form are unimportant for most of the cases which arise in practice. This method we will now give.

The approximate method for bevel wheels, just mentioned, is based on the following construction: two conical surfaces are drawn tangent to the sphere whose centre is at the intersection O of the axes and whose radius is OM, the axes of the conical surfaces coinciding with the axes OA and OB of the wheels, and their vertices coinciding with the points at which these axes pierce the plane perpendicular to the instantaneous axis OM at M. These conical surfaces GME and HMF, which we may call the *supplementary cones*, have the corresponding pitch circles ME and MF in common with the axoids, and in the vicinity of M differ but little from the elements of the spherical surface containing the correct profiles of the teeth. If we suppose this supplementary cone to be developed, and treat the developed sectors CMG and DMH as two spur wheels, that is, if we provide them with teeth by any one of the methods given for spur wheels, we shall get the required templets, which may now be wrapped on the supplementary cone and thus furnish the profiles for the teeth of the bevel wheels. Of course in working up or finishing the teeth, these profiles on the supplementary cones must be regarded as the

Fig. 281.

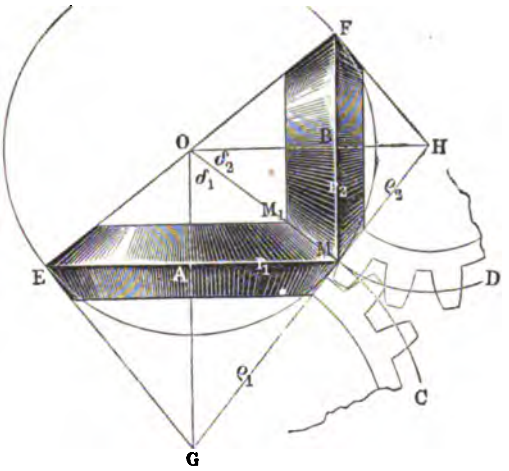
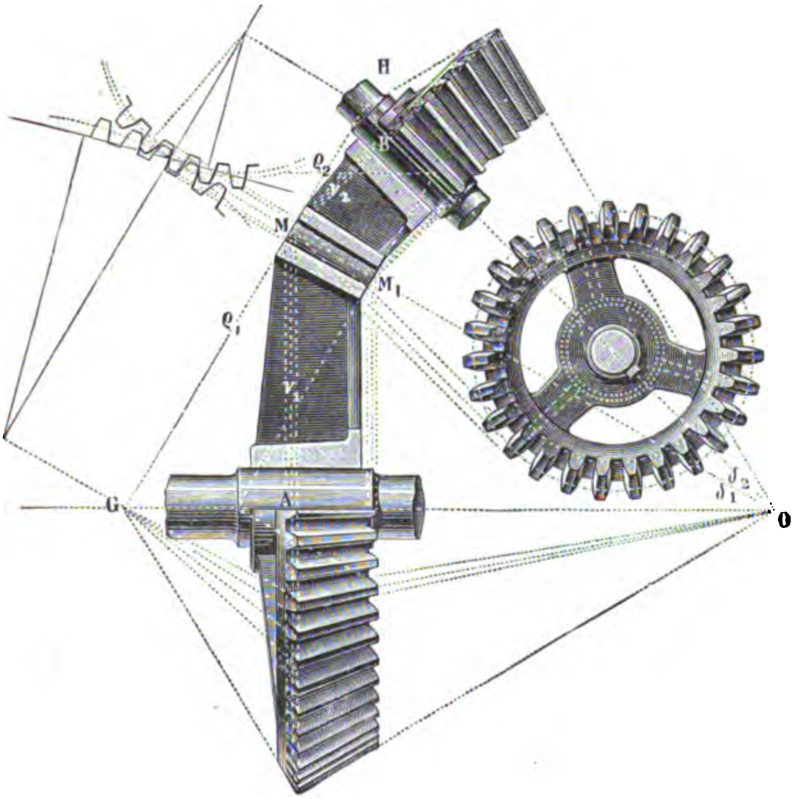


Fig. 282.



base lines of general conical surfaces, whose common vertex is at the intersection O of the axes.

It is customary to make this construction for the outer and inner pitch circles passing through M and M_1 respectively, the manner of doing this being sufficiently clear from Fig. 282. Let r_1 and r_2 again represent the radii $A M$ and $B M$, and let toe radii of the developed sectors be $G M = \rho_1$, and $H M = \rho_2$, we shall then have:

$$\rho_1 = \frac{r_1}{\cos \delta_1}, \quad \text{and} \quad \rho_2 = \frac{r_2}{\cos \delta_2},$$

where δ_1 and δ_2 again represent the angles $A O M$ and $B O M$.

§ **86. Hyperboloidal (Skew-bevel) Wheels.**—The formation of approximately correct tooth surfaces for wheels whose axes cross but do not intersect, readily follows from the general law given in § 82. For the auxiliary axoid which is to generate the required tooth surfaces, we must take an hyperboloid whose gorge circle has a radius d_3, *i.e.*, one whose axis may be taken at any distance from the instantaneous axis, provided it lies upon the *locus of axes* already mentioned; in other words, if the inclination of its axes satisfies the condition:

$$\frac{d_1}{d_3} = \frac{\tan \delta_1}{\tan \delta_3}.$$

If this hyperboloid is given a twisting motion on the axoids of the two wheels, any line connected with it will generate the two tooth surfaces which work together. It is simplest to assume an element of the hyperboloid for the generatrix. If the teeth touch externally, the twisting motion of the auxiliary axoid must take place on the outside of one wheel for the faces, and on the inside of the other for the flanks, and here also it will be desirable to have two twisting motions of the auxiliary axoid, in order that each wheel may have both faces and flanks. The tooth surfaces thus generated present certain analogies to the cycloidal forms of cylindrical and conical wheels, and similar remarks apply here as there, which, however, need not be entered upon more fully.

At this point the question naturally suggests itself, whether it is possible in hyperboloidal, as in other wheels, to obtain involute forms of teeth by allowing the auxiliary axoid to pass to the limiting case between external and internal contact, *i.e.*, to the case which in spur wheels corresponds to the rack and in bevel wheels to the face wheel. This limiting cone is evidently obtained for hyperboloidal wheels when the radius d_3 of the gorge circle of the auxiliary axoid is made larger and larger till it is infinitely large, for a positive value of d_3 corresponds to external, and a negative value to internal contact with the axoid A. Now if we make $d_3 = \infty$, we shall get from

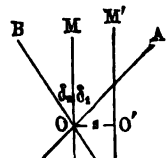
$$\frac{d_1}{d_3} = \frac{\tan \delta_1}{\tan \delta_3},$$

an infinitely large value for $\tan \delta_3$, *i.e.* $\delta_3 = 90°$. This causes the auxiliary axoid to contract to a straight line. If we now imagine this instantaneous axis $O\,M$, Fig. 283, to be itself the generating axoid, this axis will be shifted in a direction perpendicular to itself in a plane parallel to the axes A and B through a distance

$$O\,O' = s = d_1\, \alpha \cos \delta_1 = d_2\, \beta \cos \delta_2,$$

when the wheels A and B turn through the angles α and β respectively. During this motion the straight line $O\,M$ generates correct tooth surfaces relatively to the axes $A\,A_1$ and $B\,B_1$ which satisfy the general condition requiring the formation of flanks as well as faces. It will be remembered that this was not the case with cylindrical and conical wheels in external contact, when the generating plane was rolled directly on the axoids, and that the difficulty was overcome by employing the so-called base cylinders and base cones, § 73 and § 83, respectively. The surfaces thus generated on the axes A and B prove on closer

FIG. 283.



examination to be *ruled surfaces*, which can also be generated by rolling on the cylinders of the gorge circles of the hyperboloids A and B a straight line having the inclinations δ_1 and δ_2 to the axes respectively. These gorge cylinders are concentric to the axes A and B and have the radii d_1 and d_2. We thus obtain screw-shaped ruled surfaces, whose pitch is

$$h = \frac{2\pi d_1}{tan\,\delta_1} = \frac{2\pi d_2}{tan\,\delta_2},$$

the intersections of these surfaces by any plane perpendicular to their axes being involutes to the gorge circles. *Olivier** calls these surfaces *spiraloids*, and, as they are generated by the motion of the instantaneous axis, they may be employed as tooth surfaces. For it is easy to see that these surfaces do not lie wholly outside of the axoids A and B, but cut into them to the depth of the gorge-circle cylinders, so that with these surfaces we can have faces as well as flanks. To be sure, the depth of this cutting at the gorge circles is equal to zero, but it increases each way, so that a given depth of flank requires a given distance of the wheels from the shortest axial distance. For wheels so chosen that their mid-planes pass through the shortest axial distance (such wheels are often called screw wheels), these involute surfaces are as unsuitable as the involute cylinders and cones of the axoids in spur and bevel wheels. *Olivier* called attention to a peculiarity of spiraloids, which is also evident from the construction, that those cut from one side of the gorge circles can only *transmit motion in one direction*, and in order that the opposite rotation may be possible, the tooth surfaces must be cut from the spiraloids on the opposite side of the gorge circles.

Tooth surfaces with involute profiles are constructed for spur and bevel wheels with the help of certain base cylinders (§ 73) and base cones (§ 83), but an analogous construction for skew-bevel wheels is not possible, as a closer examination of the hyperboloidal axoids readily shows. If we employed for tooth surfaces the spiraloids of cylinders having the same ratio (in the sense explained in § 73) as the gorge-circle cylinders, we should at best only get *approximate* tooth surfaces for the wheels whose mid-planes coincide with the gorge circles, and whose general form does not differ essentially from cylinders, on account of their slight axial height. But these tooth surfaces cannot be regarded as exact so long as the axes include a measurable angle. The spiraloidal form just discussed can be employed as an *exact* form of tooth only for parallel axes, and consequently only for cylindrical wheels with teeth oblique to the axes, wherefore such wheels are sometimes called screw wheels (see § 88).

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* See *Olivier:* Théorie géométrique des engrenages.

REMARK. In *Olivier's* treatise, spiraloidal surfaces generated by *any* straight line (parallel to one of the axes), and consequently wholly independent of the position of the instantaneous axis, were assumed as tooth surfaces. Although surfaces thus generated without reference to the instantaneous axis may be simple in form, they cannot be regarded as correct tooth surfaces; at all events, wheels whose teeth are thus formed cannot be in contact along a straight line, as was assumed above, but can only be in contact at a point. See *Olivier:* Theorie géométrique des engrenages, on the subject of spiraloidal gearing; also *Pützer*, Zeitschrift deutscher Ingenieure. 1860.

In skew-bevel, as in bevel wheels, we can employ an approximate method, the profiles of the teeth being drawn on conical surfaces normal to the hyperboloids. Through the points M and M_1 of the instantaneous axis OM, Fig. 284, which limit the length of the tooth, pass planes normal to the instantaneous axis OM, the respective intersections G, H and G_1, H_1 of these planes with the axes of the wheels will be the vertices of the corresponding supplementary cones. By developing the cones we get such developed surfaces as $G'M'C'$ and $H'M'D'$ which may be treated as pitch circles, any of the profiling methods of spur wheels being employed to find the outer and inner profiles of the teeth. If d_1 and d_2 again represent the radii of the gorge circles, and l the length OM, we shall have for the radii of the pitch circles employed for the profiling:

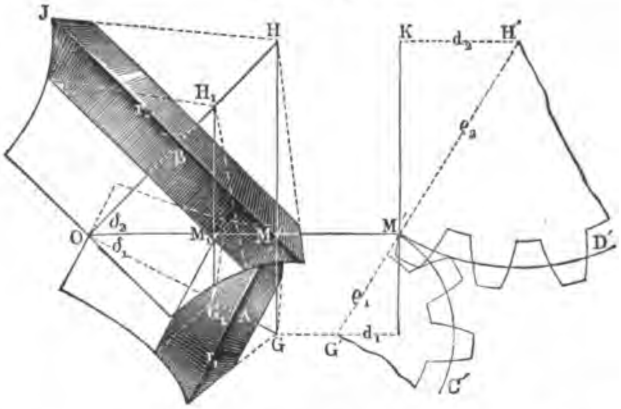
$$G'M' = \rho_1 = \sqrt{d_1^2 + l^2 \tan^2 \delta_1},$$

and

$$H'M' = \rho_2 = \sqrt{d_2^2 + l^2 \tan^2 \delta_2}.$$

We should here note that in hyperboloidal wheels the pitch is not measured in the same manner as in spur and bevel wheels. In each of the last mentioned the pitch is laid off in planes passed through the same point of the instantaneous axis perpendicular to the axes, and is the same for both wheels, but this is not the case with hyperboloidal wheels. For if we suppose the radii of the gorge circles to be d_1 and d_2, the corresponding number of teeth z_1 and z_2, and the angular velocities α and β, we have for the pitches t_a and t_b of these wheels measured along the circumferences of the gorge circles:

FIG. 284.



$$t_a = \frac{2\pi d_1}{z_1}, \quad \text{and} \quad t_b = \frac{2\pi d_2}{z_2},$$

therefore

$$\frac{t_a}{t_b} = \frac{d_1}{d_2}\,\frac{z_2}{z_1}.$$

Now since

$$\frac{d_1}{d_2} = \frac{\tan \delta_1}{\tan \delta_2}, \quad \text{and} \quad \frac{z_2}{z_1} = \frac{\alpha}{\beta} = \frac{\sin \delta_2}{\sin \delta_1},$$

δ_1 and δ_2 representing as before the inclinations of the axes to the instantaneous axis, we have, after substituting these values:

$$\frac{t_a}{t_b} = \frac{\tan \delta_1}{\tan \delta_2} \; \frac{\sin \delta_2}{\sin \delta_1} = \frac{\cos \delta_2}{\cos \delta_1},$$

or

$$t_a \cos \delta_1 = t_b \cos \delta_2 = t.$$

Evidently t here represents the lateral motion of the instantaneous axis in a plane parallel to the axes and in a direction perpendicular to itself when t_a, t_b, and t are taken infinitely small.

The circumferential pitches t'_a and t'_b of the wheels in the two planes which pass through any point of the instantaneous axis at the distance l from the point of contact of the pitch circles, and which are normal to the axes, are given by:

$$t'_a = \frac{2 \pi r_1}{z_1} = \frac{2 \pi}{z_1} \sqrt{l^2 \sin^2 \delta_1 + d_1^2},$$

and

$$t'_b = \frac{2 \pi r_2}{z_2} = \frac{2 \pi}{z_2} \sqrt{l^2 \sin^2 \delta_2 + d_2^2},$$

where r_1 and r_2 are the radii of the sections made by these planes; dividing, we get for the ratio of the pitches:

$$\frac{t'_a}{t'_b} = \frac{z_2}{z_1} \sqrt{\frac{l^2 \sin^2 \delta_1 + d_1^2}{l^2 \sin^2 \delta_2 + d_2^2}} = \frac{\sin \delta_2}{\sin \delta_1} \sqrt{\frac{l^2 \sin^2 \delta_1 + d_1^2}{l^2 \sin_2 \delta_2 + d_2^2}}.$$

Skew-bevel wheels are rarely used and only for transmitting small forces, as the friction is considerable and the construction difficult. They occur most frequently in roving machines, where it is necessary to communicate motion from a long horizontal shaft to a great number of vertical spindles.* When large forces are to be transmitted between two non-parallel and non-intersecting shafts, skew-bevel wheels are usually avoided by employing an intermediate shaft which intersects both crossed shafts, and can consequently be connected with each of the latter by bevel wheels. Such an arrangement is always possible, and is represented by Fig. 285 and Fig. 286. In the arrangement shown in Fig. 285, the intermediate shaft CD coincides with the common normal to the axes, while in that shown in Fig. 286, CD has any position whatever. It will be well to employ, whenever it is possible, the arrangement shown in Fig. 285, where each pair of bevel wheels connects two shafts at right angles.

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* See article on cotton by *Hülsse*, Technolog. Encyclopädie, Supplement, Vol. I.

Fig. 285.

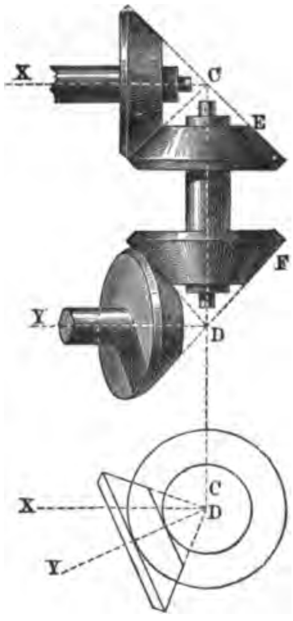
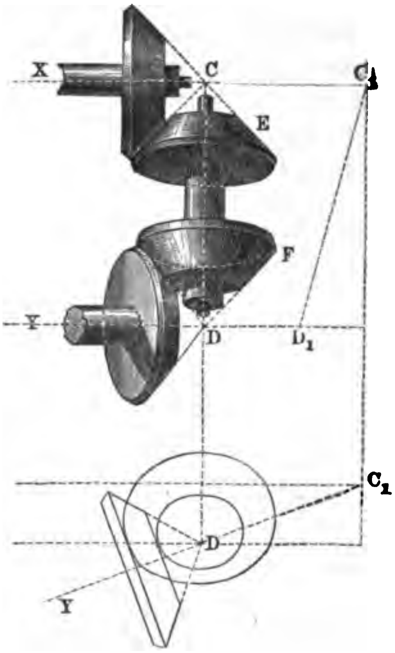


Fig. 286.

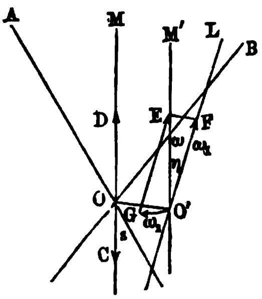


§ 87. Friction in Skew-bevel Wheels.

—To determine the friction between the teeth of hyperboloidal wheels, we may proceed as with cylindrical and conical wheels. Let us again substitute for the two rotations α and β about the axes *A* and *B*, a twisting motion of the hyperboloid *B* about the instantaneous axis, when the axis *A* is fixed, the rotation component of this motion being:

$$\omega = \alpha \frac{sin\ \delta}{sin\ \delta_2} = \beta \frac{sin\ \delta}{sin\ \delta_1},$$

FIG. 287.



and the sliding component $s = d\ .\ \alpha\ sin\ \delta_1 = d\ .\ \beta\ sin\ \delta_2$. Evidently this sliding will generate friction which will consume the work $\varphi P s$ during the rotations α and β of the shafts *A* and *B*. This friction also occurs in hyperboloidal *friction wheels*, and in corresponding tooth wheels, at the instant in which two teeth touch along the instantaneous axis *O M*. But when the teeth touch along another line *L*, Fig. 287, making the angle η with *O M* and at the distance *e* from the latter, the rotation will no longer take place around the instantaneous axis, but around the line of support *L* of the teeth, and to this rotation must be added other motions, so that the result of all the motions actually occurring shall be equivalent to the above-mentioned twisting motion about the instantaneous axis. For this purpose suppose the rotation component

$$\omega = \alpha \frac{sin\ \delta}{sin\ \delta_2}$$

of this motion about the instantaneous axis to take place about an axis *O'M'* parallel to the latter and passing through the foot *O'* of the perpendicular representing the shortest distance *e* between the axes *M* and *L*, the rotation being accompanied by a sliding motion $s' = \omega e$ perpendicular to the plane *MM'*. Now resolve the sliding motions $O C = s = d\ .\ \alpha\ sin\ \delta_1$ and $s' = \omega e$, also the rotation $O'E = \omega = \alpha \frac{sin\ \delta}{sin\ \delta_2}$

into components perpendicular and parallel to the line of contact $O'L$. Then we have the following motions:

1) A rotation $O'F = \omega_1 = \omega \cos \eta$ about the line of contact $O'L$. This rotation corresponds to a rolling of the teeth on each other, and is not accompanied by a loss of work if we, as hitherto, neglect the very small rolling friction.

2) A sliding motion $s_1 = s \cos \eta - \omega e \sin \eta$ in the direction of the line of contact $O'L$ of the teeth. This motion causes sliding along the line of contact of the teeth, like that occurring in hyperboloidal *friction wheels*, where $e = 0$. The motions given under 1) and 2) together constitute the twisting motion about the line of contact $O'L$.

3) A rotation $O'G = \omega_2 = \omega \sin \eta$ about an axis $O'G$ passing through the point O' and perpendicular to the line of contact and the shortest distance OO' between M and L. This rotation causes a *drilling motion* of the surfaces on each other, producing friction, which must be estimated as in *bevel wheels.*

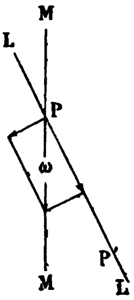
4) A sliding motion $s_2 = s \sin \eta + \omega e \cos \eta$ along the axis $O'G$, about which the drilling motion described under 3) takes place. This sliding produces friction analogous to that arising in *spur wheels*, and when combined with the rotation in 3) gives a twisting motion. It follows that the twisting motion of the wheels about the instantaneous axis may always be replaced by two other twisting motions, one of which takes place about the line of contact of two teeth, and the other about an axis perpendicular both to the line of contact and to the shortest distance between the latter and the instantaneous axis.

Moreover it is evident that in skew-bevel, as in bevel wheels, the pressure between the teeth at the line of contact will cause certain reactions at the bearings and collars of the journals, but the resulting resistances may be easily ascertained in each individual case.

§ 88. Wheels with Oblique Teeth.—We have hitherto assumed that the generatrix of the tooth surface was a *straight line* rigidly connected with the auxiliary axoid and lying on the surface of the latter. It follows, however, from the perfectly general character of the law demonstrated in § 82, that this condition is not a necessary one, and that any curved or straight line, rigidly attached to the auxiliary axoid, will generate theoretically correct tooth surfaces, even if it does not lie on the surface of the auxiliary axoid. This enables us to give a simple explanation of the forms of teeth employed in wheels with oblique teeth, examples of which have already been given in spur wheels (see § 79). There are two reasons why wheels with parallel axes are sometimes provided with teeth which are inclined to the axes. One is to create a more uniform transmission of motion by having a greater number of pairs of teeth in contact at the same instant, thus replacing the jerky and intermittent motion by a steadier one, and the other reason is to reduce as much as possible the friction of the teeth. The extent to which the latter is possible will be seen from the following discussion. A transmission of force between wheels of any kind whatever, which is to be wholly free from sliding friction (the slight rolling friction being neglected as heretofore), is only possible when the relative motion actually occurring between two teeth coincides with the corresponding motion of the instantaneous axes, and when these axes have themselves a *rolling* motion *free from all sliding*. As the latter condition is not fulfilled by axes which cross but do not intersect, there can be no transmission of force without friction in skew-bevel wheels, even when the latter are without any teeth, as in friction wheels.

In spur wheels and bevel wheels, however, the instantaneous motion consists always of a rotation about the instantaneous axis, and is free from sliding; consequently perfectly frictionless transmission of force at every instant will be possible only if at every instant the actual contact takes place at a point of the instantaneous axis, but *in no point outside of the latter*. For if M, Fig. 288, be the instantaneous axis, and L the line of contact between two teeth intersecting M at some point P, we can resolve the motion ω

FIG. 288.

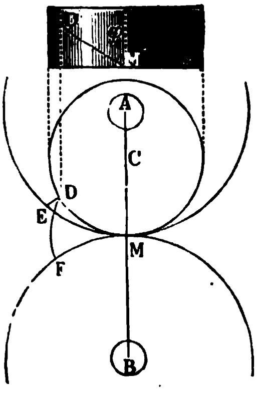


about the instantaneous axis, into two rotations, one about L and another about an axis perpendicular to L and passing through P. The last mentioned rotation will cause a drilling action, and, consequently friction, at every point of contact P' lying outside of M. In the bevel wheels hitherto considered these drilling motions always occur, for the intersection of the axes of these wheels is always common to the instantaneous axis and the line of contact of two teeth. But in spur wheels this point of intersection is infinitely distant, and the drilling motion reduces to a sliding one whose amount directly depends upon the distance e of the line of contact of the teeth from the instantaneous axis. In both cases the friction between two teeth disappears only for the instant in which the line of contact coincides with the instantaneous axis.

It is evident from this that transmission of force without tooth friction can only take place when there is *point contact.* As the latter is not suited for transmitting large forces, we must employ instead *line contact;* in other words, we must construct wheels without teeth if we wish to transmit large forces without having sliding friction at the line of contact of the wheels.

We can get an idea of frictionless toothed gearing in the following manner. Imagine a fine wire to be applied to one of the pitch cylinders of a pair of spur wheels, making any inclination whatever with the axis; for example, it may be arranged as a cylindrical helix. Now if the two pitch cylinders are pressed together with a certain force and are turned, the fine wire of one of the cylinders will leave a shallow groove on the other, whose form will depend on the form of the wire, the wire and the groove always touching at a single point on the instantaneous axis, *i.e.*, on the straight line of contact of the two cylinders. If we now provide the two wheels with teeth whose surfaces are so formed according to the course of the wire on the surface that contact between them always takes place at a point on the instantaneous axis, the wheels will evidently have a point contact which is free from frictional resistance, for the relative motion is here confined to a rolling about the point of contact. As the whole pressure to be transmitted between the teeth in this kind of contact is concentrated at a single point, large forces will soon destroy the correct form of these tooth surfaces, and then contact can no longer take place at a single point. As a consequence, friction and wear will follow, and it is plain that the assumed point contact must pass into line contact, which, according to the above, is always accompanied by friction. If, therefore, two spur wheels are to be provided with oblique teeth, say for the purpose of obtaining as uniform a motion as possible, it will be well to so form the teeth that they will have line contact at first, instead of being worn into line contact from point contact. For this case, we can easily get the surfaces which are to bound the oblique teeth from the general law governing the formation of tooth surfaces.

FIG. 289.



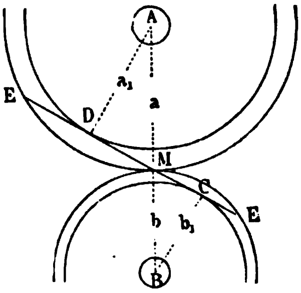
Let *A M* and *B M*, Fig. 289, be the pitch cylinders of two spur wheels whose instantaneous axis is therefore *M*, and let *C M* be any auxiliary cylinder which by rolling on *B* will generate the tooth faces of *B*, and by rolling in *A*, the flanks of *A*. Any point *D* on the surface of this cylinder will then generate the epicycloid *F D* and hypocycloid *E D*, which in normal teeth are employed as the base lines of the cylindrical tooth surfaces. Now if we do not take as generatrix the rectilinear element of the auxiliary cylinder projected at *D*, but assume instead a helix *D M* shown on the cylinder *C*, making an angle of inclination φ with the axis, we see that the tooth surfaces generated by this helix will no longer be cylindrical but screw surfaces. Their intersection by any plane perpendicular to the axes will give the cycloidal curves *F D* and *E D*. The pitch of these screw surfaces, is

$$h_a = \frac{2\pi a}{tan\ \varphi}, \quad \text{and} \quad h_b = \frac{2\pi b}{tan\ \varphi}$$

respectively, where $AM = a$ and $BM = b$ are the radii of the wheels. The surface of the generating cylinder C must now be regarded as the locus of contact, the contact of the teeth always taking place along the helical generatrix DM. As this line of contact always cuts the instantaneous axis, it is evident from Fig. 288 that the friction which arises is not due to a sliding of the tooth surfaces on each other as in normal teeth, but results from a turning or drilling motion about the point of intersection just mentioned. Remarks similar to those made in §§ 69 to 72 concerning radial flanks, pins, external and internal gearing and racks, for straight or normal teeth, apply to these oblique teeth with cycloidal sections. For these latter teeth all the forms enumerated, even the pins, are screw surfaces.

By a procedure similar to that just employed for cycloidal profiles, we may easily deduce forms of teeth of the involute type. For this purpose let us again suppose a plane E to be passed through the instantaneous axis M of two wheels A and B, Fig. 290, the plane making an angle γ of about 75° with the plane of the axes AB. Then roll this plane, as in § 73, on each of the base cylinders AD and AC; the generatrix must, however, be no longer regarded as parallel to the axes but as a line in the plane E inclined at any angle φ to the axes. It is evident that during the rolling of the plane on cylinders AD and BC, this line will generate two screw surfaces whose pitches are represented respectively by

FIG. 290.

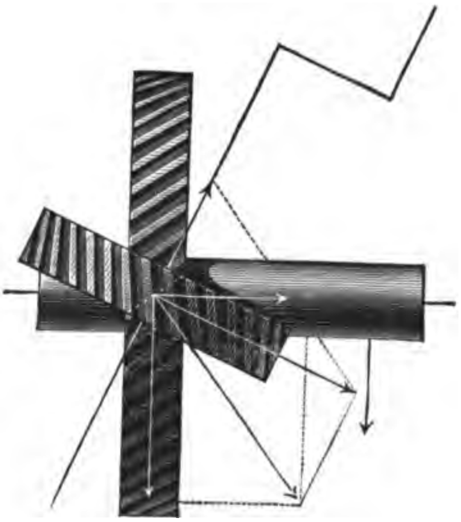


$$h_a = \frac{2\pi a_1}{tan\ \varphi}, \quad \text{and} \quad h_b = \frac{2\pi b_1}{tan\ \varphi},$$

and whose sections by planes perpendicular to the axes are involutes to the base circles *A D* and *B C*. Such surfaces were called spiraloids in § 86.

The geometrical character of these screw surfaces is the same as the *approximate* tooth surfaces employed for those wheels connecting non-parallel and non-intersecting axes, whose mid-planes are gorge circles. The difference between them was pointed out in § 86; it is, that although these screw surfaces give exact forms for the oblique teeth of wheels with parallel axes, they are only approximately correct for gorge wheels of axes neither parallel nor intersecting. The latter wheels, Fig. 291, are properly hyperboloidal wheels, and their approximate tooth forms are not screw surfaces, but surfaces generated by rolling and sliding an auxiliary hyperboloid on the axoids in the manner described in § 86. Nevertheless, it is customary in practice to call the gorge wheels screw wheels, but it is clear that this name properly belongs to the wheels with oblique teeth and parallel axes. Another difference which it is well to notice is, that the angle φ made by the oblique teeth with the parallel axes may be arbitrarily assumed, while for the spiraloidal teeth of non-parallel and non-intersecting axes, the generatrix is completely determined by the position of the instantaneous axis *M*, *i.e.*, by the relation $\alpha \sin \delta_1 = \beta \sin \delta_2$, where α and β again represent the angles of rotation, and δ_1 and δ_2 the inclinations of the instantaneous axis to the axes of the wheels.

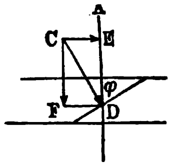
FIG. 291.



It is also evident that in wheels with oblique teeth, the obliquity φ of the latter causes the shafts to be pressed against their supports in the direction of the axes. Let $CD = P$, Fig. 292, be the normal pressure exerted by two teeth upon each other, then the component in the direction of the axes, which causes friction on the collars of the journals, will be:

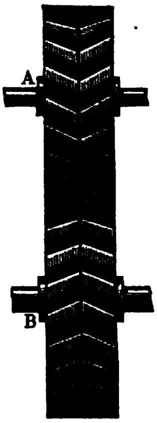
$$CF = P \sin \varphi.$$

FIG. 292.



The component turning the shaft is of course $CE = P \cos \varphi$. If the wheels are provided with teeth inclined in both directions, as shown in Fig. 293, the components in the direction of the axes will be neutralized, and the friction on the ends or collars of the journals will be removed.

FIG. 293.



We can likewise provide bevel wheels with oblique teeth by assuming for the generatrix of the tooth surfaces a conical helix on the surface of the rolling cone. In involute profiles this helical line will become a spiral on the rolling face wheel, but such oblique teeth are not employed for bevel wheels. Oblique teeth are rarely employed even for spur wheels.

§ 89. **The Construction of Toothed Wheels.**—Toothed wheels are almost always made of cast iron; only the smallest are of steel, wrought iron or brass. Up to certain limits (2 to 3 meters [6 to 10 feet] diameter), they are cast in one piece, and larger sizes are composed of several pieces, after the manner of water-wheels. At the present day wooden wheels are seldom used on account of their inaccuracy and want of durability. On the other hand, it is advisable to employ wooden teeth for one of a pair of rapidly running wheels, in order to obtain smooth and noiseless motion. The form of external teeth for a spur wheel is shown in Fig. 294, for internal teeth in Fig. 295, while Fig. 296 corresponds to a bevel wheel. Fig. 297 shows a portion of the iron rim of a wheel having wooden teeth, the latter being held in place by iron pins driven into the holes *H*. The pin wheels or trundles shown in Fig. 298 are composed of two disks $D\,E$ and $D_1\,E_1$ and intermediate pins $A\,A_1$ and $B\,B_1$. They are of wood, and are only found in old mills, but on a smaller scale they are made of metal and are then much employed in clock work. Such pin wheels are sometimes used with disk-shaped wheels having teeth or pins on their sides, like those shown in Fig. 299 and Fig. 300; the latter

FIG. 294.

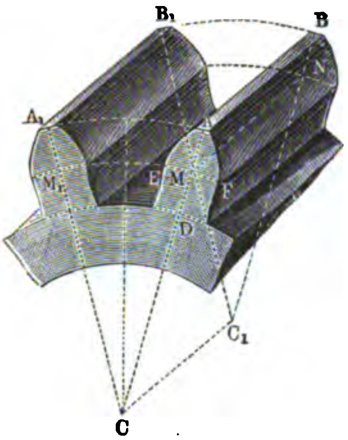


FIG. 295.

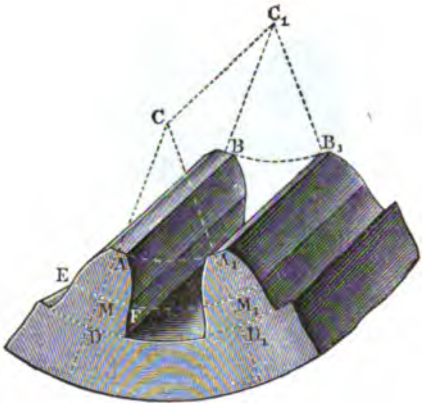


FIG. 296.

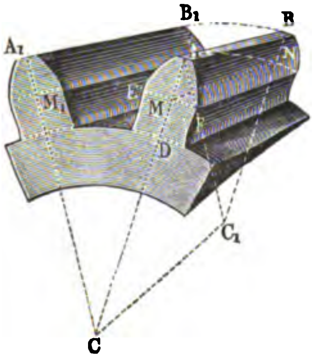
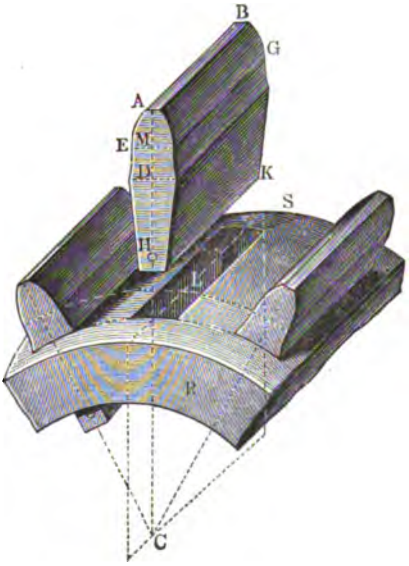


FIG. 297.



are called crown or contrate wheels, and are likewise nearly obsolete devices for connecting two intersecting shafts.

FIG. 298.

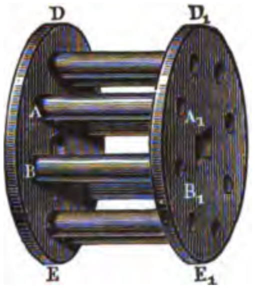
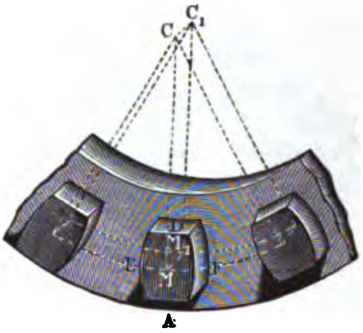


FIG. 299.



The only correct means for transmitting motion between such shafts are bevel wheels. Rim sections of cast-iron wheels are shown in Figures 301 to 307; those in Fig. 301 and Fig. 302 belong to spur wheels, that in Fig. 303 to a bevel wheel with iron teeth, and those in Fig.

FIG. 300.

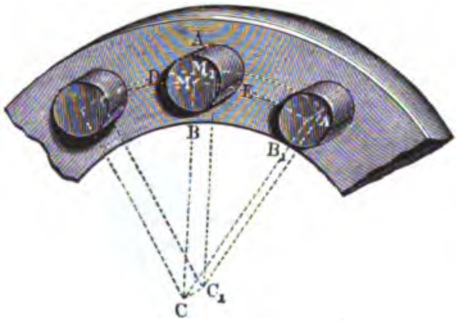


FIG. 301.

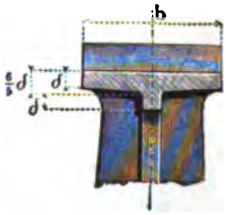


FIG. 302.

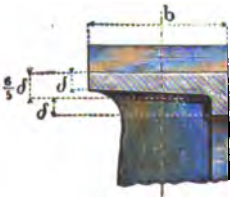


FIG. 303.

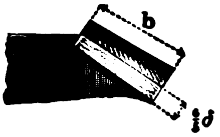
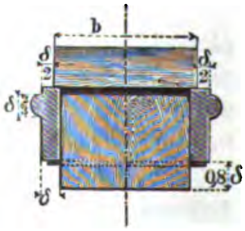


FIG. 304.



304 and Fig. 305 belong to wheels with wooden teeth. In the small gears shown in Fig. 306 and Fig. 307, the arms and rims are reduced to a simple sleeve, the strength of the teeth being increased by *casting a disk* on one or both sides. In these figures the dimensions are expressed in terms of a unit $\delta = 0.4t + 3$ mm [$\delta = 0.4t + 0.12$ in.], where t is the pitch in millimeters [inches].

FIG. 305.

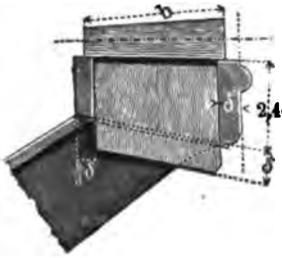
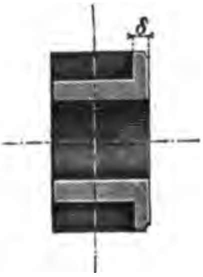


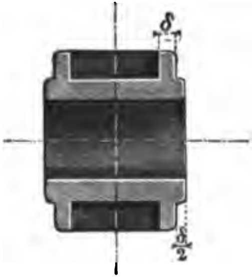
FIG. 306.



The number of arms varies from 3 to 8 according to the size of the wheels; the arms have usually a ribbed cross section, and the nave a length $L = b + 0.06r$, and a thickness (of metal about the eye) $s = \frac{1}{4}d + 5$ mm [$s = \frac{1}{4}d + 0.2$ in.], b representing the breadth of the tooth, r the radius of the wheel, and d the diameter of the shaft. For the other details of toothed wheels reference must be made to the special manuals on machine design.

Until recently the larger and more powerful wheels were cast from wooden patterns which were made as accurately as possible. But there are serious drawbacks to this method, for not only do the wooden patterns become inaccurate by warping, particularly if they are of large size, but the cost of patterns is considerable, and often leads to the employment of old patterns whose tooth forms do not satisfy the required conditions in the most advantageous manner. Consequently *wheel-moulding machines* are coming more and more into use, which produce the mould of the toothed rim by means of a pattern having one or more teeth very accurately made. This pattern is attached to a radial arm which can be revolved very accurately through

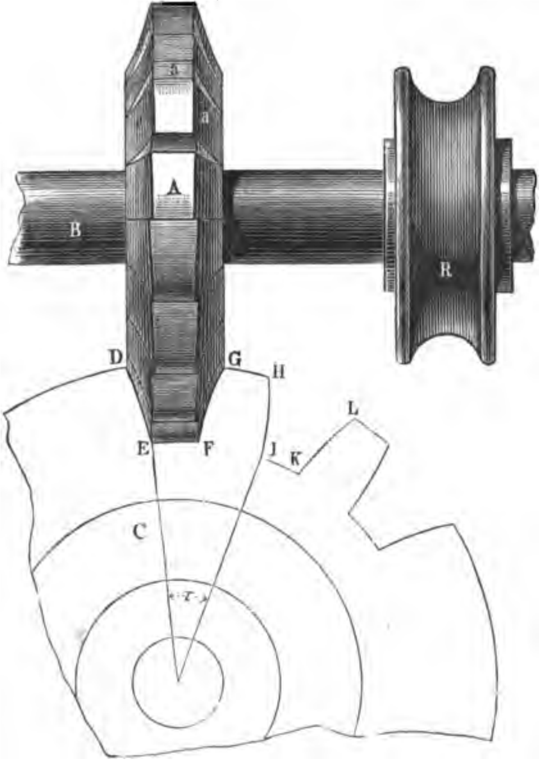
FIG. 307.



any required portion of the circumference, while the nave and arms are generally constructed from wooden patterns as formerly. In this way a very correct construction can be obtained at a slight expense for patterns. At the same time these machines render the designer independent of existing patterns, and allow him complete freedom in the choice of tooth profiles.

The teeth cast from these moulds are generally sufficiently accurate for the larger gears, but the smaller gears employed in ordinary machinery (machine tools, etc.), where the excellence of the product closely depends upon the uniform motion of all parts, require a more accurate construction of the tooth surfaces. Aside from hand work, which is always uncertain and costly, the two principal means employed for this more accurate work are *milling* and *planing*.

Fig. 308.



In milling, the tool is of steel and has the form of a solid of revolution A, Fig. 308, whose profile is exactly that of the tooth space $D E F G$. The surface of this *milling tool* or *circular cutter* is so notched that there are a great number of cutting edges on this surface of revolution. Now if the cutter is rapidly turned about the arbor *B* by its pulley *R*, and at the same time receives a slow motion parallel to the axis of the wheel *C* operated upon, the cutting edges will produce the required space *DEFG*. It is evident that when one space has been produced, a repetition of the same process, after the wheel has been turned through the pitch angle τ, will give the next tooth space *HJKL*, etc.

This method of producing teeth can be applied only to spur wheels and racks, for it presupposes the spaces to have a prismatic or cylindrical shape. These cutters cannot give a correct form for bevel wheels, for even if only one side of the space be formed at a time, the cutter making two cuts to complete the space, the finished surfaces will still have equal curvature of the profiles at every distance from the axis. This should not be the case, as these teeth ought to have a conical form. A very imperfect remedy for this consists in giving to the cutter the form of the inner profile of the space, and then oscillating the axis of the cutter so as to cut the spaces wider toward the outside.

The only means for producing exact teeth for bevel wheels is by *planing*, the point of the cutting tool being directed continually to the intersection of the axes, each cut of the tool producing an element of the required conical tooth surface. In order that the tool may successively traverse these different elements, various ingenious planing machines have been designed, in which use is made of curved *formers* similar to the required tooth profiles.* Only a few of them have been introduced, however, as they are all quite complicated. It has been proposed to plane spur wheels also by employing a special tool shaped like the tooth space†; such devices can, however, only be used for the finishing cut, as the tool can remove but a very thin layer or chip. Skew-bevel teeth are always finished by hand after the spaces have been roughly made by a circular cutter. Screw wheels are cut on the lathe in the same manner as ordinary screw threads.

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* See *Dingler's* Polyt. Journ., Vol. 209 (1873) p. 24. Civ. Ing. Vol. 10, p. 271; also Vol. 18, p. 141.

† See *Hardt*, Werkzeugmaschinen.

According to the general law governing tooth surfaces (§ 82) correct tooth forms may be given to wheels connecting two axes A and B, by employing for these forms surfaces generated relatively to the axes A and B by a line attached to a third axoid C, provided all three axoids are given the rotations belonging to them. From this follows that all tooth surfaces which can be generated by a *straight* line, can be accurately produced on any planing machine. For if we suppose the third axoid C to be associated with one of the axoids, say A, their respective angles of rotation again being γ and α, an additional rotation $-\gamma$ may be given to the whole system without altering the relative motion of A to C. This additional rotation will evidently bring the auxiliary axoid C and the generatrix to rest, while the wheel on the axis A will be subjected to two rotations, one α about the axis A and another $-\gamma$ about the axis C. If we suppose the fixed rectilinear path of the point of the tool of a planing (shaping) machine to be the generatrix, the tool will produce exactly correct tooth surfaces when the wheel to be operated upon is supported in such a way that it can receive, besides the rotation about its own axis, another about the axis C of the assumed auxiliary axoid, the two rotations being simultaneously imparted to the wheel in the ratio $\alpha : -\gamma$. For this purpose it is only necessary to provide the planing machine, which can be of ordinary construction, with a suitable rest or support for the wheel, in order to plane the tooth surfaces in a perfectly automatic manner. By this means the teeth of spur, bevel, skew-bevel and screw wheels can be produced without difficulty, and the teeth may have either cycloidal or involute forms. This method does not require any former (templet) or special cutting tool, and does not even require the drawing of the tooth profile when the generating auxiliary axoid is given. We must remark that the teeth of bevel wheels thus generated have exact forms (conical surfaces with spherical cycloids and spherical involutes as bases), while, in the customary method of constructing bevel wheels, only the approximate forms obtained by the help of the supplementary cones are employed. The rests for supporting the wheels to be planed are discussed by the author in an article in Verhandlungen des Vereins zur Beförderung des Gewerbfleisses, 1877.

§ **90. Chain Wheels.**—According to the classification given in § 40, chain wheels still remain for discussion. They resemble belt pulleys in that they do not transmit motion by direct contact of the wheel surfaces, but by means of an intermediate, namely, a chain. On the other hand, they resemble tooth wheels in having tooth-like projections which enter the links of the chain and insure the transmission of motion, the teeth preventing the chain from slipping on the circumference of its wheel. Consequently the chain does not need any special tightening, and the friction at the journals of the chain wheels is due only to the force transmitted, *i. e.*, to the pull on the chain, for the driving portion of the chain cannot be very slack.

The driving chains have different forms and are always made of wrought iron, which must be as strong and tough as possible on account of the small shocks to which the chains are continually and unavoidably subjected. The ordinary chain, Fig. 309 and Fig. 310, with oval links, which are sometimes stiffened with cast-iron studs, Fig. 311, is rarely employed with chain wheels to transmit motion. These chains are

FIG. 309.

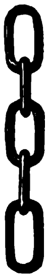


FIG. 310.

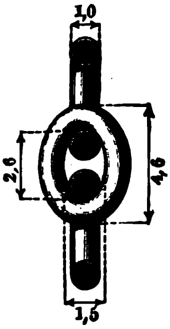
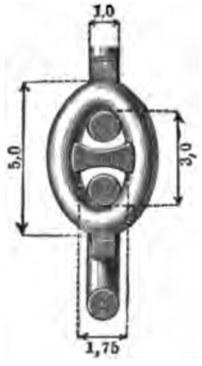


FIG. 311.



generally employed in windlasses and hoisting apparatus, where the pulleys and drums have flat surfaces without projections, the differential pulley blocks being an exception. The chains shown in Fig. 312 are employed in common tackle blocks, for example, in moving the dampers of steam boiler furnaces, etc., while the chains shown in Fig. 313 and Fig. 314 are often employed in transmitting power.

FIG. 312.

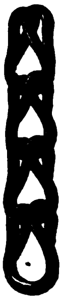


FIG. 313.

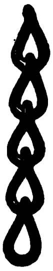
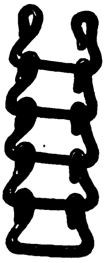


FIG. 314.



The latter chain, it is easy to see, is only useful for transmitting small forces, as the loops of the links are formed by bending the wire without welding. Nevertheless the ease with which it can be made and the secure manner in which it lies against the circumference, make this band-shaped chain very popular; for example, it is much used in carding machines, where one chain communicates motion to each of a system of rollers. This chain resembles the band chain shown in Fig. 315, which is made of welded wrought iron rings and flat links bent at the ends.

FIG. 315.

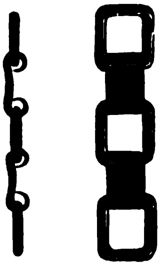


FIG. 316.

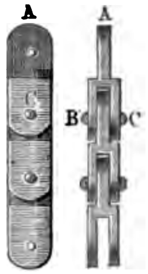


FIG. 317.

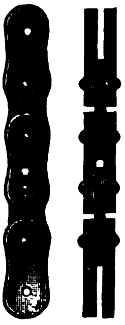
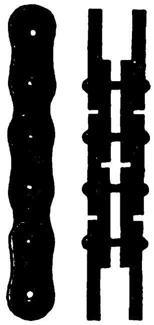


FIG. 318.



Where it is a question of transmitting large forces, it is best to use *jointed* or *flat link chains* composed of single or double links of iron united by cylindrical wrought-iron bolts. The various forms of this chain are shown in figures 316 to 318. The wrought-iron or steel bolts have rivet heads at each end, or they have a head at one end and a pin or nut at the other, to prevent them from falling out. In very careful constructions the eyes in the links are lined with steel or brass bushings.

Fig. 319.

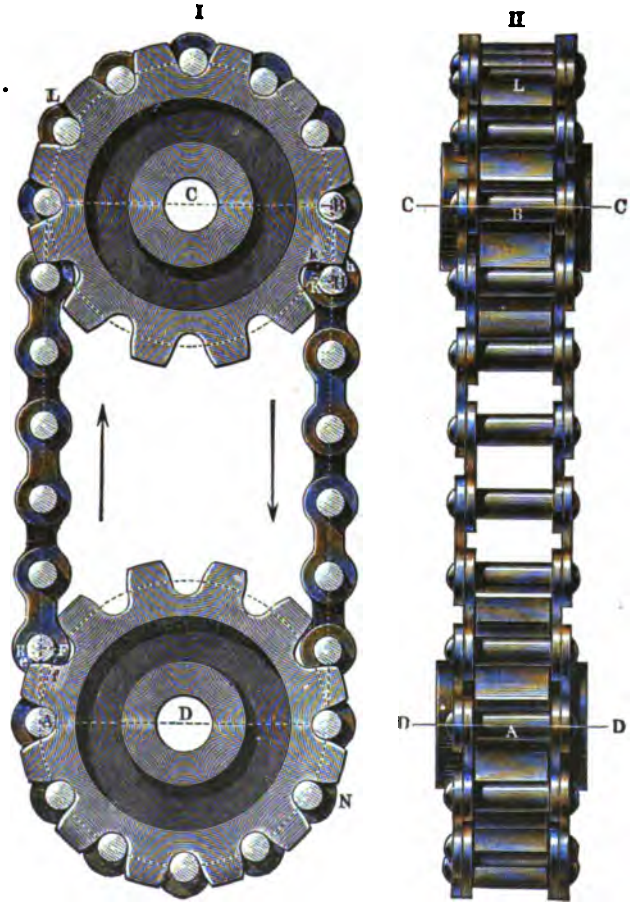
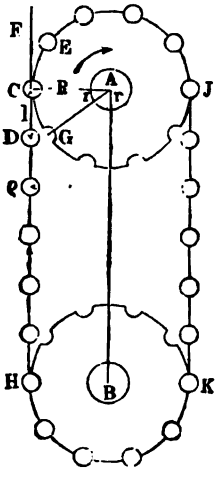


Fig. 319 represents a chain gear in which the jointed chain $A\ L\ B\ N$ transmits motion between the toothed wheels

C and *D*. The teeth enter into the hollows of the chain formed by the adjacent bolts, and act upon the latter as the teeth of a spur wheel act on the cylindrical pins of a rack. See Fig. 249, § 72. The teeth of the wheels must therefore be profiled as equidistant *e f*, *k h* to the involutes *E F*, *K H*. Chain wheels are only employed when the shafts are parallel. The great drawback to all chain wheels is that in time the links will stretch, and consequently their length will cease to agree with the pitch of the wheels. The result is *lost motion* between the teeth of the wheel and the bolts of the chain, giving rise to corresponding small shocks, whose hurtful influence becomes more marked as the velocity of the chain increases. This is why chain wheels are seldom employed and only with slow speeds. To diminish as much as possible the evil mentioned, it is advisable to make the links and bolts so strong that the stress per square inch of cross section will be comparatively small. The determination of the dimensions to be given to the links and their bolts may be found without difficulty from the formulas for tensile and shearing strength, given in Vol. I., when the greatest specific stress allowable in the material has been decided upon.

FIG. 320.



Chains are not only used as a means of transmission, but for other purposes in certain machines, for example, in *dredging machines*, *elevators*, *tenters*, etc., and will be discussed hereafter under these heads.

The friction connected with chain gear consists of the friction of the journals, the friction between the teeth and the bolts and the friction between the bolts and the eyes of the links, when the latter turn at the joints to adjust themselves to the circumference of the wheel. Let R represent the radius of the wheel $A\ C$, Fig. 320, r the radius of the journal, ρ the radius of the bolt or pin of the chain, and l the length $D\ C = C E$ of the link measured from centre to centre of the pins;

we then have for the friction on the wheel caused by the circumferential force P, the following expressions:

For the friction of the journals at A reduced to the point C:

$$F_1 = \varphi P \frac{r}{R}.$$

For the friction between the teeth and the pins, according to § 79, when n represents the number of teeth, we have

$$F_2 = \pi \varphi P \frac{1}{n}.$$

Since $\tau = \frac{l}{R}$, and the turning angle of the link equal to

$$FCE = \frac{1}{2} CAG = \frac{1}{2} \tau = \frac{1}{2} \cdot \frac{2\pi}{n}$$

when the chain moves through the distance l, for example, when the link DC moves to the position CE, we have for the friction of the links

$$F_3 = \frac{\varphi P \rho}{l} \cdot \frac{\tau}{2} = \frac{\varphi P \rho}{2R}.$$

The total resistance of friction on the wheel A is therefore

$$F = F_1 + F_2 + F_3 = \varphi P \left(\frac{r}{R} + \frac{\pi}{n} + \frac{\rho}{2R}\right).$$

In like manner the resistances of the journals of B and of the chain at the point of delivery H may be determined from

$$F' = \varphi P \left(\frac{r'}{R'} + \frac{\pi}{n'} + \frac{\rho}{2R'}\right)$$

when r', R' and n' represent corresponding magnitudes on this pulley. If for the sake of brevity we make the above parentheses

$$\frac{r}{R} + \frac{\pi}{n} + \frac{\rho}{2R} = f, \text{ and } \frac{r'}{R'} + \frac{\pi}{n'} + \frac{\rho}{2R'} = f',$$

we shall have for the total resistance of the chain gear,

$$F = \varphi P(f + f'),$$

and when the pulleys are equal,

$$F = 2\varphi P f.$$

At the point of delivery J, and at the point K, at which the chain is received by the wheel, the resistances may be neglected, as the portion of the chain JK is not under tension.

LITERATURE. The geometrical construction of the teeth of wheels was first given by the Danish astronomer *Roemer*, afterwards by *Lahire*, *Camus* and *Deparcieux*. *Euler* and *Kaestner* also sought to ascertain the best forms for the teeth. These papers are given in *Eytelwein's* Statik, Vol. I. The drawing of wheel teeth is also fully treated by *Gerstner* in the third volume of his Mechanik, by *Hachette* in his Traité élém. de machines, by *Olivier* in Théorie géométrique des engrenages (German edition by *Schnuse*), and by *Willis* in his excellent treatise on the Principles of Mechanism. See also *Verdam's* Grundsaetze der angewandten Werkzeugswissenschaft und Mechanik, Vol. II. (translated from the Dutch by *Schmidt*), *Haindl's* Maschinenkunde and *Salzenberg's* Vortraege ueber Maschinenbau; likewise *Wiebe's* Lehre von den einfachen Maschinen. A very full and thorough treatment may be found in the Constructionslehre fuer den Maschinenbau by *Moll* and *Reuleaux*. The information most needed by the designer is given in *Morin's* Aide-mémoire and in *Redtenbacher's* Resultaten fuer den Maschinenbau. Wheels are treated in almost all the works on machine construction and machine design, for example, *H. v. Reiche's*, Die Maschinenfabrikation, 2d Edition, 1876. There are, besides, numerous articles on wheels in the various technical journals.

CHAPTER III.

RODS AND THEIR GUIDES.

§ 91. **Rods.**—The machine parts discussed in the two preceding chapters served to produce turning motions, while those discussed in this chapter are employed to receive and transmit rectilinear motion; they therefore correspond to the prismatic or sliding pair described in § 28 of the Introduction. Machine parts employed for effecting rectilinear motions may be of two kinds, rigid and flectional, the former capable of resisting both tensile and compressive forces, while the latter can resist only tensile forces. The first kind may be roughly designated as *rods*, the second includes *ropes*, *chains* and other machine elements used as tension organs, and will be specially treated in the next chapter.

Rods are chiefly employed as piston rods for steam engines and pumps, receiving motion from the steam piston in the first case and communicating motion to the plunger of the pump in the second case. But they have many other applications; for instance, rod- or slide-shaped bodies are employed to give a reciprocating motion to the tool in shaping and slotting machines. Under this head come many machine parts which have reciprocating rectilinear motion without being exactly rod-shaped, for instance, the slide rests of lathes and the sliding tables of planers.

From what was said in § 28 of the Introduction concerning pairs of elements, it is evident that the rod must be so connected with other links that it will be prevented from taking any other than a rectilinear motion. The simplest means for accomplishing this consists in giving the rod the form of a solid prism and enclosing it by an exactly fitting hollow prism. This arrangement though much used is not the only one, for the constrained motion of the rod can be attained by various other means, as will appear later.

In the prismatic rod and guide just mentioned, for producing rectilinear motion, the choice of cross section for the rod is not limited by any general condition, and very different forms of cross section are used in practice. Nevertheless it is a universal rule to give a cylindrical form to those rods which must be air and water tight, as is the case in steam engines and very many pumps, for of all possible cross sections the circular is best adapted for producing a tight joint or guide.

Piston rods are almost always solid and are made of the strongest material, fine-grained iron or steel, in order to diminish the cross section and consequently the friction in the stuffing box as much as possible. When for particular reasons it is necessary to make the piston rods hollow, as in trunk-engines, hydraulic lifts and lift pumps, recourse must generally be had to cast iron.

The preceding remarks concerning cross section and material apply only to piston rods which pass through stuffing boxes; where this is not the case other forms and materials are much employed. Thus, long pump rods are often made of wood and of rectangular cross section, while the slides, etc., in the machines above mentioned are generally of cast iron, which easily permits of a massive construction of the part in question, the excellence of the work of such machines depending very largely on this massiveness. In the latter cases the part to be guided will seldom be given a cylindrical form, as this requires special means for preventing the turning of the rod; consequently angular or ribbed profiles are preferred for the portion to be guided. The turning of the cylindrical piston rod is usually prevented by its connection with the other pieces of the mechanism.

The diameters of the piston rods may be determined from the principles governing the tensile strength or the resistance to compressive forces, according as the rod is subjected to tensile stress, as in suction and lift pumps, or is alternately subjected to tensile and compressive forces as in steam engines. In the latter case the formulas for resistance to buckling (crushing by bending or breaking across) given in Vol. I., § 265, must be employed. As the rod is guided by the stuffing box and the piston, we may regard one end as fixed and the other end compelled to remain on the line of direction of the force. According as this last mentioned end is guided by prisms or by link-work (as will appear later) we must assume it to be immovably fixed, as in Fig. 321, or able to assume an inclination to the original axis, as shown in Fig. 322. Accordingly if we assume d to be the diameter and l the free length of the piston rod, we shall have for the corresponding cases III. and IV., given in § 265, Vol. I., either the formula

FIG. 321.

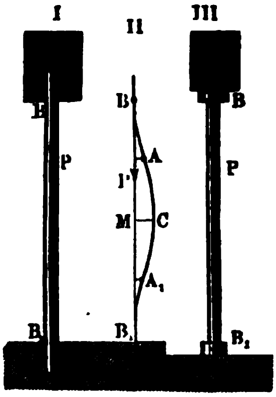
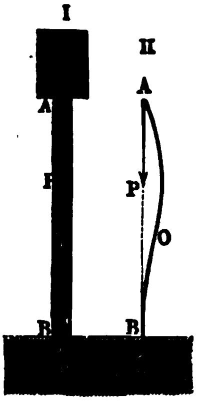


FIG. 322.



$$\text{III.}\quad P = 4\pi^2 \frac{WE}{l^2} = 4\pi^2 \frac{\pi d^4 E}{64\, l^2} = \frac{\pi^3 d^4 E}{16\, l^2} = 1.835 \frac{E}{l^2} d^4,$$

or

$$\text{IV.}\quad P = 2\pi^2 \frac{WE}{l^2} = \frac{\pi^3 d^4 E}{32\, l^2} = 0.967 \frac{E}{l^2} d^4,$$

for determining d (see example, § 265, Vol. I.).

§ 92. **Couplings.**—Pump rods are often so long that they cannot be made in one piece, in which case it is necessary to unite the separate pieces by *scarf-joints*, whose forms vary with the cross section and material of the rods. The parts of the rod outside of the stuffing box are often of wood and have a rectangular cross section, which is square when the rod is vertical and whose depth is greater than its breadth when the rod is horizontal thus allowing for the weight of the rod. Two pieces of rod are usually connected by wooden or wrought-iron *fishing-pieces* which are placed on opposite sides of the rods and rigidly united to the latter by bolts or rings. Fig. 323 represents such a wooden fish-joint, in which the rods *A* and *B* are placed end to end and notched on each side so as to receive the side plates *C*, which are 1½ to 2 meters (4.5 to 6 ft.) long, the rigid union of all the pieces being effected by two rings *D* and eight bolts *E*. The force is here principally transmitted by the projections (joggles), which must have sufficient cross section to resist the shearing stress.

FIG. 323.

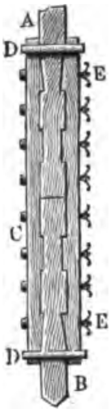


FIG. 324.

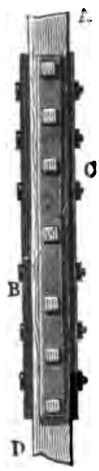
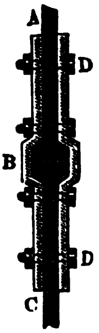


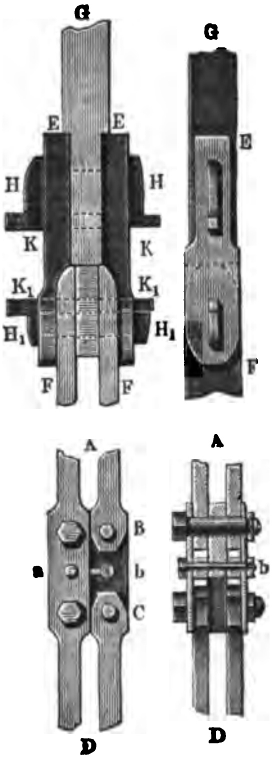
FIG. 325.



Another scarf-joint with four fish-plates is shown in Fig. 324, it occurs in the water-pressure engine at Huelgoat. The ends of the rods *A C* and *B D* are beveled off and rigidly connected by bolts and four wrought-iron fish-plates. The shanks of the bolts must here transmit the pull or push of the rods, and special care should therefore be taken that the holes in the rods and side-plates exactly fit the bolts.

Iron rods of rectangular or polygonal cross section are likewise united by fish-plates and bolts. The ends of the rods are then often upset, as shown at *B*, Fig. 325, and the side-plates are bent to correspond, thus relieving the bolts, as in Fig. 323, from transmitting the force, the transmission being effected by means of the upset portions. A compound rod composed of four single ones is shown in Fig. 326. Here each of the two pairs of rods *A* and *D* is united by the bolts *B* and *C* and three fish-plates, the fish-plates of one pair of rods being united to those of the other by the link *a b*. This figure also shows the rods joined to the end rod *G* by two fish-plates *E F*, the gibs *H H*, and the keys *K* and K_1.

Fig. 326.



Cylindrical rods are generallv united by a sleeve shoved over the ends *A* and *B*, Fig. 327, and the keys *C*, the sleeve having a tapered bore at each end. Sometimes one of the rods *B*, Fig. 328, has welded to it a sleeve whose bore fits the end *A* of the other rod, the key *C* uniting the two rods. This arrangement is not so convenient to construct as that shown in Fig. 327, but has the advantage of needing only

Fig. 327.

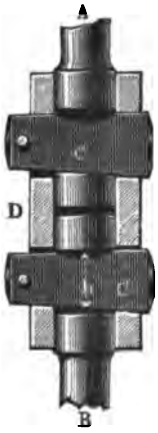
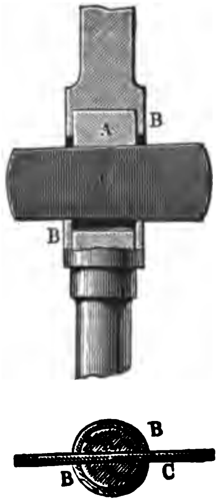


Fig. 328.



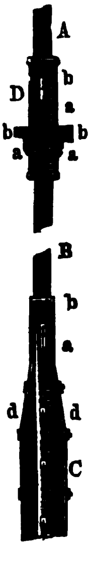
a single key. In all such keyed joints the key must be able to resist the shearing tendency of the force to be transmitted. Consequently, as each of the shearing sections has the breadth b and the height h, we must have, if we suppose the keys to be as strong as the rods:

$$2 b h k_{,,,} = \frac{\pi d^2}{4} k_1,$$

where k_1 is the permissible working stress in the rod and $k_{,,,}$ the allowable shearing stress of the key (see Fifth Ger. Ed., Vol. I., § 260). To make good the diminution of cross section by the key space, it is customary to increase the diameter d of the rods by upsetting till it has a diameter d_1 which will satisfy the equation

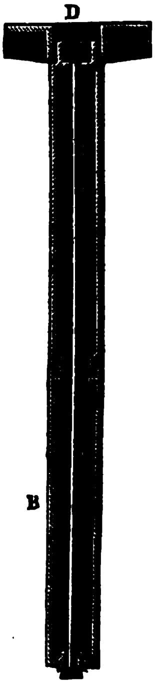
$$\frac{\pi d_1^2}{4} - b d_1 = \frac{\pi d^2}{4}.$$

FIG. 329.



The keyed joint just described, Fig. 328, is quite generally employed for fastening the piston rods to the pistons, cross heads, etc., the piston rod then having a tapered end exactly fitting the conical bore of the socket. Fig. 329 shows how a wooden rod may be united to the piston rod. Here the end C of the wooden rod is connected with an iron rod B by means of the fish-plates d and keys a and b. The other end of the iron rod B is united to the piston rod A by the sleeve D and other keys $a\,b$. In all these cases the keys should be made of the best steel, and in all the scarf-joints the bolts and keys should be secured by pins or other means, to prevent them from getting loose in consequence of the vibration caused by the alternate motion.

FIG. 330.



The stroke of the piston rod is usually small enough to avoid applying the coupling at those portions of the rods that pass through the stuffing boxes. Sometimes the stroke employed is large, as in hydraulic lifts, and then it is necessary to construct the rod of two pieces. The joint must then of course be of such a character that it will maintain the exact cylindrical form. Fig. 330 shows the joint employed for the pistons of hydraulic lifts. The platform *D* is carried by a piston about 10 meters [33 ft.] long and consists of cast-iron pipes *A* and *B* which are united by a wrought-iron bolt *C* passing through the interior. But such constructions are exceptional.

§ **93. Loss of Stroke.**—The alternate tension and compression to which the rods are subjected, causes a lengthening and shortening of the rod which involves loss of stroke, as may be seen from the following: Suppose a vertical rod to have a section *F*, and when subjected to neither tension nor compression, a length *l*, its weight being represented by *G*. Suppose the driving force to act at the upper end and the resistance at the lower end of the rod, and let P_1 represent the resistance of the plunger on the up stroke and P_2 the resistance on the down stroke. Now, when the driving force acts upward, before the resistance can be overcome the rod must be elongated in consequence of the pull P_1 and the action of the weight *G*. This elongation may be found from the following formula:

$$\lambda_1 = \frac{P_1 + \frac{1}{2} G}{F E} l$$

(see Vol. I., § 207), where *E* is the modulus of elasticity.

In like manner at the beginning of the down stroke the plunger will not be moved by the rods until its original length *l* has been shortened by the amount

$$\lambda_2 = \frac{P_2 - \frac{1}{2} G}{F E} l.$$

It is evident that the sum of these two variations of length, *i.e.*, the difference between the lengths of the rod in its extended and in its compressed condition, will be

$$\lambda = \lambda_1 + \lambda_2 = \frac{P_1 + P_2}{FE} l,$$

and will measure the loss of stroke experienced by the resistance during each stroke. For every time that the motion changes, for instance, when the rod is in its lowest position, the rod must not only recover its original length from the compression λ_2, but must be subjected to the further elongation λ_1 before its lower end can begin to rise. Moreover, the above expression shows that *the rod's own weight G does not influence the loss of stroke.*

The loss of stroke just determined may be neglected in the ordinary piston rods of steam engines and pumps, because they are short, but in long rods, composed of such elastic material as wood, the loss of stroke may be considerable; and we see, for example, that no motion of the resistance will be possible if the path s of the point of application of the force does not exceed $\lambda = \frac{P_1 + P_2}{FE} l$. For a given length l and the given resistances P_1 and P_2, we can reduce the loss of stroke by increasing the cross section F.

This loss of stroke, however, is *not* accompanied by loss of work, for if the lengthening and shortening do not exceed the limit of elasticity, the work expended in lengthening or shortening the rod will, at each change of motion, be given back again during the contraction or extension occurring in the following change. But this last remark does not apply to those changes of length in the rods which are due to yielding or slipping in the joints. For if the alternating forces produce a slight slipping in these joints, it can only be by overcoming corresponding frictional resistances. The amount of this friction and of the slipping cannot, of course, be determined in a general way, for these values intimately depend on the character of the construction; this may, however, be said, that the resistance of friction at any joint during the up stroke cannot exceed $P_1 + G'$, and during the down stroke it cannot exceed $P_2 - G'$, G' being the weight of the portion of the rod below the joint, for if the friction were greater than these values it could not be overcome. We see from this how necessary it is to screw up the plates of the joints firmly, so that the friction generated will exceed the larger of the above values, and also that a careful fitting of the bolts, joggles, etc., will greatly diminish the sliding itself and the accompanying loss of work if sliding does eventually take place.

EXAMPLE. If a wrought-iron rod 300 meters [984 ft.] long alternately exerts a thrust and a pull represented by $P_1 = P_2 = 24{,}000$ kilograms [52,910 lbs.], and has a cross section of 60 square centimeters [9.3 sq. ins.], the loss of stroke will be

$$\lambda = \frac{2 \times 24{,}000}{6{,}000 \times 19{,}700} \times 300 = 0.122 \text{ meter } [4.8 \text{ ins.}].$$

If the rod has to overcome the resistance of 24,000 kilograms in only one direction, the loss of stroke will be only 61 millimeters [2.4 ins.], *i. e.*, half that just found.

§ **94. Cylindrical Guides.**—In order that the reciprocating motion of a rod may take place in the direction of its axis, it must be provided with certain arrangements for preventing every deviation from the straight path. The arrangements employed for this purpose are known as *slide* or *parallel motions*. This object is partially accomplished by stuffing boxes, whose particular function is, however, to provide a steam-tight passage for the piston rods. Such a stuffing box is shown in Fig. 331. The accurately turned rod A passes through a suitable opening in the cylinder cover B; the body C of the box has a cylindrical bore and is cast in one piece with the cover B. The sleeve or gland D surrounds the piston rod and fits the bore, the object being to press the packing, hemp, etc., firmly against the surface of the piston rod and the internal surface

FIG. 331.

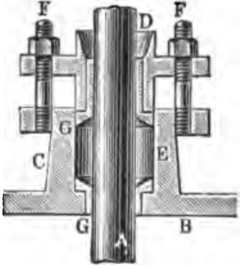
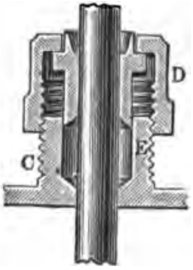
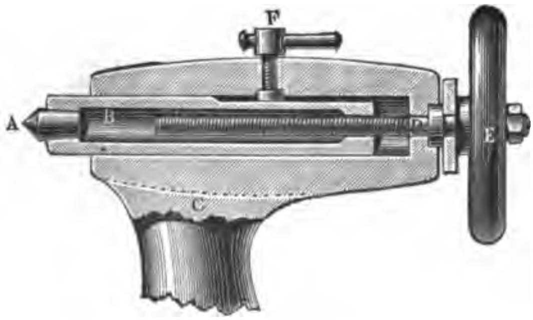


FIG. 332.



of the chamber E, and thus prevent the escape of steam or water. Two or more bolts F are employed for producing a moderate pressure. The gland and the opening in the cylinder cover are both provided with brass bushings G for the purpose of diminishing friction. For rods of small diameter the construction shown in Fig. 332 is often employed, the box C being provided with threads which fit the threads in the nut D. It is evident that when the nut is turned the packing in E will be pressed against the rod. If a piston were absolutely rigid, such a stuffing box would answer as a guide, but as the piston rod is not inelastic it will be bent when its projecting end is acted upon by lateral forces. To avoid this and the friction in the stuffing box which accompanies it, as well as to remove all danger of breaking the rod, it is usually necessary to provide an additional guide which will compel the projecting end to remain in its straight line, and which will take up any lateral forces that may arise. In cases where the stroke of the rod is small, it is customary to employ only a single guide, which must, however, be made correspondingly long. This arrangement is employed in the *tail-stock* of a lathe, and is shown in Fig. 333. The steel spindle B, which carries the back centre A, is guided by the, cylindrical chamber in the tail-stock C, motion being given to the spindle by the screw D and the hand wheel E. The set screw F serves not only to fix the spindle in any position, but also to prevent it from turning, the end of the screw F fitting into a groove on the upper surface of the spindle.

FIG. 333.



To guide the free end of a piston rod various means are employed; one of the simplest is represented in Fig. 334, where the piston rod *A* is guided outside of the cylinder by the cylindrical guide *C*, which is united to the frame of the engine by the arm *CD*. If we suppose the force to be transmitted to or from the piston by a forked connecting rod *F*, jointed to the pins or journals at a point *E* between the stuffing box *B* and the guide *C*, we shall have an arrangement which is often employed in small engines and pumps. This connecting rod *F*, when it is inclined to the piston rod, exerts a lateral pressure which bends the latter. This disadvantage, and the fact that a forked rod is not a particularly stiff form of construction, render this arrangement unsuitable for the transmission of large forces.

FIG. 334.

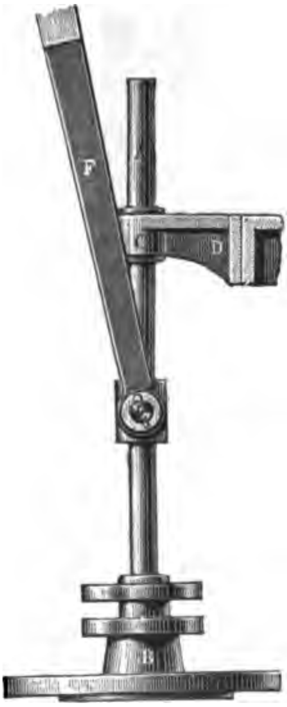
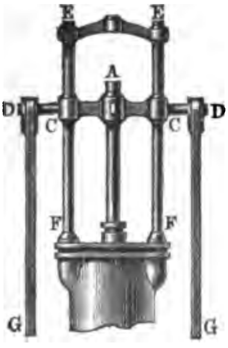
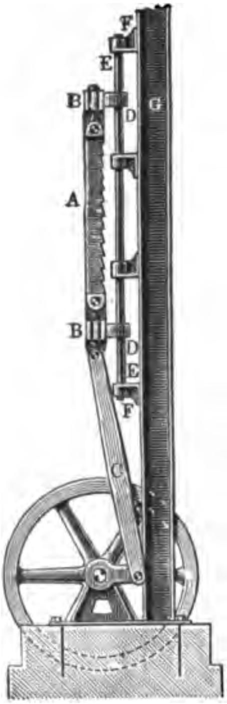


FIG. 335.



A different arrangement has therefore been employed for guiding by means of cylindrical boxes. It is shown in Fig. 335, where there are two boxes *C C* and two guide rods *E F*; in this case the guide boxes are not fixed but are attached to the cross-head *D D*, which is driven by the piston rod *A*. The guide rods *E F* are firmly screwed to the cylinder or engine bed, and the end journals *D D* of the cross-head allow the use of two connecting rods which are always parallel. It is evident that there is no kinematic difference between the case of Fig. 334, where the rod passes through fixed guides, and the case of Fig. 335, where the boxes slide on fixed guide rods. In the construction last given the piston rod is not subject to bending, the lateral force being taken up by the fixed guide rods *E F.*

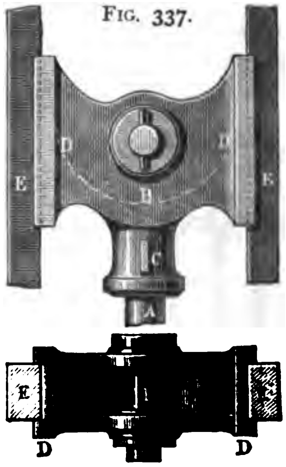
Fig. 336.



A slide motion with movable guides and fixed guide rods may also be found in the *gang-saws* of *saw-mills.* The saw blades *A* are fastened to a frame or saw-gate *B B*, Fig. 336, which is driven by the connecting rod *C*, receiving a rapid up and down motion from the latter. The saw-gate is guided in a vertical plane by metallic or lignum vitæ guides *D*, which are firmly bolted to the corners of the saw-gate and slide on two parallel guide rods *E*, fastened by their brackets *F* to the posts *G*.

§ **95. Prismatic Guides.**—When, as in large steam engines, great forces are to be transmitted, the guide bars are usually prismatic in form, and the movable guides become sliding pieces whose flat surfaces are guided by corresponding surfaces on the prismatic bars. These flat surfaces of contact are better adapted for sustaining great pressures than the cylindrical interior of the guides, and by making the sliding surfaces sufficiently large, the pressure per unit of surface, and consequently the wear, may be diminished at pleasure. An example of prismatic guiding is shown in Fig. 337, where *A* is the end of the piston rod which is to be guided, and *B* the cross-head fastened to the piston rod by the key *C* and provided with two cast-iron sliding blocks *D*, which are guided be-

Fig. 337.



tween two slide bars *E*. The cross-head is fork-shaped at the centre and arranged to receive a pin *F*, which fits the eye of the connecting rod. It is plain from the figure that the slide blocks *D* only partially enclose the slide bars, but the arrangement suffices to prevent all lateral motion of the piston rod. Instead of a rectangular cross section, the slide bars *E* are sometimes given that shown in Fig. 338, in order to maintain an excellent guiding motion, the sliding pieces *D* being made adjustable by keys or set screws. In powerful machines the cross-head *B* has a long cross bar *C*, Fig. 339, which is connected in the middle with the connecting rod, and has end journals which fit the slide blocks *D*, each block being guided between two cast-iron slide bars. In this arrangement the lateral pressure exerted by the connecting rod is well resisted by the two guide bars. That adjustable sliding blocks should also be employed is evident, and that the sliding surfaces should be kept well lubricated. For the latter purpose, the lower prisms *F*, Fig. 339, and the upper edges of the slide blocks *D* are provided with oil grooves.

FIG. 338.

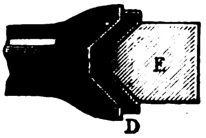
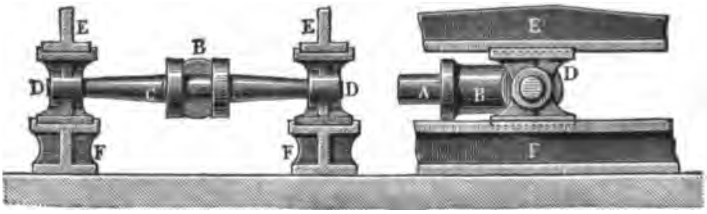


FIG. 339.



Of late the use of hollow cast-iron frames has become popular in steam engine manufacture, the guide bars *E* and *F*, Fig. 340, forming part of the frame and having cylindrical sliding surfaces. In this arrangement the guide bars are bored and the sliding surfaces of the cross-head are turned, thus securing the advantages of easy and accurate construction.

FIG. 340.

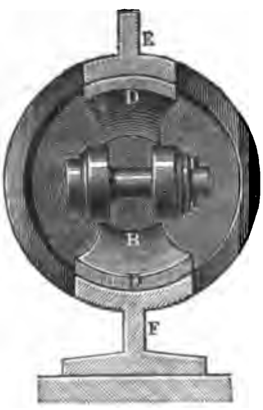
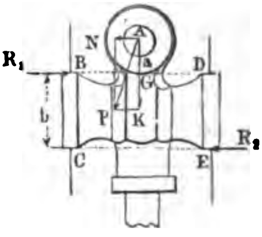


FIG. 341.



A point deserving special consideration is the relative position of the centre *A*, Fig. 341, of the cross-head pin to the sliding surfaces *B C* or *D E* and the length *b* of the latter. Let *P* be the oblique force exerted by the connecting rod on the pin *A*; let it be resolved into two components, one parallel and the other perpendicular to the direction of the piston rod, and represented by *K* and *N* respectively. The component *N* will exert a pressure on but one sliding block, *i.e.*, *B C*, when the direction of *N* passes through the sliding surfaces of the blocks, but if the direction of *N* does not pass through these surfaces but lies outside of the latter, at a distance $a = AG$, the force *N* will cause the reactions R_1 and R_2 in the guides at the points *B* and *E*, which reactions may be found from the conditions of equilibrium as follows:

$$R_1 = N\frac{a+b}{b} = N\left(1 + \frac{a}{b}\right),$$

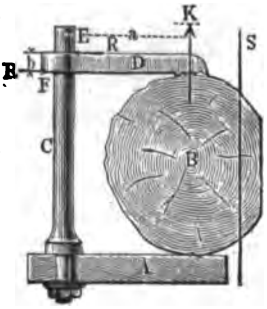
and

$$R_2 = N\frac{a}{b}.$$

As these reactions generate friction, we see that the construction is a faulty one so long as *a* has a positive value, *i.e.*, so long as the pin *A* lies outside of the rectangle *B D E C*. Therefore the pin *A* should not be placed outside of the line *B D*, if unnecessary friction is to be avoided. To more evenly distribute the pressure *N* upon the whole sliding surface *B C*, it is customary, whenever possible, to place the pin midway between the ends *B D* and *C E*. Friction then takes place on but one guide, and will be equal to that generated by N directly, *i.e.*, φN, while for every positive value of a the frictional resistance will be $\varphi N\left(1 + \frac{a}{b} + \frac{a}{b}\right)$, that is greater by $2\frac{a}{b}\varphi N$.

It deserves notice, that when the overhang a is large and the length b of the guided surface small, sliding may even become impossible under certain circumstances, namely, when the frictional resistance on the guides exceeds the component K. Extensive use is made of this binding action in practice, for example, in *saw-mill dogs* for holding down the logs. It consists of a simple wrought-iron dog D, Fig. 342, with a cylindrical eye EF fitting the fixed rod C. Suppose the log B to be acted upon by an upward force K, which causes the reactions or binding actions R between the eye of the dog and the rod C at the points E and F. The moment Rb of this couple will be equal to the moment Ka of the upward acting force. Consequently $R = K\frac{a}{b}$, and if the dog should slip, there would be the friction φR at both E and F and we should have for the instant at which sliding takes place:

FIG. 342.



$$K = 2\varphi R = 2\varphi K\frac{a}{b}, \text{ i.e., } 1 = 2\varphi\frac{a}{b}$$

Therefore so long as $2\varphi\frac{a}{b}$ is greater than 1, the resistance of friction will exceed the force K no matter how large this force may be, and the condition of clamping will be satisfied. Accordingly, by giving a small length b to the sleeve, we may always make the value $2\varphi\frac{a}{b}$ greater than unity, *i.e.*, $\frac{b}{2a} < \varphi$.

If the coefficient of friction is $\varphi = 0.1$, clamping will occur when the length a of the dog is five times the length b of the eye. We see from this what an important influence is exerted by the length of the slide blocks when the force is eccentric.

Prismatic guides are also much used in many metal- and wood-working machines, for example, in *planers*, *lathes*, etc. Thus the sliding table A of a planer, Fig. 343, is guided by two exactly parallel prisms B of trapezoidal cross section, which slide in corresponding grooves C of the frame D. The great weight of the table and of the work upon it prevents the table from rising, and the trapezoidal profile of the prisms prevents all lateral motion of the table. This simple form of support is not suited, however, to lathes and machines in which the guided part is not heavy and is subject to great lateral forces. In such cases the guide must be either enclosed by the sliding piece or must enclose it in such a way that the two cannot be separated in any direction. Fig. 344 shows an arrangement for guiding the tool-rest and similar parts. The rest A encloses the two prisms B of the frame or bed D so that by means of the wearing piece or gib C and the adjusting screws E, all lateral play, and consequently all unsteadiness of motion, may be removed.

FIG. 343.

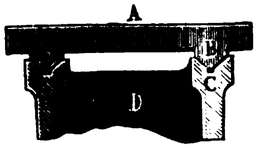
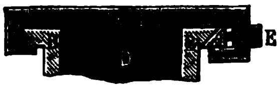


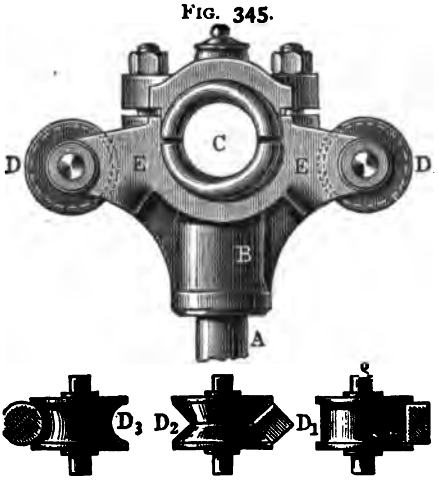
FIG. 344.



§ **96. Roller Guides.**—Whenever the connecting rod is inclined to the piston rod, as is the case in all crank mechanisms, a lateral pressure is exerted against the guides, which creates considerable friction between the latter and the slide blocks of the cross-head. To diminish this friction as much as possible, friction rollers have been employed, the object being to substitute rolling for sliding friction. The cross-head in Fig. 345 is an example; it differs from those hitherto considered only in having the two arms E provided with fork-shaped ends for receiving the rollers D.

When the slide blocks are moved along the guide bars, sliding friction will be developed by the normal pressure N, which friction must be overcome through a distance equal to the stroke of the piston; but in the present case the friction generated by the same normal pressure N on the journals must be overcome through a smaller distance, namely, $\frac{\rho}{r}$ times the stroke of the piston. These rollers play exactly the same part as the friction wheels employed as journal bearings and as the wheels of a wagon. The correct form for these rollers is evidently cylindrical, that shown at D_1, for it is only with this form that all points on the pulley have the same velocity of rotation. Every other form of profile for the rollers, for example those shown at D_2 and D_3, must be regarded as faulty, for the various points on the periphery of the roller have unequal circumferential velocities according to their distances from the axes, consequently with such forms a certain amount of sliding still remains.

FIG. 345.



In spite of these apparent advantages, friction rollers have been very little used for the cross-heads of piston rods, because the guiding thus produced is too uncertain, and the wear of the journal bearings causes unsteadiness of motion. Another drawback in this respect is that the journals of the roller soon lose their cylindrical form on account of the very variable pressure and alternating rotations to which they are subjected, and thus render the use of adjustable bearings difficult.

Rollers are oftener employed for guiding and supporting the long rods used in the shafts of mines, in order to prevent their being buckled or bent by the thrusts to which they are subjected, or by their own weight. Such a roller is shown in Fig. 346, the wooden rod *A B* having a wrought-iron rail *C C*. Fig. 347 shows the rod *A B* guided by the rollers *E* and *F*, the rod receiving its vertical motion from the lever *C D*.

FIG. 346.

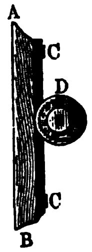
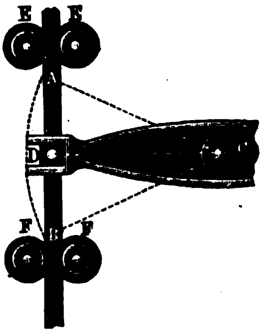
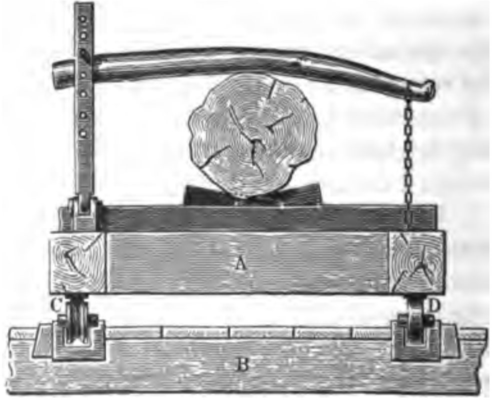


FIG. 347.



With roller guides, as with cylindrical and prismatic guides, there is no difference in the action whether the movable rod is guided by rollers with fixed axes, or whether they are connected like little wagon wheels with the rod, the latter being then moved wagon-like over the guides. Both arrangements are often employed in saw-mills for guiding the table or carriage. These tables have wooden or iron frames of great length (up to 15 meters [50 ft.]), and prismatic guides are unsuitable because with these lengths there is not sufficient stiffness, consequently roller guides are used, the rollers being given either to the carriage *A* or to the foundation *B*, Fig. 348. It is customary to make one rail *C* of triangular cross section, in order to secure as perfect a motion as pos-

FIG. 348.

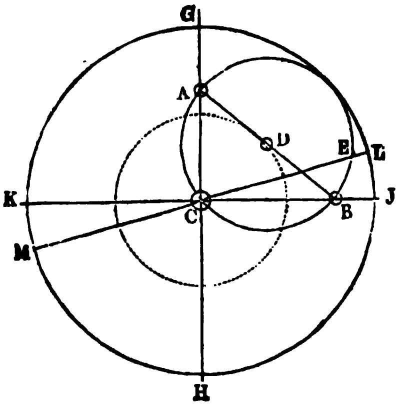


sible. The disadvantage above mentioned, that with such a form sliding friction cannot be avoided, need not be considered here, because of the slow motion of the carriage. The second rail *D*, however, is usually given a horizontal surface in order that the carriage may have a certain movability within itself, thus avoiding the binding actions and pressures which would arise if both rails had the form of *C*.

GUIDES AND LINKAGES FOR PRODUCING RECTILINEAR MOTION, COMMONLY KNOWN AS PARALLEL MOTIONS (§ 97 to § 107).

§ 97. Hypocycloidal Mechanism for producing Rectilinear Motion.—The considerable friction usually attending cylindrical and prismatic guides and the constructive difficulties often involved in the application of slide bars, have led to the construction of other straight-line motions. An examination of the motion discussed in § 11 of the Introduction, in which two determinate points of a system move on two intersecting straight lines, leads us at once to a series of straight-line motions. It was there shown that a straight line *A B*, Fig. 349, which moves so that two of its points *A* and *B* are guided along two straight lines *C G* and *C J* intersecting at right angles, has a motion whose centroids are two circles, one of which is stationary, having the centre *C* and the radius *A B*, while the other is movable and has the straight line *A B* for a diameter. Now if we suppose these circles or centroids to be embodied in actual constructions which can roll on each other, the straight line *A B* will

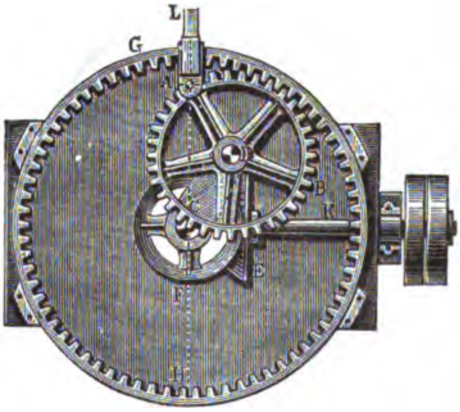
Fig. 349.



have the same motion as the assumed one. For instance, the circle about *C* may be constructed as a stationary annular wheel and the circle about *D* as an ordinary spur wheel whose centre *D* is guided in a circle around the point *C*. The point *A* will travel along the whole diameter *G H* and will return, and the point *B* will travel along the whole diameter *J K* and return, while the centre *D* makes one rotation about *C*. Moreover other points on the circumference of the rolling circle will describe diametral paths while the smaller circle is rolled within the larger and stationary one, for example, *E* will traverse the diameter *L M*.

Use has been made of this property in a few cases, for example, to give a rectilinear motion to the bed or carriage of a printing press. The mechanism is shown in Fig. 350. The internally toothed rim *G H* is bolted to the frame of the machine, while the pinion *A B* turns loosely on the pin *D* of the crank *C D*. This crank receives a uniform rotation from a vertical shaft whose axis passes through the centre of the fixed wheel. The rotation of this vertical shaft *C* is effected by bevel wheels *E F*, which in turn are driven by a belt on the pulley of the horizontal shaft *K*. The driving pin is placed at a point *A* exactly on the pitch circle of the small wheel. This driving pin traverses the diameter *A H* and communicates rectilinear motion by means of a connecting rod *A L* to the bed which carries the form. Every time the crank turns, the carriage makes one stroke forward and one back, each stroke being equal to the diameter $G H = 2 A B$ of the pitch circle. It is clear that the numbers of teeth of the two wheels

FIG. 350.

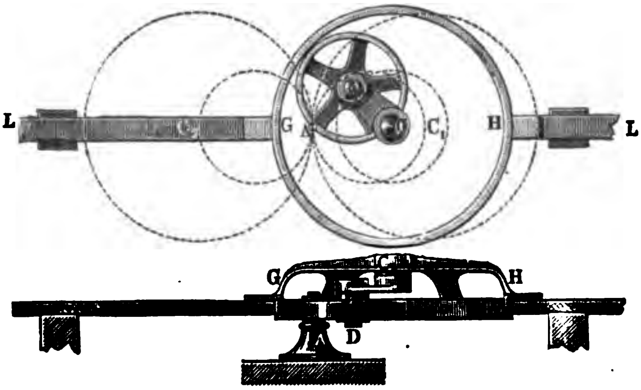


are as one to two. When the rotation of the crank *C D* is uniform, the velocity of the rectilinear motion of the point *A* is proportional to those ordinates of the fixed circle which are normal to the diameter *A H*, as was explained in § 13 of the Introduction.

To this mechanism we may give the name of hypocycloidal motion. It is rarely used except in the power printing presses just mentioned. In other cases the large diameter of the fixed toothed wheel is the principal objection to the straight-line motion.

We may also mention that this mechanism has been used in another manner as a straight-line motion, for if we suppose the whole system to receive at every instant an additional motion exactly equal and opposite to the rectilinear motion of the pin *A*, it will not alter the motion of the individual parts relatively to each other. This evidently brings the point *A* into absolute rest, while the toothed wheel *G H*, which was formerly stationary, receives rectilinear motion equal to and in the direction of *A H*. Therefore, if this wheel *G H* is united to the rod *L*, Fig. 351, and the pin *A* in the circumference of the small wheel is bolted to the frame, a rotation of the crank *C D*, in which its centre *D* rotates about the fixed centre *A*, will cause the wheel *G H* and the rod attached to it to have a reciprocating motion, the points

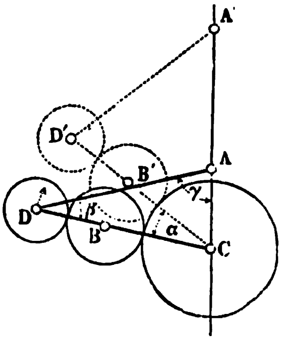
FIG. 351.



G and H alternately covering the fixed point A. The arrangement obtained by holding fast another link of a mechanism is called by *Reuleaux* an *inversion* of that mechanism. We may therefore call the mechanism last obtained an inverted hypocycloidal *straight-line motion*. This arrangement hardly has a practical value, and may be considered of theoretical interest only.

If the larger wheel CG is provided with external teeth, the smaller wheel D will have an opposite direction of rotation. But if this reversal of motion is neutralized by inserting an idle wheel between C and D of any size whatever, we obtain a straight-line motion which may be called an *epicycloidal straight-line motion*. That this will give a correct straight-line motion can be easily proved. Let the two wheels D and B be attached to the lever CBD turning about C, and let the wheel D, whose radius is again equal to half that of the wheel C, be provided with an arm DA whose length is $DA = DC$, then if the lever DBC be turned about C, the end A of the arm will remain on the prolonged diameter CA. In the isosceles triangle thus formed let the angle $ACD = CAD$ be represented by γ, then will the angle ADC at the vertex be $\beta = 2(R - \gamma)$. If the lever DBC be given a right-handed rotation through $DCD' = \alpha$, the wheels will cause the arm DA to turn to the left about D through the angle 2α, thus forming a new isosceles triangle, $A'D'C$, the angle at the vertex D' being given by

FIG. 352.



$$\beta + 2\alpha = 2(R - \gamma + \alpha).$$

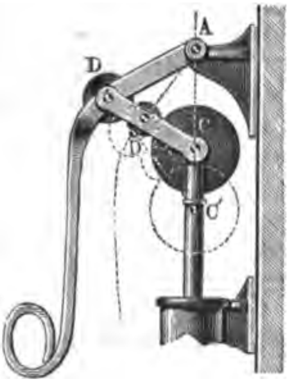
Consequently each of the angles at the base will be:

$$A'CD' = CA'D' = R - (R - \gamma + \alpha) = \gamma - \alpha.$$

This proves that A' lies in the same straight line with A

and *C*. It is easy to see that when the lever *D B C* makes a complete revolution, the end *A* of the arm will have a forward and backward stroke, each of which is equal to 4 *D C*. This mechanism, therefore, produces a straight-line motion of considerable length with comparatively small wheels.

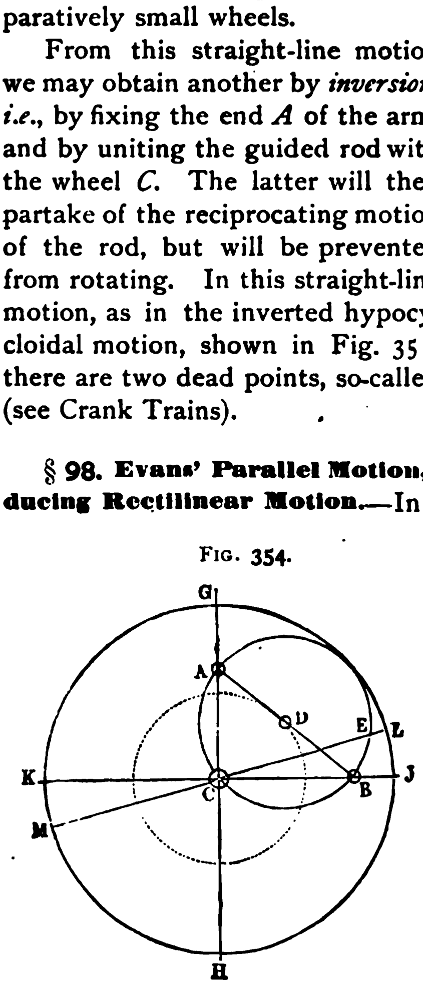
FIG. 353.



From this straight-line motion we may obtain another by *inversion*, *i.e.*, by fixing the end *A* of the arm, and by uniting the guided rod with the wheel *C*. The latter will then partake of the reciprocating motion of the rod, but will be prevented from rotating. In this straight-line motion, as in the inverted hypocycloidal motion, shown in Fig. 351, there are two dead points, so-called (see Crank Trains).

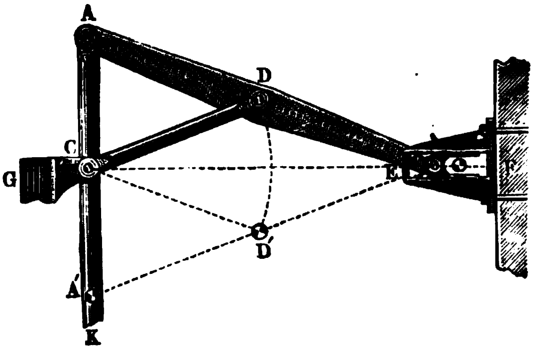
§ 98. Evans' Parallel Motion, or Mechanism for Producing Rectilinear Motion.—In the motion of a straight line discussed in § 11 of the Introduction, the ends *A* and *B*, Fig. 354, of the line, were guided along two straight lines at right angles, and this caused the centre *D* of the line to move in a circle concentric with *C* and with a radius equal to half the length of *A B*. Since the motion of a body in a plane is completely determined by the motion of two of its points, we can produce this motion of the line *A B* by guiding one end *B* along the straight line *C J* and compelling the centre *D* to

FIG. 354.



remain on the circle having C as a centre and $CD = BD = AD$ as a radius. The other end A must evidently have a rectilinear motion along the diameter GH normal to CJ. To compel D to move in a circular path about C, it is only necessary to connect D by a link to a stationary journal at C, the link having a turning motion about C. This link, provided with eyes at each end, may be called a radius bar.

FIG. 355.



This straight-line motion may be realized practically as shown in Fig. 355. The guided piston rod AK is joined at its upper and forked end to the *beam* AB, at the other end B of which there is a cross-bar with two end journals which fit into two blocks that slide along the guides E F. There are two radius bars CD, the beam lying between them at D. The other ends C of these bars are connected with two journals so attached to the stationary cross-piece G that the forked end A of the piston rod can readily pass between them. When this rod makes a complete stroke, represented by s, from A to A', and when, as is always the case in practice, the beam oscillates symmetrically on each side of the straight line CB, the maximum inclination of the beam to CB on each side of the latter will be $\alpha = ABC = A'BC$, and may be found from:

$$\sin \alpha = \frac{s}{4a},$$

where a represents half the length of the beam and the whole length of the radius bar CD. This angle is generally not greater than 20°, sometimes it is as high as 25°; the ordinary ratio of the length $2a$ of the beam to the stroke s is $\frac{2a}{s} = 1.5$, hence we have:

$$\sin \alpha = 0.333; \qquad \alpha = 19° 30' = \text{about } 20°.$$

During a single stroke of the piston rod from A to A' or in the opposite direction, the point B traverses the path $BB' = \sigma$, which may be found from

$$\sigma = 2a(1 - \cos \alpha).$$

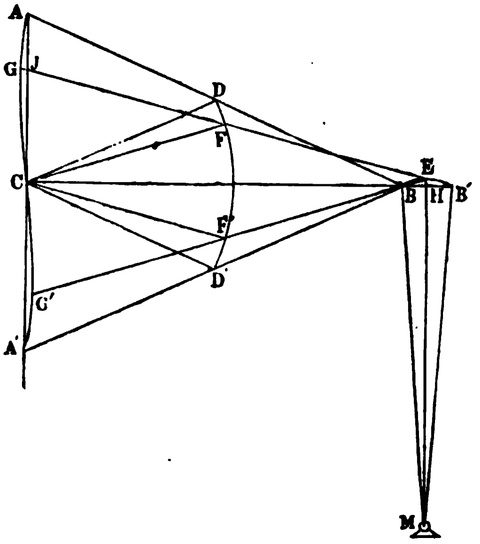
This is always much smaller than the stroke s, for example, when $\alpha = 20°$, we have:

$$\sigma = 2a \times 0.060 = 0.12a = 0.09s.$$

This justifies the use of this straight-line motion for guiding the point A, although the motion of one point B is still controlled by prismatic guides.

Generally, however, the prismatic guiding of the point B is avoided by carrying it on a vibrating pillar MB with a fixed centre at M. In this way B is guided along an arc which differs but little from the straight line BB'. The motion of A does not then take place exactly in a straight line, but its deviation from the latter may be neglected in practice if the length t of the pillar is not taken too small, say equal to the

FIG. 356.



length s of the stroke. In order to ascertain the character and magnitude of this deviation, we must note that since the point B moves in the circular arc $B E B'$, the other end A of the beam must pass through the three points A, C and A' of the straight line $A A'$ in which the piston rod is to be guided, for these three points correspond to the three positions B, B' and B of the back end of the beam on $C B$. In all other positions of the beam, the point A deviates from the straight line and is on opposite sides of the line $A A'$ during the two halves of the stroke, the end describing a curve, say like $A G C G' A'$, which has a point of inflexion at C. The greatest deviation f occurs at the instant when the point B is at the vertex of the arc $B E B'$ or in its immediate neighborhood, and a very close approximation to these deviations may be found as follows: Let us first suppose the exact straight-line motion by means of slide bars to exist, and consider that position of the beam in which its end is at H half way between B and B', then the other end will be in the straight line $A A'$, and the angle $\alpha' = JHC$, which the beam makes with the line $C B$, may be found from

$$\cos \alpha' = \frac{2a - \frac{1}{2}\sigma}{2a}.$$

Employing the above ratio, $\frac{2a}{s} = 1.5$, which gives $\sigma = 0.120a$, we have

$$\cos \alpha' = \frac{1.940}{2} = 0.970; \qquad \alpha' = 14°.$$

Now if we suppose the beam to receive a slight rotation about its centre D, Fig. 357, so that the end describes the small arc $H E'$ from H to the guiding arc $B E B'$, then the other end will describe an equal arc $J J'$. The deviation of this point from the straight line $A A'$ is therefore given by $J' K = E E'$. Now let h represent the height $H E$ of the arc $B E B'$, then we have, approximately,

$$f = E E' = h \sin \alpha'.$$

The height *HE* is also approximately

$$h = \frac{\left(\frac{\sigma}{2}\right)^2}{2t} = \frac{\sigma^2}{8t}.$$

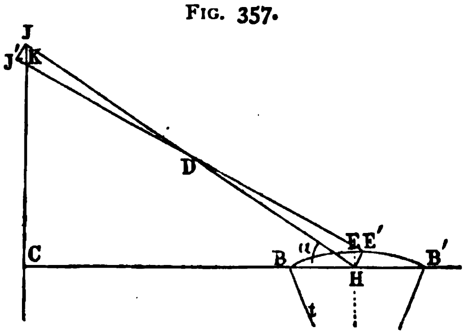


FIG. 357.

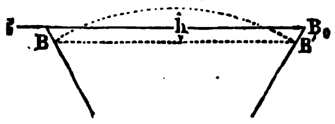
If we assume the length of the pillar, for example, $t = s$, we get:

$$h = \frac{(0.090\, s)^2}{8s} = 0.001\, s,$$

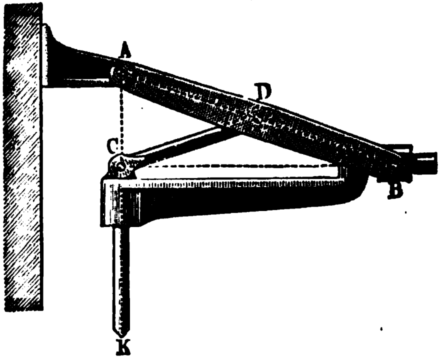
consequently the deviation will be

$$f = 0.001\, s \sin 14^\circ = 0.00024\, s.$$

We may also remark that this practically infinitesimal deviation (less than ¼th millimeter for 1 meter stroke [less than $\frac{1}{100}$th of an inch for 40 inches of stroke]) may be still further reduced by so placing the pillar *MB* that the height *h* of the arc *BB'* will be bisected by the straight line CB_0, Fig. 358, in which case the guided point *A* enters the straight line *AA'* four times every stroke.

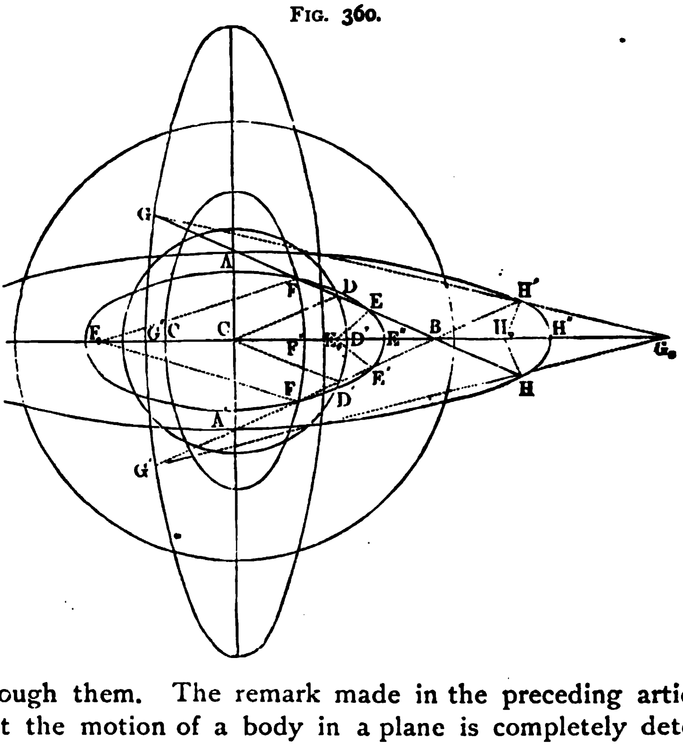
FIG. 358.


It is moreover evident that we may obtain a correct straight-line motion by so inverting the *Evans* mechanism that the end *A* of the beam, Fig. 359, will be fixed, and the centre *C* of the radius bar and the prismatic guide will be rigidly connected with the piston rod.

FIG. 359.


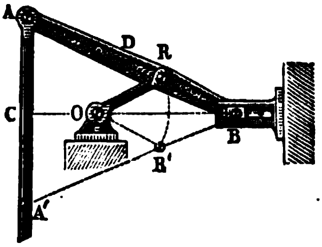
§ 99. Elliptical Guides and Linkages for Producing Rectilinear Motion.—*Evans'* parallel motion discussed in the preceding article is only a particular case of a series of straight-line motions which may be obtained from the often-mentioned motion treated in § 11 of the Introduction. It was there proved that every point of the straight line *A B*, Fig. 360, describes an ellipse whose principal axes coincide in direction with the straight lines *A C* and *B C*, along which the points *A* and *B* are guided. The lengths of the semi-major and semi-minor axes are respectively $a + e$ and $a - e$, where a represents half the length of *A B*, and e the distance of the describing point from the middle of *A B*. The points *E*, *F*, *G* and *H* describe the elliptical paths which are drawn through them. The remark made in the preceding article that the motion of a body in a plane is completely determined by the motion of two of its points, holds here also. It follows from this that the point *A* will move in the straight

FIG. 360.



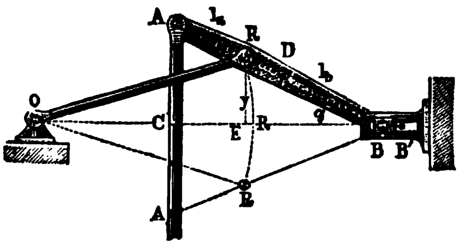
line $A\ C$ if the point B is guided along the straight line $C\,B$, and any one of the points E, F, G or H is guided in the elliptical path belonging to it. The latter may be done by employing elliptical surfaces, or surfaces which are equidistants to the latter; but such a construction would involve many practical difficulties, and as only a small part of the periphery of the ellipse in the vicinity of the vertex is employed, it will be sufficiently accurate to replace the elliptical arc by a circular arc having three points in common with the latter, say the middle and end points. If, therefore, we find the centre E_0 of the circular arc passing through E, E' and E'', and guide the end B of the lever $A\ B$ by prismatic guides along the straight line $C\,B$, and also the point E of the lever by a radius bar $E_0\,E$ whose axis is at E_0, the other end A will be guided in a line which approximates to the straight line $A\,A'$. In this way a number of straight-line motions can be obtained which differ from each other in having different positions for the point which is guided by the radius bar. These mechanisms may be called *elliptical* (or approximate) *straight-line motions.* We may accordingly distinguish three different kinds of elliptical straight-line motions:

FIG. 361.



1) In the *first* kind of elliptical motions, the radius bar guides a point like E or F, which is situated between A and B. Corresponding constructions are shown in Fig. 361 and Fig. 362, which differ only in this, that the radius bar takes hold of $A\ B$ at R on different sides of the centre D. We see from the figures that the axis O of the radius bar in these two cases

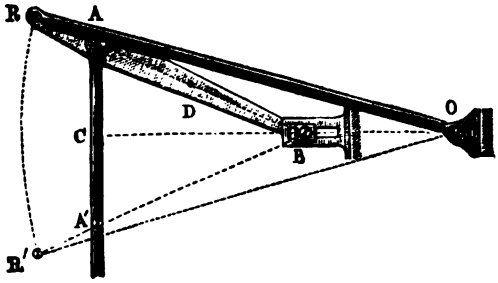
FIG. 362.



lies upon different sides of the required straight line $A\ A'$, and that the radius bar $O\ R$ is shorter the nearer the point R to the point B guided by the slide bars. Evidently *Evans'* straight-line motion lies half way between these two cases, the radius bar for this motion having a length equal to a and having its axis at C.

2) In the *second* kind of elliptical straight-line motion, the radius bar guides a point R lying outside of $A\ B$ beyond A, and consequently the axis O of the bar lies on the opposite side beyond B.

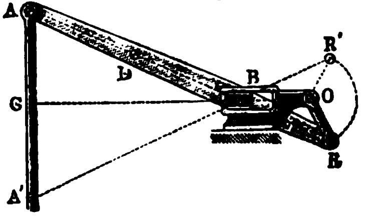
FIG. 363.



3) The *third* kind of elliptical motion is shown in Fig. 364. Here the point R, guided by the radius bar, lies outside of $A\ B$ beyond B, the axis O of the bar lying on the same side of B.

Moreover, it is clear that in all these mechanisms, instead of guiding the point B by slide bars, we may approximately guide it by a lever or vibrating pillar, as in Fig. 365, and to this arrangement the discussion accompanying the analogous case in *Evans'* motion is likewise applicable.

FIG. 364.



The position of the axis O of the radius bar and the length r of the latter may in every case be easily found by a graphical construction when we know the stroke s, the length $A\ B = l$ of the beam and the position relatively to A and B of the point R guided by the radius bar, *i.e.*, the distances $R\ A = l_a$ and $RB = l_b$. It will then only be necessary to draw the beam in its two extreme positions, $A\ B, A'\ B$ and in its middle position CB', Fig. 366, in order to determine the centre of the circle which passes through the corresponding points R, R' and R''. The length of the radius bar may also be very easily found by calculation. For if in Fig. 366 we represent the co-ordinates RE by y, $R''E$ by x and the radius OR by r, we shall have the known relation

$$y^2 = (2r - x)x; \quad \text{or} \quad r = \frac{y^2}{2x} + \frac{x}{2}.$$

But according to the figure

$$y = l_b \sin\alpha; \quad x = l_a(1 - \cos\alpha);$$

FIG. 365.

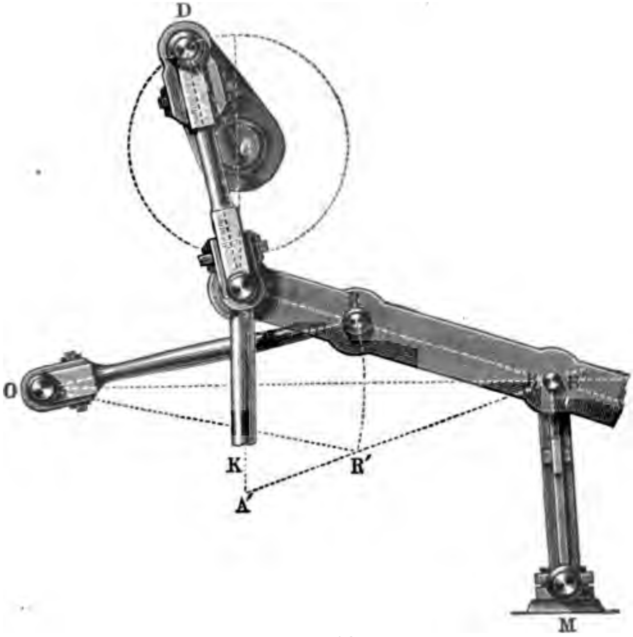
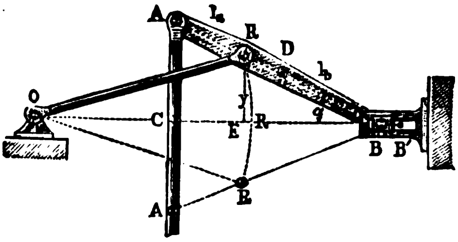


FIG. 366.



hence

$$r = \frac{l_b^2 \sin^2 \alpha}{2 l_a(1 - \cos \alpha)} + l_a \frac{1 - \cos \alpha}{2} = \frac{l_b^2}{l_a} \frac{1 + \cos \alpha}{2} + l_a \frac{1 - \cos \alpha}{2}.$$

If we again assume half the angle of oscillation of the beam to be 20°, we shall have for the radius bar, the length

$$r = 0.970 \frac{l_b^2}{l_a} + 0.030\, l_a .$$

The above expression for r is applicable to all three kinds of elliptical straight-line motions, and when R is taken at the middle point D, we have $l_a = l_b$ and $r = l_a = l_b$, which corresponds to the case of *Evans*' straight-line motion.

The circular path of R will not coincide with the osculatory circle of the corresponding ellipse, for the circle found does not contain three consecutive points of the ellipse. But we may recognize the deviation of the radius of curvature ρ from length r of the radius bar from the above formula $r = \frac{1 + \cos \alpha}{2} \frac{l_b^2}{l_a} + \frac{1 - \cos \alpha}{2} l_a$, while the radius of curvature at the vertex of the ellipse is $\rho = \frac{l_b^2}{l_a}$, where l_a and l_b represent the semi-axes.

It is evident that of these straight-line motions the only one which is exact is that of *Evans*, for which the ellipse becomes a circle. For all other positions of the point R guided by the radius bar, there is a deviation of the circular path from the proper or elliptical one, which deviation must be greater the greater the eccentricity of the ellipse. Without entering into any exact and complex calculation, we may conclude that the motion of the point A will be more nearly rectilinear the less the eccentricity of the point guided by the radius bar $O R$, *i.e.*, the nearer this point is to the middle of the beam. For this reason, levers of the first kind are best suited for straight-line motions, and for another reason also, namely, that for a given stroke s and angle of deviation α, the straight line motions of the first class require the least length of beam. When this kind is employed, a more exact straight-line motion is obtained when the point R of the radius bar lies on the portion DA, than when it lies on the half DB of the beam which is nearer to the slide BB', for in the first case the arc described by the radius bar approximates more closely to the portion of the ellipse which it replaces ($FF''F'$ in Fig. 360) than in the latter ($EE''E'$). At any rate, when the length r of the radius bar is correctly chosen and the angle of deviation α is small the deviation of the point A from the straight line AA' will be very slight.

In the arrangements just described, the circular path of the point R is so chosen that it coincides at three points R, R'' and R', Fig. 367, with the ellipse; consequently the end of the piston rod must pass through the exact rectilinear path at the corresponding points A, C and A'. Whenever the circular arc deviates from the ellipse, the end A of the piston rod will deviate from the exact rectilinear path. We see from the figure that both halves RUR'' and $R'U'R''$ of the elliptical arc lie either inside or outside of the circular arc RTR', being on the inside when R'' is the vertex of the major axis (as E'' and H'' in Fig. 360), and on the outside when R'' is the vertex of the minor axis (as F'' and G'' in Fig. 360). Consequently the end A of the beam will lie on the same side of the desired rectilinear path during both halves of its motion. If we suppose a circular arc $r\,r''\,r'$ to be described with O as a centre and such a radius $O\,r$ that it will bisect the greatest deviation UT, the end of the beam will be so guided as to meet the exact rectilinear path in four points corresponding to the intersections S, S', S', S of the circle $r\,r''\,r'$ with the ellipse. The deviations of the piston rod at intermediate points will then be alternately on opposite sides, and only about half as large as they would be if RTR' were the guiding circle. These approximate straight-line motions may, like all other mechanisms, be inverted. For instance, we may

FIG. 367.

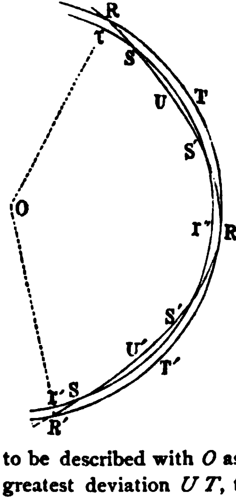
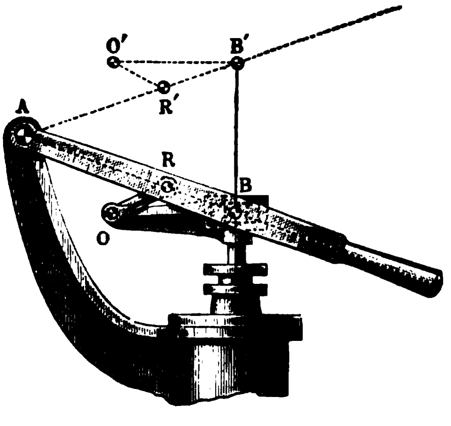
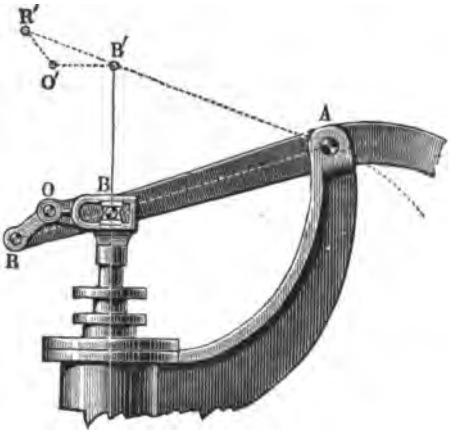


FIG. 368.



suppose the whole system to receive an additional motion equal and opposite to the stroke *s* of the piston rod, this will cause the end *A* of the beam to become a fixed point of the frame. Then it is only necessary to unite the piston rod *K* to the eye *O* of the radius bar and to the slide bars at the other end of the beam.

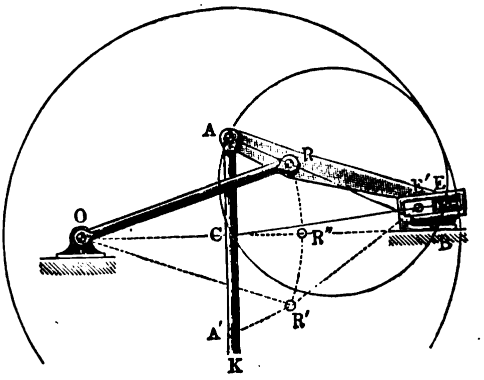
FIG. 369.



These parts, which were formerly at rest, now take on the additional reciprocal sliding motion communicated to the system. *Nehrlich's* straight-line motion, Fig. 368, is an example of this, and must be regarded as an inversion of the approximate elliptical motion of the first kind. An inversion of the elliptical motion of the third kind is shown in Fig. 369. After what has been said, no further explanation of these figures is necessary. In some cases these straight-line motions may be useful; indeed they have been employed for pump rods.

§ 100. Oblique Elliptical Guides and Linkages for Producing Rectilinear Motion.—The parallel motion discussed in the last article does not correspond to the most general case, for it was tacitly assumed that the end *B* of the beam moved in a straight line (or a close approximation to it) which was perpendicular to the direction of motion of the other end *A*. But this condition is not essential, for it was shown in § 11 of the Introduction that when the ends of a line *A B* are guided on two straight lines at right angles

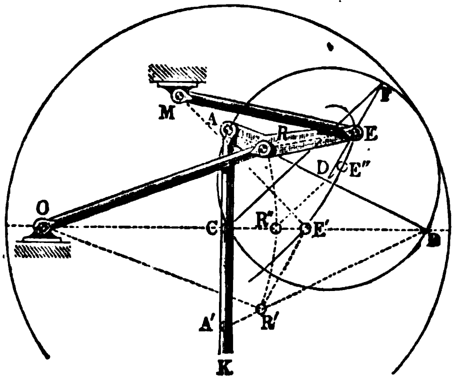
FIG. 370.



to each other, every point E, Fig. 370, on the circumference of a circle described on $A\,B$ as a diameter will likewise move in a straight line $C\,E$ passing through C. Therefore, instead of guiding the point B in a straight line $C\,B$ perpendicular to $A\,A'$, we may guide any point E on the circumference mentioned, in a straight line $C\,E$ which is oblique to $A\,A'$. By this means, and by guiding the point R with the help of the radius bar $O\,R$, we obtain a mechanism which may be called an *elliptical straight-line motion with oblique guides.* The beam or principal lever here assumes the triangular form $A\,R\,E$.

This straight-line motion may also be arranged when the point E does not lie on the circumference of the circle described on $A\,B$, Fig. 371, in which case this point is, of course, no longer guided in a straight line, but in the arc of an ellipse. The ellipse representing the exact path of this point E may easily be found from what was said in § 11 of the Introduction, by drawing through E the radius $D\,E\,F$ of the rolling circle. For the point F travels in the straight line $C\,F$, which is the direction of the transverse axis of the required ellipse. This ellipse is easily drawn, for the semi-axes $a + e$ and $a - e$ are given, e representing the distance $E\,D$ of the point E from the middle of $A\,B$, and a the radius $A\,D = B\,D$ of the rolling circle. Now when this ellipse has been drawn, we may take the elliptical arc $E\,E''\,E'$ which is traversed by the point E during the stroke of the piston rod $A\,K$, and find as before the centre M of a circular approximation to this elliptical arc. In this straight-line motion, therefore, the

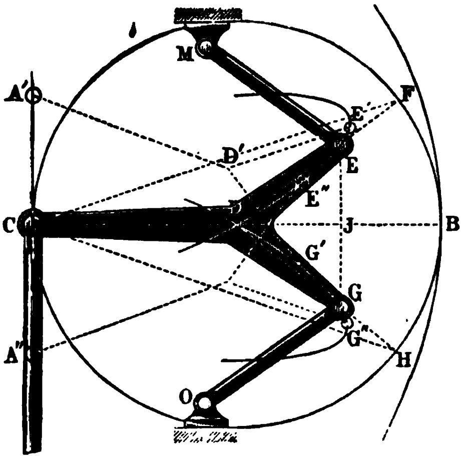
FIG. 371.



beam is guided by two radius bars *O R* and *M E* in two circular arcs which act as substitutes for the elliptical paths of the points *R* and *E*.

Finally, if, instead of assuming the point *R* on the straight line *A B* as hitherto, we choose it at any point outside of this line, and if we substitute a circular arc for the elliptical path of this point *R*, we shall obtain the most general form of the elliptical straight-line motion. Among the infinite number of straight-line motions belonging to this case there is a particularly interesting one which can be obtained by assuming the points guided by the radius bars symmetrical with respect to the straight line *C B*, the latter being perpendicular to the piston stroke *A A'* at its centre *C*. This gives us *Robert's* straight-line motion, Fig. 372. Here the points *E* and *G* are guided by the radius bars *M E* and *O G*, and are symmetrical to the line *C B*. In order to find the length of these radius bars and the positions of their centres *M* and *O*, it is only necessary to draw in the manner described above the ellipses representing the exact paths of the points *E* and *G*. To do this we draw radii through *E* and *G* from the centre of the rolling circle, and join the two points *F* and *H* on the circumference of the rolling circle with *C* by straight lines *C F* and *C H*; the latter are then the major axes of the required ellipses. The semi-axes of the latter are again $a + e$ and $a - e$, where a represents the radius *C D* of the rolling circle, and e the distance of the points *E* and *G* from *D*. From these ellipses we may now find by the method used above the centres *M* and *O* of the radius

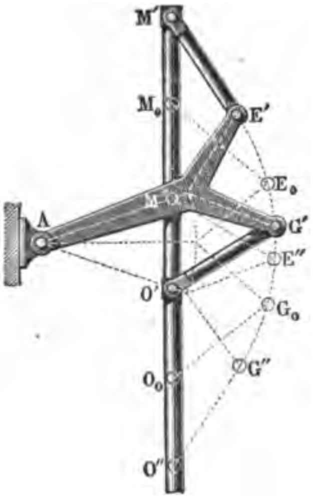
FIG. 372.



bars and their length r. According to *Robert*, this straight-line motion will be very accurate when the base $b = EG$ of the triangle is equal to $b = 0.593 s$ and the height $h = CJ = 1.112 s$, which data correspond to an angle $ECG = 30°$ at the vertex of the triangle. These assumptions give for the length of the radius bars

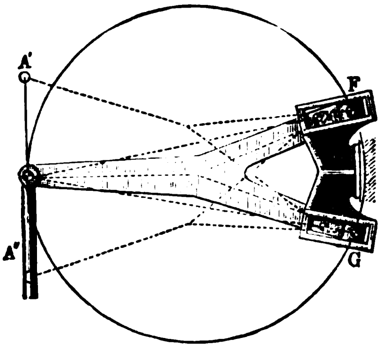
$$r = ME = OG = 0.584 s.$$

FIG. 373.



By inverting *Robert's* mechanism we get a straight-line motion like that shown in Fig. 373. Here the vertex A of the triangle is fixed and the centres M' and O' of the radius bars are united with the piston rod. These centres were formerly fixed, but now have a reciprocating motion equal to the stroke s.

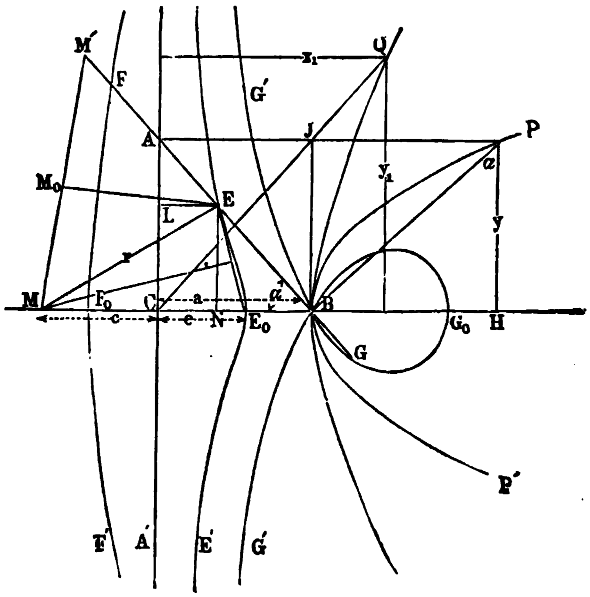
FIG. 374.



A particular case of the last mentioned mechanism is *Reuleaux's* straight-line motion, Fig. 374, in which the guided points F and G are assumed on the circumference of the rolling circle; as a consequence, these points move along rectilinear paths passing through C. The result is that the radius bars may be replaced by slide bars.

§ 101. Conchoidal Guides and Linkages for Producing Rectilinear Motion.—In the previous articles we discussed the motion obtained by moving the ends A and B of a straight line AB on two straight lines at right angles to each other. If we now suppose this form of motion to be so modified that the straight line *A B*, Fig. 375, has only one end *A* guided along the straight line *A A'*, while the line itself is constrained to always pass through a fixed point *B*, we shall have an arrangement which likewise gives rise to different forms of straight-line motion. To determine the centroids for this motion of the straight line *A B*, we will suppose this straight line to be in a position in which it makes the angle *A B C* = α with the normal distance *B C* of the fixed point *B* from the guiding line *A A'*. The instantaneous centre for the elementary motion occurring at this instant is found at the intersection *P* of the two normals *A B* and *B P* erected to the paths of the points *A* and *B*; *i.e.*, by drawing *A B* perpendicular to *A A'* and *B P* perpendicular to *A B*. The instantaneous centres thus obtained determine the fixed centroid *P B P'*, which, it is easy to see, is a parabola whose

FIG. 375.



vertex is at the fixed point B, and whose parameter is equal to half the distance BC. If C is taken as the origin of co-ordinates and the constant distance CB is represented by a, we see from the figure that for the co-ordinates of any point P we shall have:

$$tan\, ABC = tan\, \alpha = \frac{y}{a},$$

and

$$tan\, BPH = tan\, \alpha = \frac{x-a}{y};$$

hence

$$y^2 = a(x-a).$$

In order to find the moving centroid, let us suppose the point which comes in contact with P to lie at Q when the point which is guided along the straight AA' is at C. Now if we suppose the straight line QC to be attached to the moving system, we see that this line will take the position PA at the instant when Q coincides with P, so that we have $CQ = AP = x$. But the moving system has taken this position by leaving its initial location, which corresponds to the position CB of the guided straight line, and turning through the angle $CBA = \alpha$, thus bringing the guided straight line into the position AB. While the moving system was turning through the angle α, the straight line CQ attached to the system moved with it, so that we may conclude that CQ makes the angle α with both AP and CB. Now if x_1 and y_1 represent the co-ordinates of the point Q, we have

$$tan\, \alpha = \frac{y_1}{x_1} = \frac{y}{a}, \quad \text{or} \quad \frac{y^2}{a} = a\frac{y_1^2}{x_1^2},$$

also

$$CQ = AP = x = \sqrt{x_1^2 + y_1^2}\,; \text{ hence } x - a = \sqrt{x_1^2 + y_1^2} - a.$$

Now, since the fixed centroid is represented by the equation

$$\frac{y^2}{a} = (x-a),$$

we have for the moving centroid the equation:

$$a\frac{y_1^2}{x_1^2} = \sqrt{x_1^2 + y_1^2} - a.$$

When the motion just described takes place, the path of any point, E, F or G, of the straight line $A\,B$ will give certain curves $E\,E'$, $F\,F'$ or $G\,G'$, which may be employed for straight-line motions. These curves can be easily constructed by points, by laying off from the guided point A on a sufficient number of positions of the straight line $A\,B$, a distance (say $A\,E = e$) equal to the distance of the describing point (E) from the guided point A. We see from this that these curves, whatever the position of the describing point E, will approach more and more nearly to the straight line $A\,A'$ without ever reaching it, and that these curves turn their concave sides to the guiding line $A\,A'$ at their vertices $E_0\,F_0\,G_0$. If one of these vertices, as G_0, lies beyond the fixed point B (the distance being measured from the guiding line $A\,A'$), *i.e.*, when e is assumed greater than a, this point will pass through the fixed point B twice during its motion, and will therefore describe the loop-shaped curve G. These curves are called *conchoids.*

We can obtain the equations of these curves in the following manner. Let $x = E\,L$ and $y = E\,N$ be the co-ordinates of any point E at the distance $A\,E = e$, and let the inclination of the guided line to the axis $C\,B$ be $A\,B\,C = \alpha$, we shall then have:

$$y = (a - x)\ tan\ \alpha,$$

and since

$$tan\ \alpha = \frac{A\,L}{L\,E} = \frac{\sqrt{e^2 - x^2}}{x},$$

we have for the equation of the conchoid:

$$y\,x = (a - x)\sqrt{e^2 - x^2}.$$

This equation holds for all points E, F and G, and the three cases represented by these points differ in this, that the distance e of the point E is positive and smaller than a, the distance e of F is negative and the distance e of G is positive and greater than a. For $x = e$ we get for all the curves $y = 0$, which corresponds to the vertex of the curve; for every value of x numerically smaller than e we get two equal real and opposite values for y, from which follows that the curves are symmetrical to the axis CB. If x is taken greater than e, we have an imaginary square root and therefore no point of the curve. Hence x can only vary between 0 and e. If in an equation for the curve G, *i.e.*, when $e > a$, we substitute $x = a$, we shall get $y = 0$; *i.e.*, all conchoids of the third kind for which $e > a$ form loops which intersect at the fixed point B. When $e = a$ the loop reduces to the point B.

The present case may be discussed in the same way as the elliptical guide, *i.e.*, we can obtain a straight-line motion along the line AA' for the point A of the beam AB if we guide any one point E, F or G of the beam in its own conchoid, and at the same time compel the beam to pass through the fixed point B; hence it will suffice for most practical cases to replace a portion of the conchoid extending to both sides of the vertex by a circular arc having three points in common with the exact conchoid. For this we may employ the vertex of the curve and the two extreme points corresponding to the greatest angle of deviation of the beam on each side of CB. This arc agrees sufficiently well with the conchoid, as long as the angle of deviation α is not too great, say not more than 20° to 25°. As the deviation of the beam on both sides of the axis CB is the same, the centre of the circular approximation will fall into the line CB and will serve as the axis of the radius bar. Therefore if we know the greatest angle of deviation $\alpha = ABC$ of the beam, and the distance $e = AE$, AF or AG of the point guided by the radius bar, we can find the axis M of the latter by finding the centre of the circle passing through the three points EE_0E', FF_0F' or GG_0G' respectively. To do this, for example for the point E, it is only necessary to draw the chord EE_0 and erect a perpendicular at its middle point;

the intersection of this perpendicular with the axis BC will give the required centre M.

On the other hand, if, instead of knowing the position of the point E guided by the radius bar, we know the centre M by its distance $CM = c$ from the straight path AA', we may find the position of E by laying off from A on the outermost position of the beam, the distance $AM' = CM = c$, and erecting a perpendicular at the middle M_0 of the line MM'; this perpendicular will intersect the beam at the point E. That this construction is correct will be evident if we suppose the radius bar turned about E till it coincides in direction with the beam AB, for then the radius bar will project beyond the beam a distance $CM = c$.

The required length $r = ME$ of the radius bar can be easily determined by calculation. The equation of a circle whose centre is at M and whose radius $r = ME_0 = c + e$ when the origin of coordinates is at C, is as follows:

$$(x - c)^2 + y^2 = (e - c)^2,$$

or

$$x^2 - 2cx + y^2 = e^2 - 2ce.$$

If this circle is to pass through the point E, the co-ordinates

$$x = e \cos \alpha, \quad \text{and} \quad y = a \tan \alpha - e \sin \alpha \text{ (see figure)}$$

of the latter must satisfy the equation of the circle, that is:

$$e^2 \cos^2 \alpha - 2ce \cos \alpha + a^2 \tan^2 \alpha - 2ae \tan \alpha \sin \alpha + e^2 \sin^2 \alpha = e^2 - 2ce;$$

or after reduction

$$c = \frac{2ae \tan \alpha \sin \alpha - a^2 \tan^2 \alpha}{2e(1 - \cos \alpha)}.$$

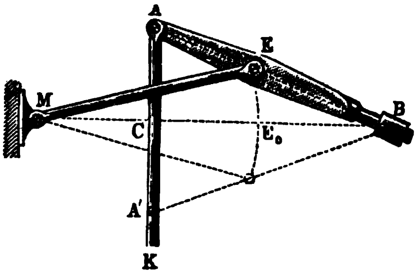
If in this we substitute $2a \tan \alpha = s$, where s is the stroke of the piston, we shall have

$$c = \frac{4se \sin \alpha - s^2}{8e(1 - \cos \alpha)}, \quad \text{and} \quad r = e - c.$$

The centre M of the radius bar falls on the straight line AA' when $c = 0$, *i.e.*, when $a^2 \tan^2 \alpha = 2\,a\,e \tan \alpha \sin \alpha$, or $e = \dfrac{a}{2 \cos \alpha}$.

According as the position of the point guided by the radius bar lies between A and B, beyond A or beyond B, we may, as in the case of elliptical guides, distinguish three kinds of conchoidal guides represented by the figures 376, 377 and 378. The oldest of these is shown in Fig. 378, and is due to *Reichenbach*. The straight-line motion obtained by its means is said to be particularly accurate when MG or r is taken equal to MB or $c - a$, *i.e.*, $c = \dfrac{a + e}{2}$. If we place this value equal to the above expression for c we shall get

FIG. 376.

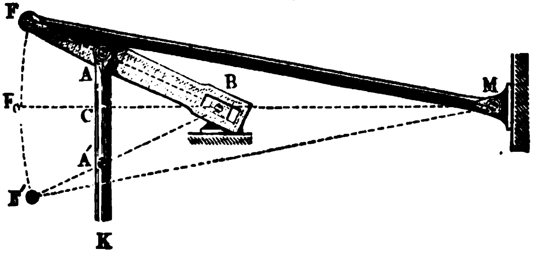


$$\frac{a + e}{2} = \frac{2\,a\,e \tan \alpha \sin \alpha - a^2 \tan^2 \alpha}{2\,e\,(1 - \cos \alpha)},$$

as the equation of condition for the ratio of $a : e$.

The second kind of guides, Fig. 377, is but little used, because the approximately straight path which it describes is short. The first kind of guide, Fig. 376, is better suited for practical construction than the second, provided the radius bar r is not taken too short, *i.e.*, provided the point E is not assumed too far

FIG. 377.



from A. For the case in which the axis of the radius bar falls on the line $A\,A'$, *i.e.*, at C, we have, as was before mentioned:

$$c = \frac{2\,a\,e\,tan\,\alpha\,sin\,\alpha - a^2\,tan^2\,\alpha}{2\,e\,(1 - cos\,\alpha)} = 0:$$

from which follows $e = \frac{a}{2\,cos\,\alpha}$, in other words, the point E guided by the radius bar lies half way between A and B, as in *Evans*' straight-line motion, when the beam is in its outermost position. If E is assumed nearer to A, Fig. 376, M moves away from C, for c then becomes negative; if we assume a larger value for e the axis M will fall between C and B, but such an arrangement is not to be recommended.

FIG. 378.

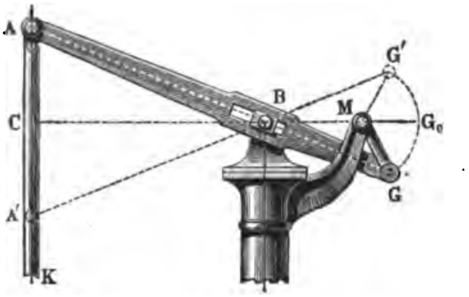


FIG. 379.

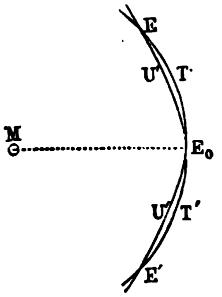
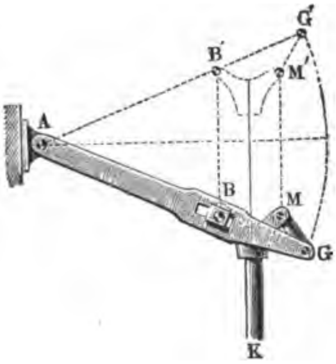


FIG. 380.

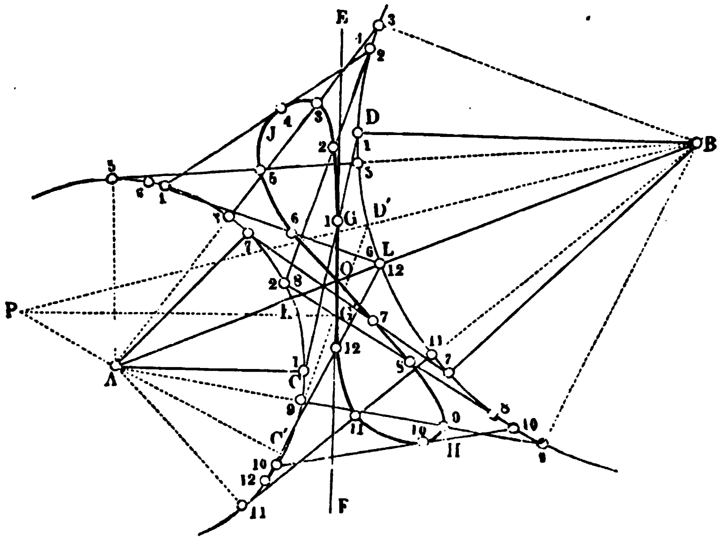


As regards the arrangement of the radius bar, remarks similar to those made in connection with the elliptical guides are again applicable. For if we substitute a circular arc for the conchoidal arc, the two arcs having the middle point and the two extreme points in common, Fig. 379, the circular arc will deviate to the same side of the conchoid both at T and at T'. Consequently if the circular arc were passed through, say the middle of the deviations UT and $U'T'$, the arc would cut the conchoid in four points and would deviate slightly to both sides of the latter.

It is evident that the conchoidal guides may be inverted, the point A being brought to rest by the additional motion $-s$ and the guided piston rod united to the bearings of B and M. In this way the arrangement shown in Fig. 380 may be obtained from *Reichenbach's* straight-line motion, the beam AG being double in order that the piston rod K and its cross-head BM may work in the intermediate space.

§ 102. Lemniscate Guides and Linkages for Producing Rectilinear Motion.—We may likewise find an approximate parallel motion by discussing a certain motion of a straight line. For if we suppose a straight line or rod CD, Fig. 381, to be so moved that two of its points C and D move on two fixed circles having centres A and B, say by uniting the points C and D by radial arms turning about A and B respectively, then will any point G of this straight line

FIG. 381.

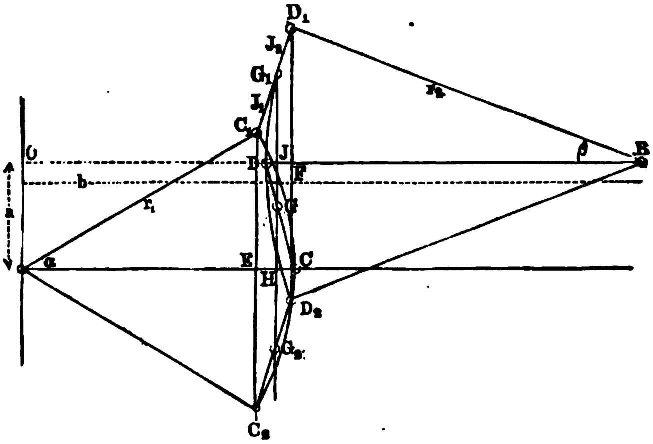


describe a peculiar looped line *G G′ H* J having a resemblance to the figure 8, and known in geometry as a lemniscate. This curve is a line of the fourth degree ;· the development of its equation is a long operation, and can be avoided for the present purpose, for only a portion of the curve is needed to produce this straight-line motion. In the figure, successive positions of the straight line *C D* are represented by 1, 1, 2, 2 . . . 12, 12. If we join the centres of the guiding circles by the straight line *A B*, we see that this line must be an axis of symmetry of the curve, because for every position of the straight line *C D*, for example that shown in 1, 1, there is another 7, 7, which with the first is symmetrical with respect to the line of centres, *i.e.*, so situated that the radii drawn from *A* and *B* to the points 1 and 7 make equal angles on opposite sides of the line of centres *A B*. Of course the two corresponding positions of the describing point *G* are also symmetrically situated with respect to *A B*, the positions of *G* being represented in the figure by the same numbers (in this case by 1 and 7). It is also plain that during every complete period of the motion, the describing point *G*, whatever its position between *C* and *D*, must pass twice through the line of centres, crossing the latter on that portion *K L* included between the two circumferences, so long as the point *G* lies between *A* and *B*. This assumption that *G* lies between *C* and *D* will be maintained for the present. From the symmetrical form of the curve with respect to the line of centres, it follows that the two passages of the describing point *G* across the line of centres must take place at one and the same point *O*. Consequently every lemniscate must have a double point at which the curve intersects itself, all the double points *O* of lemniscates described by the different points *G* lying upon the line of centres *A B*. When the describing point *G* coincides with either of the guided points *C* or *D*, the lemniscate reduces to the corresponding circular arc, both branches of the curve coinciding and the double points lying at intersections *K* and *L* of the line of centres with these circles. These points *K* and *L* are twice traversed by *C* and *D*, while the straight line *C D* is passing through positions which are symmetrical. These positions for the point *L* are represented by 6, 6 and 12, 12. The figure shows moreover that the extreme points of the arcs described by *C* and *D*, namely, 5, 11, 3 and 9, are obtained when the straight line *C D* coincides in direction with the opposite radial arm, *i.e.*, with *B* 5, *B* 11, *A* 3 and *A* 9 respectively.

That position of the guided straight line *C D* in which the two radial arms *A C* and *B D* are parallel is of particular interest. These positions are represented in the figure by 1, 1 and 7, 7. An examination of the position 1, 1 shows that at this instant the straight line *C D* has an infinitesimal motion about an instantaneous centre infinitely distant in the direction of the radii *A* 1 and *B* 1, for according to § 7 of the Introduction, the instantaneous centre for the motion of the rod is found at the intersection of the normals to the paths of the guided points *C* and *D*. Now, a rotation about an infinitely distant point lying in the direction of the parallel radii is equivalent to a rectilinear motion in a direction *E F* perpendicular to the radii, and we may therefore conclude that at the instant when the radial arms are parallel, all points of the rod *C D* describe paths which have one and the same direction *E F* perpendicular to the radial arms. The tangent *E F* is drawn for the point *G* in its position 1. It should be noticed that *four* different parallel tangents may be drawn to the path of any point, and parallel to a given direction, for example, to the direction *E F*, for the lemniscate is of the fourth degree. We can easily find the position of these points graphically, if we draw the corresponding positions of the instantaneous centres. For example, the point *G'* of the path drawn is such that its tangent is parallel to *E F* when *G'*, which corresponds to the position *A C' D' B* of the system, lies in such a position that the instantaneous radius *P G'* becomes perpendicular to the direction *E F*. The instantaneous centre *P* is found at the intersection of the two radial arms *A C'* and *B D'*, and it is evident from what has been said, that at the instant in question, the point *G'* describes the element of a path perpendicular to the radius *P G'* and therefore parallel to *E F*. In like manner we could find two other points of the curve (in the vicinity of 5 and 9) whose tangents are parallel to $E F$, but it is unnecessary, since only that portion of the curve lying near the double point O is used for the straight-line motion.

A closer examination of the lemniscate shows that a portion of the curve on both sides of the point G differs but little from the straight tangent at this point, so long as the angles of deviation in both directions of the radial arms $A C$ and $B D$ are not large. Therefore, if we regard the parallel positions of the radial arms as their middle positions, we may with sufficient accuracy for practical purposes regard the path of the point G as a rectilinear one, and may unite a piston rod to the rod $C D$ at this point G. Until now the point G has been arbitrarily assumed upon $C D$, but it is evident that the paths of all the points of $C D$ do not lie equally close to a straight line parallel to $E F$, and that these paths approximate more closely to the circular arcs described about A and B respectively, the nearer the point G is to the guided point C or D. In order, therefore, to find the most suitable position for G, *i.e.*, to find the position for which the deviations of the actual path from the ideal rectilinear one are as small as possible, we must add one more condition. The condition generally chosen is that *the rectilinear path along which the end of the piston rod is guided, must not only pass through the middle position of the point G, but also through its two extreme positions.* This is the same condition to which the elliptical and conchoidal guides are usually subjected. This condition renders the position of the point G between C and D no longer indeterminate, and also prescribes a certain position of the points A and B relatively to each other—in other words, a certain ratio of the lengths of two radial arms $A C$ and $B D$, as will be shown in what follows. Let B, Fig. 382, be the axis of one radial arm and $B D$ its middle position, so that the extreme positions $B D_1$ and $B D_2$ form the equal angles $D_1 B D = D_2 B D = \beta$ with the middle position $B D$. Draw perpendicularly to $B D$ through a point J, arbitrarily chosen for the present, the direction of the stroke $G_1 G_2$, and at the points D_1, D and D_2 lay off the link $D C$ so that its point G will fall on the direction of the stroke; as the distances of G from C and D are represented respectively by $CG = l_1$ and $DG = l_2$, we have $D_1 G_1 = D G = D_2 G_2 = l_2$. This gives us the three positions C_1, C and C_2 of the other

FIG. 382.



end of the link $D C$, and through these three points C_1, C and C_2 we can pass a circle whose centre A is the axis for the second radial arm $A C$. It is required of this second radial arm that $A C$ be its middle position, and that it be parallel to $B D$. It is clear that when this is the case the chord $C_1 C_2$ will be perpendicular to $A C$, and therefore parallel to the chord $D_1 D_2$ of the first radial arm, and also parallel to the direction of the stroke $G_1 G_2$. Now, since $D_1 G_1 = D_2 G_2 = l_2$, the extreme positions $D_1 C_1$ and $D_2 C_2$ of the link $D C$ must be parallel, and it follows from this that $D_1 C_1 C_2 D_2$ is a parallelogram, and that the *chords of the two arcs are equal to each other and to the piston stroke s, i.e.*, $C_1 C_2 = D_1 D_2 = G_1 G_2 = s$. If, therefore, α represents the angle of deviation $C_1 A C = C_2 A C$ of the second radial arm and r_1 and r_2 the lengths of the radial arms, we shall have

$$r_1 \sin \alpha = r_2 \sin \beta = \frac{s}{2}.$$

Let a and b be the co-ordinates of the centres A and B referred to a system of rectangular co-ordinates whose origin is at O and whose axes pass through A and B, and let γ_1, γ_0 and γ_2 be the angles made by the link with OB, the axis of X, when the link is in positions C_1D_1, CD and C_2D_2, then if we make $l_1 + l_2 = l$ we shall have from the figure:

$$OA = a = l \sin \gamma_0;$$

also

$$a = EC_1 + l \sin \gamma_1 - D_1F = l \sin \gamma_1.$$

From this immediately follows that $\gamma_0 = \gamma_1$, i.e., $l_2 \cos \gamma_0 = l_2 \cos \gamma_1$, or $DJ = JF$. The assumed arrangement is therefore possible only when the direction of the stroke passes through the middle of the rise FD of the arc D_1DD_2, and in this case the rise EC of the arc described by the other radial arm is also bisected at H by the direction of the stroke, for we have

$$l_1 \cos \gamma_0 = l_1 \cos \gamma_1 = CH = HE.$$

For the rises $EC = e_1$ and $FD = e_2$ we have:

$$e_1 = r_1(1 - \cos \alpha); \quad e_2 = r_2(1 - \cos \beta),$$

and

$$\frac{e_1}{e_2} = \frac{l_1}{l_2}.$$

From the condition governing the relative position of the two radial arms we may also find a perfectly determinate relation between their lengths r_1 and r_2. According to a well-known geometrical property of the circle, we have:

$$C_1E^2 = EC(2r_1 - EC);$$

or since

$$C_1E = \frac{s}{2} = r_1 \sin \alpha = r_2 \sin \beta,$$

we have

$$(r_2 \sin \beta)^2 = e_1(2r_1 - e_1).$$

Since, moreover,

$$e_1 = \frac{l_1}{l_2} e_2 = \frac{l_1}{l_2} r_2 (1 - \cos\beta),$$

it follows that

$$r_1 = \frac{(r_2 \sin\beta)^2 + e_1^2}{2 e_1} = \frac{r_2^2 (1 - \cos^2\beta) + \left(\frac{l_1}{l_2}\right)^2 r_2^2 (1 - \cos\beta)^2}{2 \frac{l_1}{l_2} r_2 (1 - \cos\beta)}$$

$$= r_2 \frac{\frac{l_2}{l_1}(1 + \cos\beta) + \frac{l_1}{l_2}(1 - \cos\beta)}{2} = r_2 \left(\frac{l_2}{l_1} \cos^2\frac{\beta}{2} + \frac{l_1}{l_2} \sin^2\frac{\beta}{2}\right).$$

By means of this equation any one of the four quantities r_1, r_2, β and $\frac{l_1}{l_2}$ may be determined when the other three are known. For example, if r_2, β and r_1 are given, we can find the ratio $\frac{l_1}{l_2} = n$ of the distances on the link from

$$\frac{r_1}{r_2} = \frac{1}{n} \cos^2\frac{\beta}{2} + n \sin^2\frac{\beta}{2}$$

by means of

$$n^2 - \frac{n}{\sin^2\frac{\beta}{2}} \cdot \frac{r_1}{r_2} + \cot^2\frac{\beta}{2} = 0,$$

or

$$n = \frac{l_1}{l_2} = \frac{r_1}{2 r_2 \sin^2\frac{\beta}{2}} - \sqrt{\frac{r_1^2}{4 r_2^2 \sin^4\frac{\beta}{2}} - \cot^2\frac{\beta}{2}}$$

$$= \frac{\frac{r_1}{r_2} - \sqrt{\left(\frac{r_1}{r_2}\right)^2 - \sin^2\beta}}{2 \sin^2\frac{\beta}{2}}.$$

When, as is generally the case, the stroke s is given instead of the angle β, we place

$$\cos\beta = \frac{\sqrt{r_2^2 - \frac{s^2}{4}}}{r_2} = \sqrt{1 - \frac{s^2}{4r_2^2}},$$

and get

$$r_1 = \frac{r_2}{2}\left[\frac{l_2}{l_1}\left(1 + \sqrt{1 - \frac{s^2}{4r_2^2}}\right) + \frac{l_1}{l_2}\left(1 - \sqrt{1 - \frac{s^2}{4r_2^2}}\right)\right].$$

In like manner when s, β and $\frac{l_1}{l_2}$ are given, we find the radii

$$r_2 = \frac{s}{2\sin\beta}$$

and

$$r_1 = \frac{s}{2\sin\beta}\left(\frac{l_2}{l_1}\cos^2\frac{\beta}{2} + \frac{l_1}{l_2}\sin^2\frac{\beta}{2}\right).$$

For the case in which $r_1 = r_2$ we obtain $\alpha = \beta$ from

$s = 2r_1 \sin\alpha = 2r_2 \sin\beta$, also $e_1 = e_2$, and therefore $l_1 = l_2$.

This last arrangement is the most useful, because, as will be shown later, it gives the closest approximation to the exact rectilinear path. We easily see from Fig. 381 that with this arrangement the middle position of the point G falls on the line of centres $A\ B$ and coincides with the double point O of the lemniscate, which is impossible when the radial arms have unequal lengths r_1 and r_2. When the angle of deviation β is small, we can assume that $\sin^2\frac{\beta}{2} = 0$ and $\cos^2\frac{\beta}{2} = 1$ approximately, and then get:

$$\frac{r_1}{r_2} = \frac{l_1}{l_2},$$

i.e., the lengths of the radius rods vary inversely as the corresponding distances on the link, when the angle of oscillation is small.

The co-ordinates of the centres *A* and *B* may be easily found from

$$a = \sqrt{(l_1 + l_2)^2 - \tfrac{1}{4}(e_1 + e_2)^2} = (1 + n)\sqrt{l_2^2 - \tfrac{1}{4} r_2^2 (1 - \cos\beta)^2}$$

where n is the ratio $\frac{l_1}{l_2}$, and

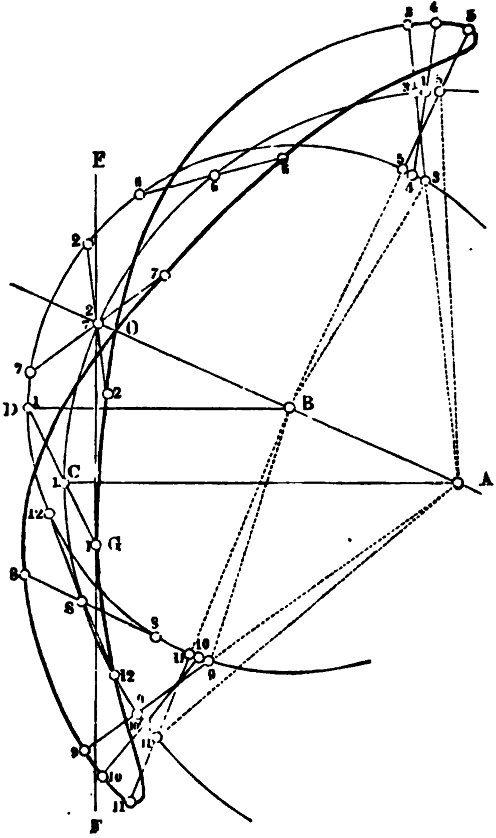
$$b = r_1 + r_2 - \tfrac{1}{2}(e_1 + e_2) = r_1 + r_2 - \tfrac{1}{2}(1 + n)\, r_2 (1 - \cos\beta).$$

Although, as has already been stated, the greatest accuracy in the straight-line motion is attainable when the two radius rods are of equal length, it is sometimes necessary to deviate from this condition, and sometimes considerations of economy of space require that both centres should lie on the same side of the piston rod that is to be guided, in which case all consideration of equality of length in the radius rods is excluded. For such an arrangement of the radius rods the same considerations obtain as in the arrangement hitherto discussed, as may be seen from the following.

Let *A C* and *B D*, Fig. 383, be two radius rods whose centres are at *A* and *B* and whose ends *C* and *D* are again connected by the link *C D*, then will any point of the latter likewise describe a lemniscate, and to it apply remarks similar to those made with reference to the case of Fig. 381. The line of centres *A B* is again an axis of symmetry for every curve whose double point *O* falls upon it (*A B*), also every point moves in a direction perpendicular to the radius rods at the instant when the latter are parallel. Therefore, if we again start with the supposition that the stroke of the piston rod is perpendicular to the parallel position of the radius rods, and again require that the condition be fulfilled that the point *G* in its highest, middle and lowest position shall fall in the direction of the stroke, then by a construction, Fig. 384, perfectly analogous to that employed in Fig. 382, we can find the position of the point *A* and the length *A C* when

$B\ D$, the stroke $s = D_1 D_2$ and the distance $D_1 G_1 = D\ G = D_2 G_2 = l_2$ of the guided point G from D are given. The present case differs from the preceding one in that the guided point G no longer lies between the points C and D, but outside of $C\ D$; for if it did not lie outside the two centres A and B could not lie on the same side of the required straight line.

FIG. 383.



Further investigation shows that here also the chords of the arcs described by the radius rods are equal to each other and to the stroke:

$$D_1 D_2 = C_1 C_2$$
$$= 2\, r_1 \sin \alpha$$
$$= 2\, r_2 \sin \beta = s,$$

and that the rises of these arcs $C\ E = e_1$ and $D F = e_2$ are bisected at H and J by the direction $G_1\ G_2$ of the stroke. If, as before, we represent the distances of the guided point G from C and D by $G\ C = l_1$ and $G\ D = l_2$ we shall also have the relation:

$$\frac{e_1}{e_2} = \frac{l_1}{l_2}.$$

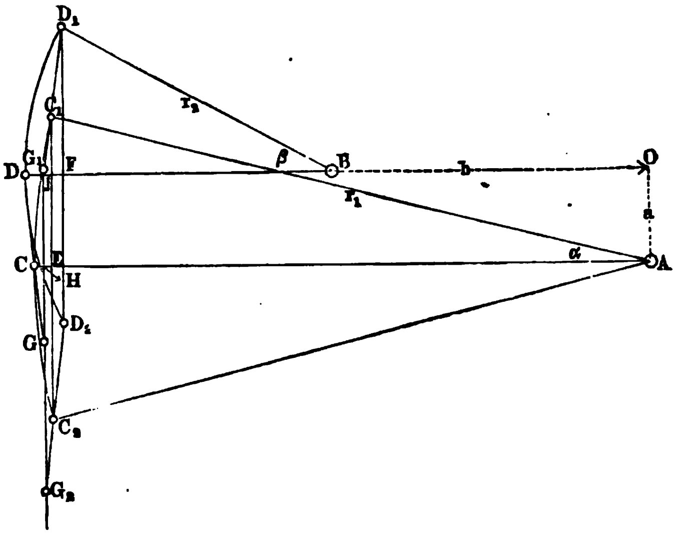
Finally, we obtain exactly as before from

$$C_1\ E^2 = E\ C(2r_1 - E\ C)$$

the equation

$$r_1 = r_2\left(\frac{l_2}{l_1}\cos^2\frac{\beta}{2} + \frac{l_1}{l_2}\sin^2\frac{\beta}{2}\right).$$

FIG. 384.



For the distance of the centres we have here:

$$a = \sqrt{(l_2 - l_1)^2 - \tfrac{1}{4}(e_2 - e_1)^2}$$

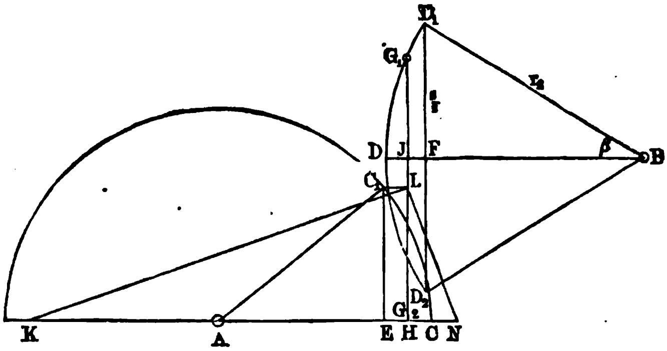
$$= (1 - n)\sqrt{l_2^2 - \tfrac{1}{4} r_2^2(1 - \cos\beta)^2},$$

$$b = r_1 - r_2 + \tfrac{1}{2}(e_2 - e_1) = r_1 - r_2 + \tfrac{1}{2}(1 - n)\, r_2(1 - \cos\beta).$$

By means of this equation, as was shown above, we may find any one of the four quantities r_1, r_2, β, and $\frac{l_1}{l_2}$, when the other three are given, the calculation offering no difficulties when the stroke s is given instead of r or β. From what has already been said, it is evident how the radius r_1 may be found graphically when r_2 β, and l_2 are given. In an equally simple manner the unknown quantities may be found when any three other quantities, independent of each other, are given.

For example, if besides r_2 and β we know the position of the centre A to B, Fig. 385, we have in $D_1 D_2$ the stroke s and in DF the chord e_2. Therefore if we draw through the

FIG. 385.



middle J of DF the direction $G_1 G_2$ of the stroke, it will evidently cut off from AC, which is parallel to BD, a portion $AH = r_1 - \frac{e_1}{2}$. Now, since

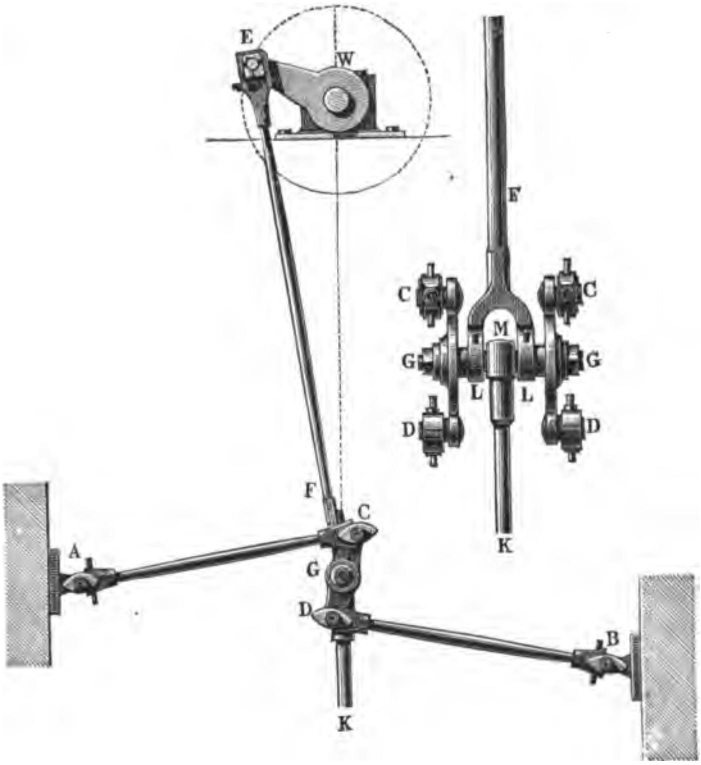
$$C_1E^2 = \left(\frac{s}{2}\right)^2 = (2r_1 - e_1)e_1 = 2\left(r_1 - \frac{e_1}{2}\right)e_1 = 2AH \, . \, e_1,$$

we can easily construct $c_1 = \frac{C_1E^2}{2AH}$ by making $HL = FD_1 = \frac{s}{2}$, then laying off $AK = HA$ and erecting a perpendicular LN to KL at L, we shall have in HN the rise c_1, and in $\frac{KN}{2}$ the radius r_1, etc. In like manner the construction may be easily made when r_1, r_2, and β are given. For the chord of the sector whose radius is r_2 and angle at the centre 2β gives us immediately the stroke s, which when applied to a circle having the the radius r_1, gives the angle α and the rise c_1. Then if the two sectors be so arranged that the middle radii are parallel and the middle points of the rises lie on the same normal to the radii, all the relations will be determined.

The distance *a* of the parallel radius rods of course depends on the magnitude $l_2 + l_1$ or $l_2 - l_1$, according as the radius rods lie on opposite sides or on the same side of the stroke.

A simple lemniscate guide is shown in Fig. 386. The two equal radius rods *A C* and *B D* are united by a link *C D*, whose journals *G G* are connected to the piston rod *K* and the connecting rod *E F*. The latter serves to convert the rotary motion of the crank *W E* into a reciprocating sliding motion of the piston rod. From the right hand portion of Fig. 386, we can readily see the arrangement of the mechanism, and particularly how the forked end *L F L* of the con-

Fig. 386.



necting rod encloses the pin *G*, whose central portion is keyed to the piston rod *M K*. The link, as well as the radius rods, is double, thus avoiding one-sided action, and diminishing the resulting lateral motion of the rods.

§ 103. **Deviation from Exact Rectilinear Motion.**—The motion of the point which is guided by radius rods is not perfectly rectilinear, but the deviations which take place, when the proportions have been correctly chosen and the angle of oscillation of the radius rods is not assumed too large, are so insignificant that they may be neglected in practical constructions. This is the more allowable as the unavoidable inaccuracies occurring in the construction, and particularly in the setting up of the machinery, are usually much more considerable. From the form of the lemniscate, it is evident that the deviations from a straight line will be greater the greater the angle of oscillation of the radius rods, consequently the lengths of the radius rods are usually made so large in comparison with the stroke that the angles of deviation α and β remain small. For example, a rule often followed is to make the length of a radius rod not less than one and a half times the stroke, and this rule $r = 1.5\, s$ gives for the angle of deviation on each side of the middle position:

$$\sin \alpha = \frac{s}{2r} = \frac{1}{3}, \text{ or } \alpha = 19^\circ\, 30' \doteq \text{about } 20^\circ.$$

Now although with such proportions the deviations of the guided point from a rectilinear path are so slight that it is seldom necessary to even determine them, nevertheless the investigation of these deviations is of interest, because it enables us to recognize the influence exerted by the different elements of the construction on the magnitude of such deviations.

Let the axis *A*,* Fig. 387, be taken as the origin of a rectangular system of co-ordinates; let $A\,X$, the positive axis of X, coincide with the middle position $A\,C$ of the radius rod; and let $A\,Y$ be the direction of the positive axis of Y, so that

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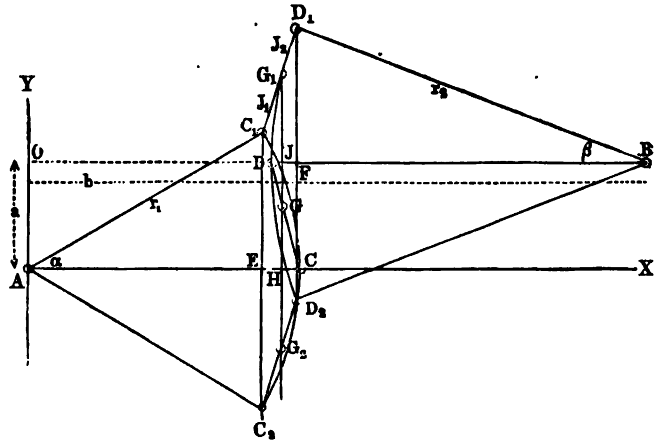
* In the following investigation use was made of an article by *J. Lueders*, Zeitschr. deutsch. Ing., 1860, p. 83.

Fig. 387.



the ordinates of B may be represented by a and b. Moreover, let α, γ, and β represent the angles made by the directions $A\,C$, $C\,D$, and $D\,B$ with the axis of X, so that these angles can be taken positive or negative according as the positions of the points C, D, and B deviate from the points A, C, and D respectively to the positive or negative side of the axis of Y, and let these angles in the highest, middle and lowest position be represented respectively by $\alpha_1\ \alpha_0\ \alpha_2$, $\gamma_1\ \gamma_0\ \gamma_2$, and $\beta_1\ \beta_0\ \beta_2$. We have accordingly, $\alpha_1 = -\alpha_2$; $\alpha_0 = 0$; $\beta_1 = \beta_2$; $\beta_0 = 0$; $\gamma_1 = \gamma_2$; $\gamma_0 + \gamma_1 = 180°$ (this last equation is true only when $C_1\,G_1 = G_1\,D_1$). The angle γ, so far as it is here to be considered, is always positive, and is but little greater or less than 90°. Let the distances of the guided point G from C and D be again represented by $l_1 = C\,G$ and $l_2 = G\,D$, also $l = l_2 \pm l_1 = C\,D$.

We now have for the co-ordinates x and y of any point of the lemniscate described by G the equations:

$$x = r_1 \cos \alpha + l_1 \cos \gamma \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = r_1 \sin \alpha + l_1 \sin \gamma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Moreover the arrangement of the whole system is represented by the equations:

$$r_1 \cos \alpha + r_2 \cos \beta = b - l \cos \gamma = B \quad . \quad . \quad . \quad (3)$$

and

$$r_1 \sin \alpha + r_2 \sin \beta = a - l \sin \gamma = A \quad . \quad . \quad . \quad (4)$$

representing $b - l \cos \gamma$ by B, and $a - l \sin \gamma$ by A.

For the distance of the line $G_1 G_2$ from A, we have from the preceding article:

$$x_0 = r_1 \cos \alpha_1 + \frac{e_1}{2} = r_1 \cos \alpha_1 + \frac{r_1}{2}(1 - \cos \alpha_1)$$

$$= \frac{r_1}{2}(1 + \cos \alpha_1).$$

If, therefore, f represents the error or deviation from the straight line $G_1 G_2$ of any point of the lemniscate whose coordinates are x and y, we have for this deviation,

$$f = x - x_0,$$

or, according to 1:

$$f = r_1 \cos \alpha + l_1 \cos \gamma - r_1 \frac{1 + \cos \alpha_1}{2} \quad . \quad . \quad . \quad (5)$$

It is only of interest to know the greatest deviation. The latter evidently obtains where the lemniscate has a tangent parallel to the direction of the stroke $G_1 G_2$, and consequently also parallel to the axis of Y. This position, as is well known, is given by:

$$\frac{\delta x}{\delta y} = 0.$$

This value of $\frac{\delta x}{\delta y}$ is obtained by differentiating 1 and 2:

$$\frac{\delta x}{\delta y} = - \frac{r_1 \sin \alpha \, \delta \alpha + l_1 \sin \gamma \, \delta \gamma}{r_1 \cos \alpha \, \delta \alpha + l_1 \cos \gamma \, \delta \gamma} = 0,$$

i.e., for the position of greatest deviation we have:

$$r_1 \sin\alpha\,\delta\alpha + l_1 \sin\gamma\,\delta\gamma = 0,$$

or

$$\frac{\delta\alpha}{\delta\gamma} = -\frac{l_1 \sin\gamma}{r_1 \sin\alpha}.$$

The general expression for $\frac{\delta\alpha}{\delta\gamma}$ is found by differentiating 3 and 4, thus getting:

$$-r_1 \sin\alpha\,\delta\alpha - r_2 \sin\beta\,\delta\beta = l \sin\gamma\,\delta\gamma,$$

and

$$r_1 \cos\alpha\,\delta\alpha + r_2 \cos\beta\,\delta\beta = -l \cos\gamma\,\delta\gamma.$$

Deducing two expressions for $\delta\beta$ from these equations and placing them equal, we get:

$$\delta\beta = \frac{-l \sin\gamma\,\delta\gamma - r_1 \sin\alpha\,\delta\alpha}{r_2 \sin\beta}$$

$$= \frac{-l \cos\gamma\,\delta\gamma - r_1 \cos\alpha\,\delta\alpha}{r_2 \cos\beta},$$

from which we obtain:

$$\delta\alpha \,.\, r_1\left(\frac{\cos\alpha}{\cos\beta} - \frac{\sin\alpha}{\sin\beta}\right) = \delta\gamma \,.\, l\left(\frac{\sin\gamma}{\sin\beta} - \frac{\cos\gamma}{\cos\beta}\right),$$

hence

$$\frac{\delta\alpha}{\delta\gamma} = \frac{l \sin(\gamma - \beta)}{r_1 \sin(\beta - \alpha)}.$$

This general expression for $\frac{\delta\alpha}{\delta\gamma}$ must now be placed equal to the expression calculated above for the greatest deviation, and we then have:

$$\frac{l \sin(\gamma - \beta)}{\sin(\beta - \alpha)} = -\frac{l_1 \sin\gamma}{\sin\alpha}.\text{*}$$

Expanding and dividing both sides by $\sin\alpha$, $\sin\beta$, $\sin\gamma$, we get:

$$l \cot\gamma = l_1 \cot\alpha + l_2 \cot\beta \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is now a question of expressing $\cot\alpha$ and $\cot\beta$ in terms of γ, for which purpose we employ equations 3 and 4, deducing from them:

$$\sin^2\beta + \cos^2\beta, \quad \text{and} \quad \sin^2\alpha + \text{co[illegible]} \cos\alpha_1)$$

and placing these expressions equal to 1. W[illegible]

$$(r_2 \sin\beta)^2 + (r_2 \cos\beta)^2 = r_2^2 = (A - r_1 \sin\alpha)^2 + (B - r_1 \cos\alpha)^2,$$

expanded this becomes:

$$2 r_1 (A \sin\alpha + B \cos\alpha) = A^2 + B^2 + r_1^2 - r_2^2 = p;$$

in like manner we get from

$$(r_1 \sin\alpha)^2 + (r_1 \cos\alpha)^2 = r_1^2 = (A - r_2 \sin\beta)^2 + (B - r_2 \cos\beta)^2,$$

after expansion:

$$2 r_2 (A \sin\beta + B \cos\beta) = A^2 + B^2 + r_2^2 - r_1^2 = q,$$

when for the sake of brevity we make

$$A^2 + B^2 + r_1^2 - r_2^2 = p,$$

and

$$A^2 + B^2 + r_2^2 - r_1^2 = q.$$

---

[*] This equation may easily be deduced from Fig. 381, and has a significance that is evident from the figure. For if G' is the point of the curve which has a vertical tangent, corresponding to the position $A\,C'D'B$ of the system, the instantaneous radius $P\,G'$ must be parallel to $B\,D$. Therefore, in the triangle $P\,C'D'$ we have the angle $C'P\,D' = \beta - \alpha$, and $P\,D'C' = \gamma - \beta$; moreover $P\,C' = l_1 \frac{\sin\gamma}{\sin\alpha}$, and $C'D' = l$, therefore $\frac{P\,C'}{C'\,D'} = \frac{l_1 \sin\gamma}{l \sin\alpha} = \frac{\sin(\gamma - \beta)}{\sin(\beta - \alpha)}$

If we substitute in these expressions

$$cos = \sqrt{1 - sin^2} \text{ and } sin = \sqrt{1 - cos^2},$$

we get by solving the quadratic equations:

$$sin\,\alpha = \frac{pA \pm B\sqrt{4r_1^2(A^2+B^2) - p^2}}{2r_1(A^2+B^2)} \quad . \; . \; . \quad (7)$$

$$cos\,\alpha = \frac{pB \mp A\sqrt{4r_1^2(A^2+B^2) - p^2}}{2r_1(A^2+B^2)} \quad . \; . \; . \quad (8)$$

$$sin\,\beta = \frac{qA \mp B\sqrt{4r_2^2(A^2+B^2) - q^2}}{2r_2(A^2+B^2)} \quad . \; . \; . \quad (9)$$

$$cos\,\beta = \frac{qB \pm A\sqrt{4r_2^2(A^2+B^2) - q^2}}{2r_2(A^2+B^2)} \quad . \; . \; . \quad (10)$$

hence

$$cot\,\alpha = \frac{pB \mp A\sqrt{4r_1^2(A^2+B^2) - p^2}}{pA \pm B\sqrt{4r_1^2(A^2+B^2) - p^2}} \quad . \; . \; . \quad (11)$$

$$cot\,\beta = \frac{qB \pm A\sqrt{4r_2^2(A^2+B^2) - q^2}}{qA \mp B\sqrt{4r_2^2(A^2+B^2) - q^2}} \quad . \; . \; . \quad (12)$$

These formulas give the angles α and β as functions of the angle γ formed by the connecting link with the axis of X, for the quantities A, B, p and q also contain γ. If we assume any angle for γ, these expressions will give two values for α and two for β, as is evident from the double sign before the radicals, and Fig. 387 shows how these values are related. The angle γ_1 made by the connecting link in its highest position $C_1 D_1$ is less than 90°, and becomes greater as it moves gradually downward, having in its middle position a value γ_0 greater than 90°, namely, $\gamma_0 = 180° - \gamma_1$, then diminishing until it again has the value $\gamma_2 = \gamma_1$ in its lowest position. During this stroke the angle γ twice assumes every value between γ and γ_0, once above and once below the middle position of the system. To every value of γ, therefore, there are two pairs of values, one for α and one for β, and these values of α and β may also be paired so as to correspond to the upper and lower positions of the system. In order that there may be no doubt about this latter pairing, we should notice that, in accordance with the supposition made at first concerning positive and negative angles, those values of α and β belonging together always have opposite signs. This explains the difference in the double signs before the radicals in the preceding formulas, and here we must remember that in all these expressions for α and β, the upper signs belong to one pair of values and the lower to the other pair. We can easily prove by calculation the correctness of the signs in the Equations from 7 to 12; for example, if we substitute in Equation 4 the values of *sin* α and *sin* β obtained from 7 and 9 we get

$$r_1 \sin \alpha + r_2 \sin \beta = A.$$

This is only the case when the radicals have opposite signs, for since the quantities under the radicals

$$4 r_1^2 (A^2 + B^2) - p^2, \quad \text{and} \quad 4 r_2^2 (A^2 + B^2) - q^2$$

are equal and disappear, we have

$$r_1 \sin \alpha + r_2 \sin \beta = \frac{pA + qA}{2(A^2 + B^2)} = A,$$

since $2(A^2 + B^2) = p + q$.

If we now substitute the values of *cot* α and *cot* β of 11 and 12 in the equation of condition for the maximum deviation given by No. 6,

$$l \cot \gamma = l_1 \cot \alpha + l_2 \cot \beta,$$

we shall get an expression containing besides known dimensions only γ, and from the resulting equation we can obtain the two values of γ corresponding to the two* greatest deviations. These values substituted in the Equations 7 and 8 will give the corresponding values of α' and α''; we then get from Equation 5 the maximum deviations f, and from 2 the corresponding ordinates y, *i.e.*, the heights at which these deviations occur. But such a calculation would be exceedingly long, and therefore devoid of all practical value. In every case, therefore, the desired result can be more easily obtained by an approximate method. For this it is only necessary to assume a probable value for γ, and by means of it find from 11 and 12 the quantities $\cot \alpha$ and $\cot \beta$, and then see whether the equation of condition, 6, is satisfied by the assumed value of γ and the corresponding calculated values of α and β. If this is not the case, it will be necessary to obtain a closer approximation by making a change in the assumption of γ. When γ has been found in this way, the determination of f offers no difficulties. This approximate calculation is rendered easier by the fact that the value of γ in the present case varies within very narrow limits, namely, between γ_1 and $\gamma_0 = 180° - \gamma_1$. From the figure it is evident that γ_1 is determined by

$$\cos \gamma_1 = \frac{1}{2} \frac{CE + DF}{CD} = \frac{1}{2} \frac{e_1 + e_2}{l}.$$

Now if we assume, as is generally done, $r = 1.5\, s$, we have $\sin \alpha_1 = 0.333$, $\alpha_1 = 19° \, 30'$, and therefore $e = r(1 - \cos \alpha_1) = 0.057\, r = 0.086\, s$. Moreover, the length l of the connecting link is not usually taken less than $\frac{4}{7} s$. If, therefore, we put $l = 0.6\, s$, and assume $r_1 = r_2$, $e_1 = e_2 = e$, we shall get for the outermost angle γ_1:

$$\cos \gamma_1 = \frac{e}{l} = \frac{0.086}{0.6} = 0.143,$$

---

* Since the lemniscate is a curve of high degree, we have to do, strictly speaking, with several maximum deviations, only two of which, however, are of interest in the present case, as only two occur in the portion of the curve under consideration.

or $\gamma_1 = 82° 50'$, and $\gamma_0 = 97° 10'$. The angles γ' and γ'', which correspond to the greatest deviations, therefore differ but little from 90°.

If we assume the frequently occurring case $r_1 = r_2 = r$, we have

$$p = q = A^2 + B^2;$$

we therefore get from 8 and 10:

$$\cos\alpha = \frac{B \mp A\sqrt{\frac{4r^2}{p} - 1}}{2r},$$

and

$$\cos\beta = \frac{B \pm A\sqrt{\frac{4r^2}{p} - 1}}{2r}.$$

As the direction of the stroke is in this case half way between the centres, we have $x_0 = \frac{b}{2}$, and $l_1 = l_2 = \frac{l}{2}$. Substituting these values in 5, we get for the deviation:

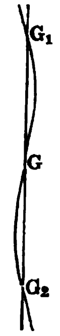
$$f = r\cos\alpha + l_1\cos\gamma - x_0 = \frac{B}{2} \mp \frac{A}{2}\sqrt{\frac{4r^2}{p} - 1} + l_1\cos\gamma - \frac{b}{2},$$

or, since $B = b - l\cos\gamma$, we have:

$$f = \mp\frac{A}{2}\sqrt{\frac{4r^2}{p} - 1} \quad \ldots\ldots \quad (13)$$

For equal radius rods, therefore, we have two equal and opposite deviations from the straight line, and it is evident that the middle point G must be a point of inflexion, for according to the preceding article the path of G is parallel to the stroke, the points G_1 and G_2 lie on this path, and the portions of the curve $G G_1$ and $G G_2$, Fig. 388, deviate to opposite sides of the straight line. It may also be noted that when the radius rods are equal, we have $l_1 = l_2$, and the middle position of G falls on the line of centres $A B$, so that the point of inflexion G is at the same time the double point of the lemniscate. Equation No. 6, for the case when $l_1 = l_2$, becomes $2 \cot \gamma = \cot \alpha + \cot \beta$.

Fig. 388.



Since the maximum deviation f takes place when the connecting link makes an angle γ with the axis of x, which differs but little from 90°, we can assume $\gamma = 90°$, and determine an approximate value for f from Equation 13. For this supposition gives us:

$$A = a - l \sin 90° = a - l,$$

$$B = b - l \cos 90° = b;$$

and since A is very small in comparison with B we can place

$$p = A^2 + B^2 = b^2.$$

Hence

$$f = \mp \frac{a - l}{2b} \sqrt{4r^2 - b^2}.$$

Now, according to Fig. 387, we have for the rise of the radius rod:

$$e^2 = l^2 - a^2 = (l + a)(l - a);$$

hence

$$a - l = - \frac{e^2}{l + a} = - \frac{e^2}{2l}$$

when we assume $l = a$ approximately. In like manner $e + b = 2r$; therefore $e^2 + 2eb = 4r^2 - b^2$, in which for the purpose of approximation we may neglect e^2; then substituting these values we obtain:

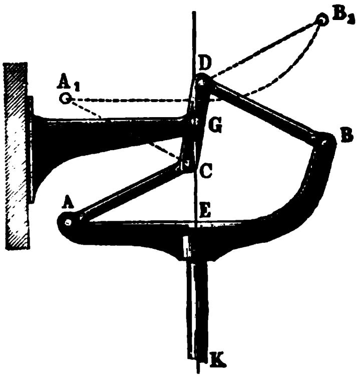
$$f = \pm \frac{e^2}{4lb} \sqrt{2be} = \pm \frac{e^2}{4l} \sqrt{\frac{2e}{b}},$$

and if we assume $b = 2r$ approximately, we get:

$$f = \pm \frac{e^2}{4l} \sqrt{\frac{e}{r}} \quad \ldots\ldots\ldots \quad (14)$$

This approximate formula holds, as was remarked above, only when the radius rods are equal, and does not correspond exactly to the maximum deviation, but to the deviation which occurs when the connecting link is vertical ($\gamma = 90^\circ$).

FIG. 389.



We may here, as in preceding straight-line motions, easily obtain another by inversion of the lemniscate guide. To do this, Fig. 389, we rigidly unite to the piston rod the journals of the two radius rods *A* and *B* by the yoke *A E B*. The point *G*, formerly guided in a straight line, is now replaced by two con-axial pins whose bearings are fixed to lateral supports so arranged that the piston rod with its cross-head *E* and the (double) radius rods *A C* can work freely between the bearings of *G*. Probably this kind of guide will be seldom used.

EXAMPLES. 1) If for a lemniscate guide with equal radius rods we assume the angle of deviation $\alpha_1 = \beta_1 = \pm 30^\circ$, and the length of the connecting link $l = l_1 + l_2 = 0.5\,s$, we have first of all $r_1 = r_2 = \frac{s}{2 \sin 30^\circ} = s$, and the rise $e = s(1 - \cos 30^\circ) = 0.134\,s$. Moreover, the horizontal distance between the two centres is $b = 2r - e = 1.866\,s$, and their vertical distance

$$a = \sqrt{l_2 - e_2} = s\sqrt{0.5^2 - 0.134^2} = 0.4817\,s.$$

According to the above approximate formula, the lateral deviation when the connecting link is vertical ($\gamma = 90^\circ$) is

$$f \pm = \frac{0.134^2\,s^2}{4 \times 0.5\,s}\sqrt{\frac{0.134\,s}{s}} = 0.00898\,\sqrt{0.134}\,s = 0.00329\,s,$$

therefore, when the stroke is one meter [39.37 ins.] the deviation is 3.29 millimeters [0.13 in.].

2) Let the length of the principal radius rod be taken $r_2 = \frac{3}{2}s$, then from

$sin\ \beta = \frac{s}{2r} = \frac{1}{3}$ we find the angle of deviation $\beta_1 = 19° 28' 17''$. The rise e_2 of this arc will consequently be $e_2 = r_2 (1 - cos\ \beta_1) = (1 - 0.9428) r_2 = 0.0572\ r_2$. If we now assume that the other rod has a radius $r_1 = \frac{2}{3} r_2 = s$, we shall get for the ratio $\frac{l_1}{l_2} = n$ of the distances on the connecting link:

$$\frac{l_1}{l_2} = n = \frac{\frac{2}{3} - \sqrt{\left(\frac{2}{3}\right)^2 - sin^2 19° 28' 17''}}{2\, sin^2 9° 44' 8''} = 1.563.$$

Therefore if we assume $l_2 = \frac{1}{4} s = \frac{1}{6} r_2$, we shall have $l_1 = n\, l_2 = 1.5627 \frac{r_2}{6}$ $= 0.2603\ r_2$, and hence, $l = l_1 + l_2 = 0.427\ r_2$.

For the horizontal and vertical distances of the centres when the latter lie on opposite sides of the stroke (Fig. 381) we have:

$$b = r_2 + 0.6667\ r_2 - (1 + 1.563) \frac{0.0572}{2} r_2 = (1.6667 - 0.0733) r_2 = 1.5934\ r_2$$

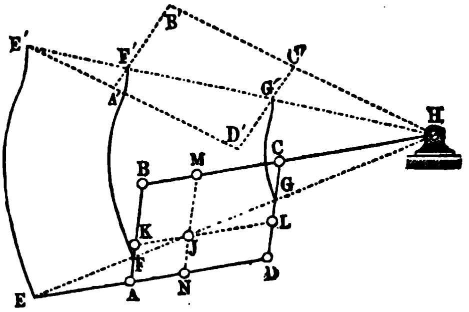
and

$$a = r_2 \sqrt{0.4270^2 - 0.0733^2} = 0{,}4206\ r_2.$$

§ 104. Pantographic Linkages for Duplicating Rectilinear Motions.

—In steam engines there is often a question of guiding several piston rods, namely, the steam piston rod and the pump rods. In such cases it is sufficient to give a straight-line motion to one of the rods by any one of the means discussed above, for it is very easy to communicate this motion to other rods by means of the *pantograph* mechanism. An idea of this mechanism may be obtained as follows: Let *A B C D*, Fig. 390 or Fig. 391, be a parallelogram formed of straight rods jointed at its four corners, and let us suppose it to be intersected by any transversal *F H*, which is not parallel to any of the sides. This transversal will

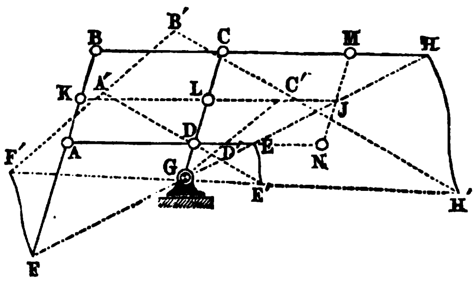
FIG. 390.



have four points of intersection E, F, G and H, which possess the remarkable property that, when any one of these points is fixed, so that the whole system can turn upon it, the paths of the other three points will be similar curves, having the fixed axis for the external or internal centre of similitude according as these points lie on the same side or opposite sides of the axis. It is immaterial whether the transversal cuts two sides of the parallelogram directly and the other two sides in their prolongations, as in Fig. 390, or whether, as in Fig. 391, all the sides of the parallelogram are intersected in their prolongations. The transversal may therefore also pass through a corner of the parallelogram.

The correctness of this assumption becomes evident when we remember that the parallelism of the opposite sides $A\ B$ and $D\ C$, and of $A\ D$ and $B\ C$ is maintained during the assumed motion about a point of the transversal, however great this motion may be. From this follows, in accordance with simple geometrical laws, that the distances of the stationary point from the other intersections of the transversal with the sides of the parallelogram have an invariable ratio to each other, which proves that the stationary point has the properties of a centre of similitude of the paths of the other points of intersection. For example, if we suppose the point G, Fig. 391, to be fixed, and the whole system $F A B C H E D$ to be brought into any position $F' A' B' C' H' E' D'$, we have, on account of the continued parallelism of the rods:

FIG. 391.



$$GE : GH = DE : CH = e : h$$

and

$$GE' : GH' = D'E' : C'H' = e : h.$$

In like manner we also have

$$GF : GH = CB : CH = b : h$$

and

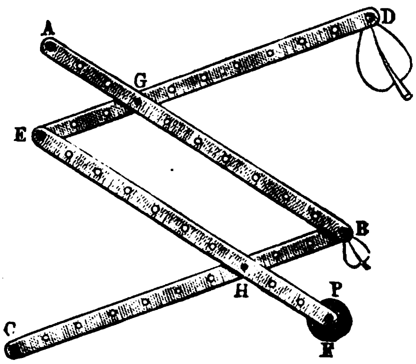
$$GF' : GH' = C'B' : C'H' = b : h.$$

While the point G acts as an external centre of similitude of the curves described by E and H, it also acts as an internal centre of similitude for the paths of F and H and for the paths of F and E. In Fig. 390 H is the external centre of similitude for the paths of all the other points of intersection.

It is also very evident that any other point J of this transversal can be made to describe curves which are similar to those already mentioned and have the same centre of similitude. For this it is sufficient to unite the point J with one pair of sides of the parallelogram by means of a link KLJ or MJN parallel to the other sides. That this remark is correct will be seen if we start with, say the parallelogram $KBCL$, for which the transversal gives the points of intersection F, G, J, H, etc.

On this property of the jointed parallelogram depends the instrument known as the ***pantograph***, which is often employed for copying drawings and models on a different scale. In its simplest form, as employed by copper engravers and others, it consists of four thin bars of metal or wood, AB and BC at B, Fig. 392, and DE and FE at E being jointed like a pair of compasses. These two pairs of bars are then united to form a jointed parallelogram by means of pins at G and H, so that its vertex B will lie on the straight line joining the ends F and D. This is easily

FIG. 392.

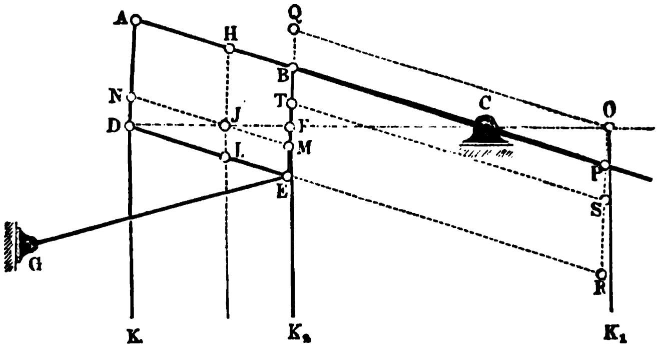


done by making $DG : DE = BG : FE$; for this purpose the bars are provided with a series of equidistant holes which are designated by numbers. The end F can turn about a pin projecting from a plate P fixed to the table, while D is a tracer and B a pencil or pricker. Now, if the point D is guided over a drawing it is plain that the pencil B will make a copy of the drawing, reducing it in the ratio $FH : FE$ or $BH : DE$. In *embroidering* machines this apparatus is employed, the point B being united to the frame in which the material to be embroidered is stretched, so that the needles moving up and down in fixed guides may all produce reduced copies of the pattern along the lines of which the tracer is guided by the workman. Finally, if we suppose a ball and socket joint to be placed at F permitting any motion of the apparatus about F, the latter may be employed for copying solid articles like busts, etc., the pencil at B being replaced by a cutting tool, say a circular cutter.

In the present case the problem is to copy rectilinear motions, and that this is possible is evident from what has preceded. If in the pantograph, Fig. 391, we suppose one of the moved points of the transversal, say E, to be guided in a straight line by any of the straight-line motions that have been described, all the other points H, J, F, etc., will traverse straight lines parallel to the path of E. It is also evident from the similarity of the figures that the strokes of all these points will be proportional to their distances from the fixed axis G. Moreover, if the path of the guided point E is only approximately straight, it of course follows that the paths of all the other guided points of the pantograph, as H, J and F, will describe approximate straight lines. It again follows from the similarity of the paths that the linear deviations from a straight line at the different points will also be proportional to the distances of these points from the fixed axis.

One of the best straight-line motions for large beam engines is the *Watt parallel motion*. This arrangement consists essentially of a combination of a lemniscate linkage with a parallelogram, and allows a straight-line motion to be given to several piston rods. For, if AC, Fig. 393, is an arm of the oscillating beam which receives at its end A the reciprocating rectilinear motion of the piston rod $D\ K$, the pair of radius rods $C\ B$ and $G\ E$ and the connecting link $B\ E$ will guide

FIG. 393.



the point F in a line which is approximately straight. This point is usually connected with the piston rod of the air pump, while the steam piston rod is connected with the fourth vertex D of the parallelogram constructed on $A\ B\ E$. From what has preceded it is evident that this parallelogram must be so chosen that the fourth vertex D will lie on a straight line passing through F and C. If, as in a *Woolf* engine with two cylinders, a second piston rod is to be guided, we can connect the latter with any point J of the transversal $C\ D$, provided J is carried by a rod like $H\ L$ parallel to the links, or by a rod like $M\ N$ parallel to the beam. If the rod to be guided lies beyond the centre of the beam, for example at $O\ K_1$, the point O of the transversal $D\ C$ may be guided in a straight line by connecting the rod $O\ P$ parallel to $B\ E$ with the beam, and the rod $O\ Q$ parallel to the beam with the produced back-link $E\ B$. But it is plain that instead of employing the links $O\ P$ and $O\ Q$ we can employ $O\ P\ R$ and $E\ R$ or $O\ P\ S$ and $S\ T$. Generally speaking, all that is necessary to give a straight-line motion to the point O is to constrain it to move so that the distance $C\ O$ will always have the same ratio to $C\ F$.

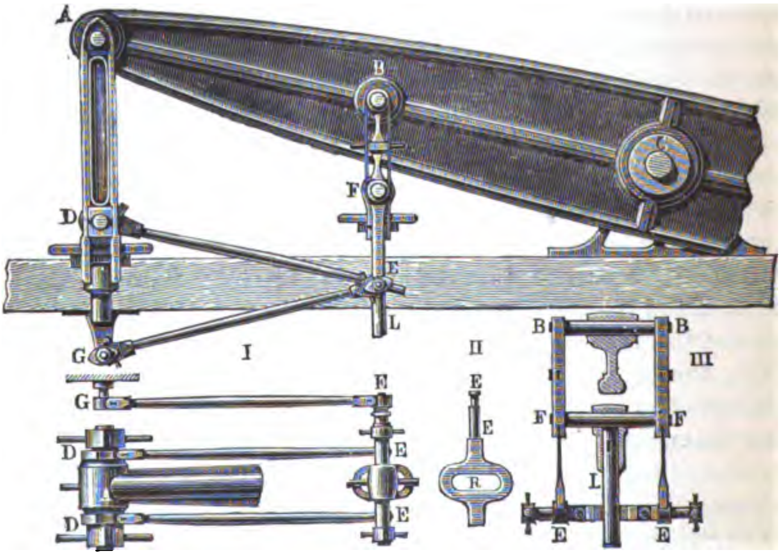
If s again represents the stroke of the rod at F and S the stroke of the rod at D we have

$$s : S = B\,C : A\,C = r : R,$$

the lengths of the arms $B\,C$ and $A\ C$ being represented by r and R. The deviation f of the rod at D is of course $f = \frac{R}{r} f_2$, where f_2 is the deviation of the rod $F\,K_2$. The mode of determining the length of the radius rod $G\,E$ and the ratio of the segments $E\,F$ and $B\,F$ of the back-link have already been discussed in the preceding article. We also see that the chord of the arc described by any point A, P, H of the beam is equal to the stroke of the rod which it moves, and that the stroke of this rod bisects the rise of the arc.

In *Watt's* parallel motion it is customary to make the

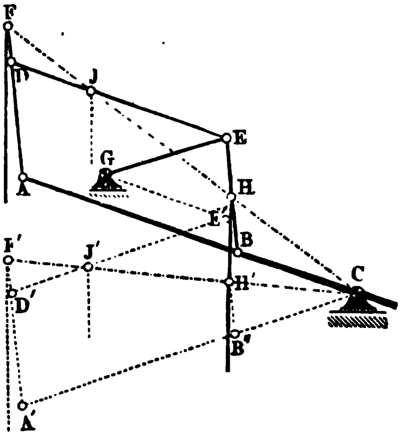
FIG. 394.



radius rods $C\,B$ and $G\,E$ of equal length, so that F lies at the centre of the back-link $E\,B$. Moreover, the arm $C\,A$ is usually made twice as large as *C B*, *i.e.*, the air pump at *F* is given only half the stroke of the steam piston at *D*. These assumptions correspond to the pantographic guide shown in Fig. 394. The cast-iron beam *C A* is supported at its centre by two bearings, one on each side of the beam, resting on frame work which at the same time serves as a support for the journals *G* of the double radius rod *G E*. To connect the latter with the beam the back-link *E B* is employed; this link, the main link *A D* and the parallel bar *D E* are all constructed double to exclude all lateral action. The piston rods *K* and *L* are keyed to the transverse axes *D D* and *F F*, the cross-bar *E* being provided at its centre with an elongated opening *R*, shown in II. and III., for allowing a free passage to the piston rod *L*.

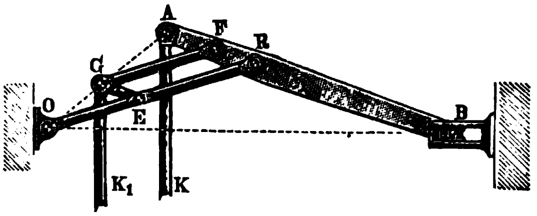
In marine engines the beam is often placed below, and is constructed double, so that it lies on both sides of the steam cylinder and air pump; an arrangement chosen because it is economical of space. The arrangement for this case is shown in Fig. 395. *C A* is the beam and *C B* and *G E* are the two radius rods which guide the point *H* of the back-link in a straight line. The parallelogram *A D E B* causes the points *F* and *J*, lying on the straight line passing through the points *C* and *H*, to move in straight lines; *F* can therefore be connected with the piston rod of the steam cylinder, *J* with the rod of the air pump and *H* with the rod of the feed pump.

Fig. 395.



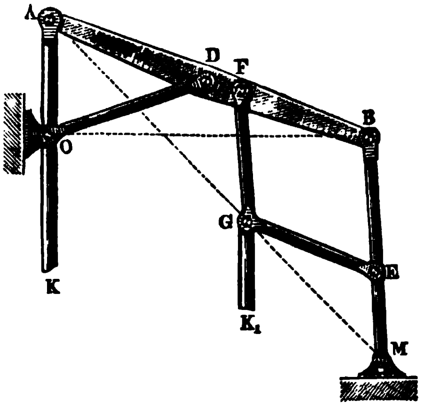
If a rod is guided by any other mechanism than the lemniscate linkage assumed in the preceding examples, the parallelogram may still be employed to communicate the existing rectilinear motion of one point to any number of others. Those straight-line motions are particularly suited to the application of the parallelogram which are provided with a lever oscillating about a fixed point, for the lever can be employed as the oscillating side of the parallelogram. This is the case with the lemniscate linkage where the beam forms one side and the connecting link another side of the parallelogram, so that only two sides are needed to complete it, namely, the main links and the parallel bars. Elliptical linkages are also suitable, for the oscillating beam and the radius bar swinging about a fixed point permit a simple arrangement of the parallelogram. The rule according to which the parallelogram may be designed follows readily from what has been said. If *A*, Fig. 396, is the point guided along a straight line by the elliptical linkage *O R A B*, it is only necessary to pass through this point *A* and the fixed axis *O* of the radius rod a straight line *A O* and then connect any desired point *G* of the latter by parallel links *G E* and *G F*, with the radius rod and with the beam. If the slide bars of the elliptical linkage are replaced by the swinging pillar *B M*, Fig. 397, the construction is in many cases still more convenient, if the centre *M* be taken as the centre of oscillation of the parallelogram *E B F G*. We may then draw the transversal *A M*, of which one point *G* is guided in a straight line by the parallelogram.

FIG. 396.



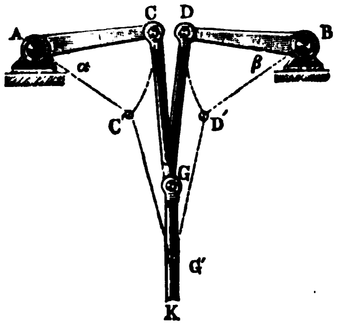
§ **105. Other Linkages for Producing Rectilinear Motion.**—We may now mention a few other mechanisms likewise consisting of a certain number of levers and links, by means of which an approximate rectilinear motion may be obtained. In the lemniscate linkage there were two rods oscillating about fixed axes whose relative motions were made to depend upon each other by the connecting link. Since the ends of the connecting link describe paths of different curvature, it is easy to infer that some third point will describe a path approximately without curvature, *i.e.*, rectilinear, constituting in a certain sense the transition between the oppositely, or at least unequally, curved paths of the ends of this link. Also in the approximate elliptical linkage with oscillating pillar there are two swinging levers whose free ends are united by an oscillating beam; and here, too, the problem is to find that point of the connecting link whose path is approximately a straight line. In solving the problem it is therefore principally a question of establishing the relative position of the different points.

FIG. 397.



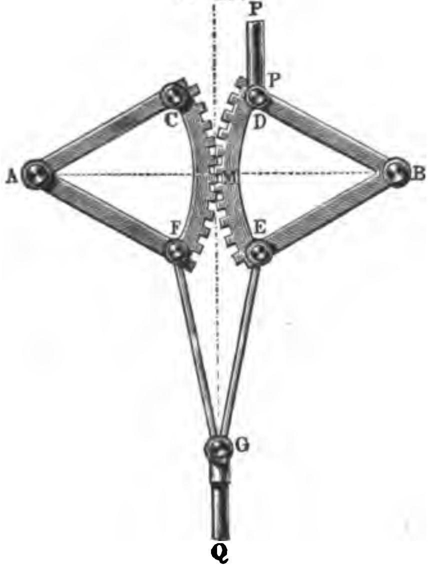
By another train of reasoning we may obtain a series of straight-line motions. Let us suppose that there are two levers *A C* and *B D*, Fig. 398, turning about fixed centres, the free ends of the levers being indirectly connected by two jointed rods *C G* and *D G*, instead of being directly connected by one rigid rod as heretofore. It is clear that thus far the motion of one of the links, *A C* for example, is not accompanied by a determinate motion of the other *B D*, for instance the latter may be held stationary and a certain amount of motion will still be possible for the lever *A C* and *vice versa*. If we suppose the connecting point *G*

FIG. 398.



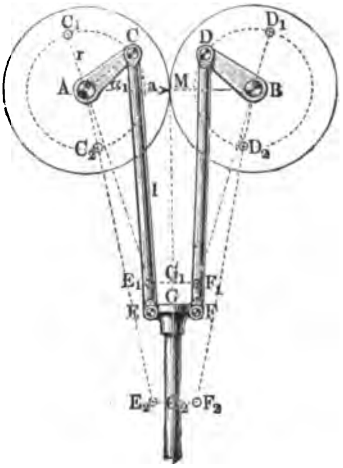
of the two links $G\,C$ and $G\,D$ to be given a certain rectilinear motion, say from G to G', the two levers will rotate through certain angles, say $C\,A\,C' = \alpha$ and $D\,B\,D' = \beta$, which may readily be found graphically, although the analytical determination is lengthy. Let us suppose that we know the relation of these rotations α and β for every instant of the rectilinear motion of the point G in the direction $G\,G'$. Then *vice versa* it is clear that we may also constrain the point G to have a rectilinear motion in the direction $G\,G'$ by giving to the levers at each instant the proper rotations α and β. Since the relative lengths and positions of the levers and links as well as the direction of the rectilinear motion of G may be assumed at pleasure, it follows that the arrangement just described is capable of an infinite number of variations, it being only necessary in each case to cause the levers $A\,C$ and $B\,D$ to turn at each instant with the proper relative motion. It is this latter condition which constitutes the difficulty of the construction, for we must notice that in general the ratio of the rotations of α and β for every instant of the motion is variable.

FIG 399.



There is but one case where the rotations during the whole motion of the point G have a constant ratio, the rotations being equal to each other and opposite in direction when the arrangement of the system is symmetrical with respect to the straight line $G\,G'$. In this case the two axes A and B, Fig. 399, can be made dependent on each other by means of two equal, cylindrical, toothed sectors $A\,C\,F$ and $B\,D\,E$. It is evident from the figure that in the motion of the system thus caused, the end G of the piston rod must move in the straight line MG perpendicular to the line of centres at M. The motion of the sectors may be communicated by a connecting rod PP acting at any point of one of the sectors. It is also clear that the arrangement may be, as in Fig. 400, such that the levers will have not a reciprocating but a continuous rotation. The length of stroke is readily determined in all these cases. For example, if $AC = BD = r$ is the length of the lever, $CE = DF = l$ of the links, $AM = BM = a$ half the distance between the axes, and $EG = FG = e$ the half distance between the joints of the links, since G_1 and G_2 are the extreme positions of G, we have:

FIG. 400.



$$s = MG_2 - MG_1 = \sqrt{(l+r)^2 - (a-e)^2} - \sqrt{(l-r)^2 - (a-e)^2}.$$

In like manner when the levers have an oscillating motion, the stroke may be found from the angle of oscillation. Let α_1 represent the angle $MAC = MBD$, *i.e.*, the angle of deviation of each lever above the line of centres, and for this position let γ_1 represent the inclination of the link CE to the direction of stroke MG, and γ_0 its inclination to the stroke when the lever is in its middle position or in the line of centres AB. We shall now find for the rise s_1 from the middle position corresponding to this angle α_1:

$$s_1 + l \cos \gamma_1 = r \sin \alpha_1 + l \cos \gamma_0$$

i.e.

$$s_1 = r \sin \alpha_1 + l (\cos \gamma_0 - \cos \gamma_1),$$

where the angles γ_0 and γ_1 are given by

$$\sin \gamma_0 = \frac{a - r - e}{l} \quad \text{and} \quad \sin \gamma_1 = \frac{a - r \cos \alpha_1 - e}{l}.$$

In like manner we obtain the portion s_2 of the stroke corresponding to the swing α_2 of the lever below the position *A B* from

$$s_2 + l \cos \gamma_0 = r \sin \alpha_2 + l \cos \gamma_2,$$

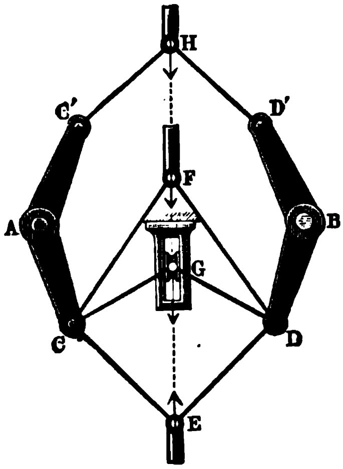
i.e.,

$$s_2 = r \sin \alpha_2 + l (\cos \gamma_2 - \cos \gamma_0),$$

when the angle of inclination γ_2 of the link in its lowest position is determined from

$$\sin \gamma_2 = \frac{a - r \cos \alpha_2 - e}{l}.$$

Fig. 401.



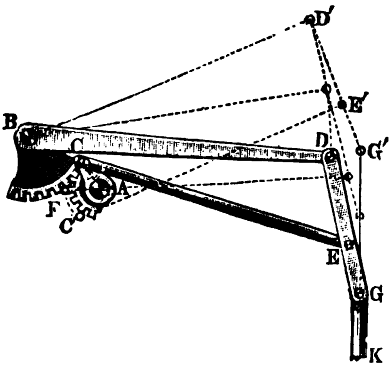
In Fig. 401, the rotations of the two levers *A C* and *B D* are not made dependent on each other by toothed sectors but by guiding the connecting point *G* of the two links by a special straight-line motion, say by slide bars, the point *G* being guided along the straight line *G H* which is perpendicular to *A B* at its middle point. This mechanism may be employed for communicating rectilinear motion to any other points *E* and *F*. These points are also connected with the levers by the links *CE*, *CF*, *DE* and *DF*. We see that it depends upon the position of the point *E* or *F* whether the direction of motion is the same as that of *G* or opposite. Nor is it necessary to connect the links of the guided point directly with the journals *C* and *D*, for any other levers *A C'* and *BD'* may be employed, provided the latter have the same axes as $A\,C$ and $B\,D$ and are symmetrically arranged with respect to the stroke $G\,H$. Neither is it necessary that these levers $A\,C'$, $B\,D'$ should lie in the same plane with $A\,C$ and $B\,D$, for we may regard the axes A and B as having any desired length and can suppose them to carry any number of such shear-like systems as $A\,C'\,H\,D'\,B$. We see from this that the mechanism under discussion may, like the pantograph, be employed to transmit the rectilinear motion obtained by other means to any number of rods. One essential difference of the two mechanisms is that while in both the rectilinear motions all lie in one plane, this plane is parallel to the lever axes and half way between them in the mechanism just discussed, while in the pantograph the plane is perpendicular to the axes of the levers. The mechanism just described is rarely employed.

If the two levers and the links are not arranged symmetrically with respect to the required straight line, we must effect the rectilinear motion by bringing the rotations of the levers into such a dependence upon each other as this motion of the point demands. It has already been remarked that in this case the ratio of the rotations of the levers is no longer constant, consequently cylindrical toothed wheels can no longer be employed for connecting the levers. If toothed gears are to be employed we must provide the two axes of the levers with non-circular wheels or sectors, whose varying velocity ratio will agree with the varying ratios of the lever rotations α and β, resulting from the mechanism. Since toothed wheels are too uncertain in their action where very exact motions are required, the levers are connected in a simpler manner, namely, by special coupling rods. It is evident in advance that in this way the conditions of the problem can be only approximately met. Nevertheless, a suitable choice of proportions and positions will enable us to obtain a degree of accuracy which is entirely sufficient for practical purposes, in some circumstances even exceeding the accuracy of the straight-line motions of the elliptical, conchoidal and lemniscate linkages already discussed. The analytical investigation of this kind of straight-line motion is extremely lengthy, and cannot be carried out practically because of the great number of jointed links involved. The exact paths of points guided by such mechanisms are curves of very high degree. This renders it possible to obtain a closer approximation of the path to a straight line, for a curve can have as many points in common with a straight line as there are units in the number representing its degree. In most of the straight-line motions hitherto discussed, the path coincides with the required straight line in three points; in the present mechanism, however, the exact path is a curve of the eighth degree and coincides at five points with the required straight line. This constitutes a certain advantage of these mechanisms over those discussed before, and in cases where it is important to have as exact a motion as possible such an arrangement may prove desirable, although its complexity will prevent its application in ordinary cases.

Since the analytical investigation is impracticable, these straight-line motions when designed must be treated by graphical methods, from which the required results may be obtained with comparative ease. We will now describe a few of the most interesting of these mechanisms.

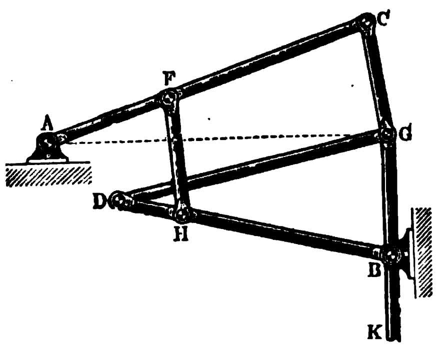
In *Maudslay's* straight-line motion, Fig. 402, the axes *A* and *B* of the levers are connected by toothed sectors *A F* and *B F*. Here the two levers *A C* and *B D* are unequal in length, and the piston rod *G K* is not hung from the connecting point *E* of the two links *C E* and *D E*, but from a point *G* on the prolongation of *D E*. If we suppose the point *G* of the piston rod to be moved into different positions on the straight line *K G*, and if we determine the angles α and β of the levers which correspond to these successive positions, we shall get the corresponding velocity

FIG. 402.



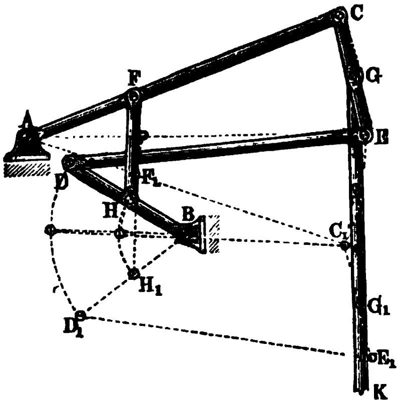
ratios of the toothed sectors *A F* and *B F*, which may then be designed as non-circular wheels, in accordance with the rules given in § 49.

FIG. 403.



Two other mechanisms of the same kind, whose levers are united by a coupling-rod *F H*, are shown in Fig. 403 and Fig. 404. There is an unimportant difference between these two, namely, in the mode of suspension, the piston rod *G K* in *Tschebysheff's* straight-line motion (Fig. 403) being suspended from the connecting point *G* of the two links, while in *Harvey's* (Fig. 404) mechanism, the piston rod is hung from one of the intermediate points of the link *C E*. In both mechanisms the levers are connected by a link *F H* whose joints *F* and *H* must be found from the drawing. For this purpose we must draw the whole mechanism in different positions corresponding to the different positions of the guided point *G* on the given straight line *G K*. We must then find by trial those two points *F* and *H* whose distance is most nearly constant for all the different positions of the levers. It is evident that by varying the lengths of the levers and the links, and by changing the positions of the centres relatively to each other and to the given straight line, we can obtain an infinite number of different combinations.

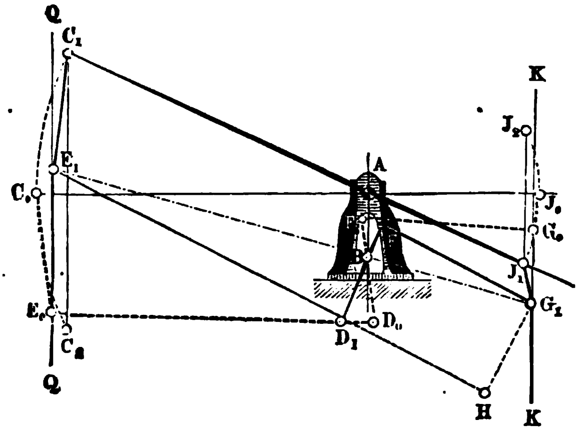
FIG. 404.



Remark. If we wish to obtain good proportions for the straight-line motion shown in Fig. 403, we must, according to the inventor, take $DB = DG = 0.809\,AC$, also $FC = HB = 0.618\,AC$, and $CG = FH = \frac{s}{2}$, where s represents the whole stroke of the piston rod GK. Moreover, the direction GK of the stroke must pass through the axis B and be tangent to the arc described by the point C while the lever AC swings about A.

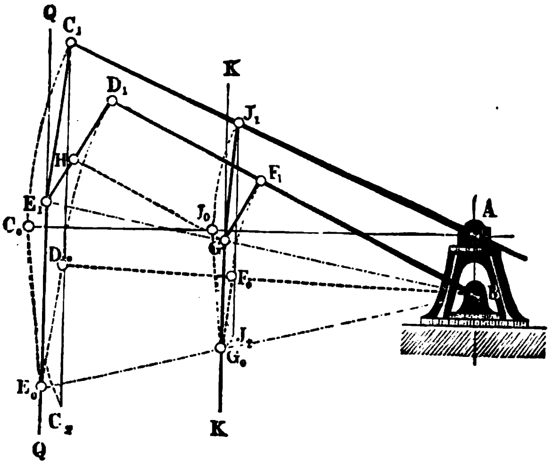
This straight-line motion may be employed to communicate the given rectilinear motion of one point to any other points in a manner similar to the pantograph. For instead of connecting the levers AC and BD by a coupling-rod or toothed sector so that a certain point, say the connecting point of the two links, shall be guided in a straight line, we may obtain the same determinate motion of the levers by actually guiding the point mentioned in the given straight line. For example, let AC, Fig. 405 and Fig. 406, be the

Fig. 405.



beam of a steam engine, and let the point E of the link be guided by any means, say by a lemniscate linkage, along the straight line QQ. From what has preceded we know that this line bisects the rise of the arc $C_1\,C_0\,C_2$ so that the angles γ_1 and γ_0 of the link CE are numerically equal when the latter is in its extreme and middle positions. Let us now suppose that the rectilinear motion of a point *G* is to be deduced from that of *E*. We may employ for this the pantograph mechanism *E B D F G H*, the point *B* on the straight line *E G* being taken as a stationary axis on which is placed a second lever *D B F*. On this lever, and on the guided point *G* draw a parallelogram as shown in the figures. It is evidently unnecessary to complete the parallelogram, for we may employ the just discussed straight-line motion with two levers and two connecting links to guide *G* in a straight line. It is plain that we have here two levers, *A C* and *B D*, whose ends *C* and *D* are not only united by the links *C E* and *D E* but are also constrained to assume perfectly determinate motions relatively to each other, for the reason that the point

FIG. 406.



E is already guided in a straight line. Therefore, if the point *G*, which is to be guided, is connected with such points of the two levers (the levers being supposed to have the determinate motions just mentioned) as will cause the point *G* to describe a straight line, the rectilinear motion of *E* will be communicated to *G*. Now the point *G* is already connected with the lever *B D* at *F*; it is therefore only a question of finding that point *J* of the other lever or beam *A C* which is to be connected by a link with G. The position of this point J, if we assume that the point G in its highest, middle and lowest positions falls on the given straight line $K K$, may be found as follows: Let r and r_1 represent the lever arms $A C$ and $A J$ of the beam, l and l_1 the lengths of the links $C E$ and $J G$, γ and δ their inclinations to the straight lines, γ_1 and δ_1 corresponding to the highest and γ_0 and δ_0 to the middle positions and finally let s and s_1 represent the strokes of the points E and G. Designating the arms $D B$ and $F B$ by ρ and ρ_1 respectively, we now have

$$\frac{s}{s_1} = \frac{E B}{G B} = \frac{D B}{F B} = \frac{\rho}{\rho_1}.$$

Moreover, from the figure we get

$$\tfrac{1}{2} s = r \sin \alpha_1 + l \cos \gamma_0 - l \cos \gamma_1 = r \sin \alpha_1,$$

for $\cos \gamma_0 = \cos \gamma_1$. In like manner we have

$$\tfrac{1}{2} s_1 = r_1 \sin \alpha_1 + l_1 \cos \delta_0 - l_1 \cos \delta_1,$$

and consequently

$$\frac{s}{s_1} = \frac{r \sin \alpha_1}{r_1 \sin \alpha_1 + l_1 (\cos \delta_0 - \cos \delta_1)}.$$

When the direction $K K$ bisects the rise of the arc described by J we also have $\cos \delta_0 = \cos \delta_1$ and

$$\frac{s}{s_1} = \frac{r \sin \alpha_1}{r_1 \sin \alpha_1} = \frac{r}{r_1}.$$

Now, since we also have

$$\frac{s}{s_1} = \frac{\rho}{\rho_1},$$

the equation of condition for guiding the point G in a straight line becomes:

$$\frac{\rho}{\rho_1} = \frac{r}{r_1}.$$

It is easy to see that this condition can only be fulfilled if the line of centres $A\,B$ is parallel to the direction of the stroke. The point J must then be so assumed on the beam that it will describe an arc whose rise is bisected by the given line $K\,K$.

§ 106. Peaucellier's Linkage for Producing Rectilinear Motion.

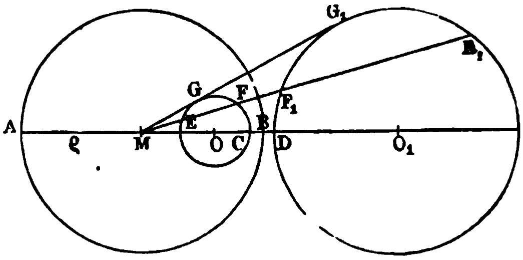
—In 1864 a captain of engineers, *Peaucellier*, described in the *Nouvelles Annales* des Mathématiques, Vol. III., 2d Series, an instrument called the ***universal*** or ***compound compass***, by means of which a circular motion could be converted into an exact rectilinear one without the aid of slide bars. Later, in 1871, *Lipkin* in *Petersburg** discovered a straight-line motion based on the same principle as *Peaucellier's* compound compass. This straight-line motion may, with the help of a few known properties of the circle, be explained as follows:

Let the diameter $A\,B$ of the circle whose radius is $M\,A = \rho$, Fig. 407, be divided internally at C and externally at D in the same ratio, that is so that

$$A\,C : B\,C = A\,D : B\,D;$$

then will $A\,B$ be divided harmonically at the *conjugate points* C and D, and A, B, C and D will be four harmonic points. It is evident that for every two such conjugate points we shall have the equation:

FIG. 407.



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* See Zeitschrift deutscher Ingenieure, 1877, p. 11.

$$MC \,.\, MD = \overline{MA}^2,$$

or if we make $MG = c$ and $MD = d$ we have:

$$cd = \rho^2.$$

This equation follows immediately from the above harmonic proportion:

$$\rho + c : \rho - c = d + \rho : d - \rho.$$

Now, if we suppose any circle $OFGE$ to be passed through C, and determine for all of its points, as E, F, G, the corresponding conjugate points E_1, F_1, G_1 for the first circle M, these latter points all lie on the circumference of a second circle whose centre is O_1. To prove this, draw a tangent MG from M to the circle O and also any secant MEF, then, under the supposition that E_1, F_1 and G_1 are the conjugate points to E, F and G for the circle M with radius ρ, we have, according to the above:

$$ME \,.\, ME_1 = MF \,.\, MF_1 = MG \,.\, MG_1 = \rho^2.$$

But since we also have

$$ME \,.\, MF = \overline{MG}^2$$

division gives us

$$MF : ME_1 = ME : MF_1 = MG : MG_1.$$

From this follows that the two figures EGF and $F_1G_1E_1$ are similar and similarly situated, and that the centre M of the first circle is a centre of similitude, an *external* one when M, as in Fig. 407, lies without the circle O, and an *internal* centre of similitude when M, as in Fig. 408, lies within the circle O, in which case the shortest half chord MG takes the place of the tangent. It is likewise evident that, for the limiting case in which the circumference of the circle O passes through the centre of similitude M, the circle O_1 reduces to a straight line O_1D perpendicular to the line of centres MO, Fig. 409.

On these geometrical properties of the circle[*] depends the compound compass of *Peaucellier* and the corresponding straight-line motion.

FIG. 408.

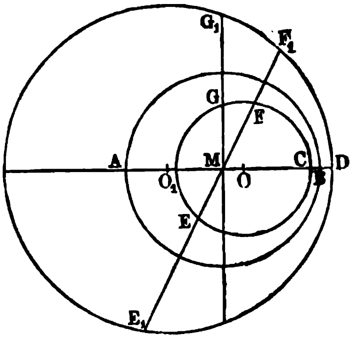
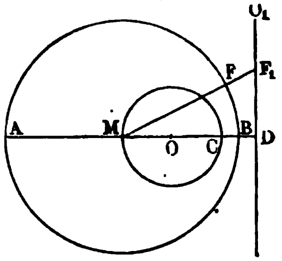
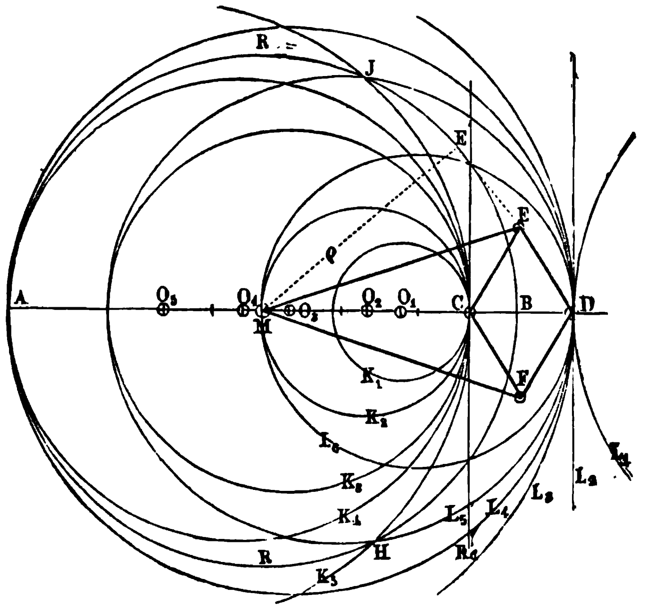


FIG. 409.



This mechanism consists essentially of a *jointed rhombus* *CEDF*, Fig. 410, two opposite points of which, *E* and *F*, are jointed to the fixed centre *M* by the long links *ME* and *MF*.

FIG. 410.



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[*] See Chauvenet's Geometry.

This combination of levers and links has thus far no constrained motion (see § 30, Introduction), and whatever motion is given to it, the vertices C and D and the fixed axis M will lie on the straight line bisecting the angles at M, C and D, for the links are symmetrically arranged. If we represent the length of the levers by $ME = MF = L$, the length of a side of the rhombus by l, and by φ and γ the half angles at M and C when the system is in any position, we have for the distances of the vertices C and D from the fixed axis M in this arbitrarily chosen position :

$$MC = c = L \cos \varphi - l \cos \gamma,$$

$$MD = d = L \cos \varphi + l \cos \gamma ;$$

hence

$$MC . MD = cd = L^2 \cos^2 \varphi - l^2 \cos^2 \gamma$$

$$= L^2 \cos^2 \varphi - l^2 (1 - \sin^2 \gamma),$$

or since $l \sin \gamma = L \sin \varphi$:

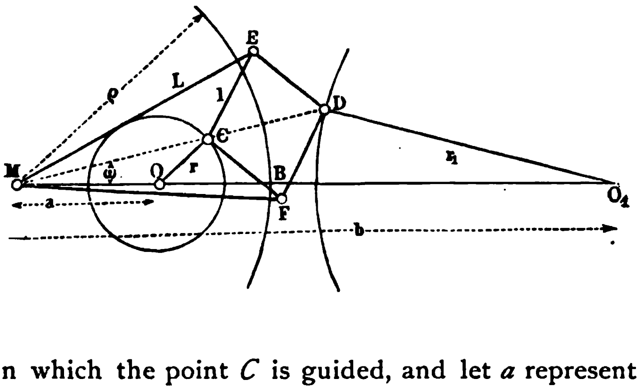
$$cd = L^2 - l^2 = \rho^2,$$

when the constant $L^2 - l^2 = ME^2 - CE^2$ is represented by ρ^2. The mechanism described has, therefore, the property that the two vertices of the rhombus not directly connected with the levers move so that the product of their distances c and d from the axis M is a constant ρ^2. If we therefore lay off ρ as one side, ME', of a right-angled triangle MEE', whose hypothenuse $ME = L$ and whose other side $EE' = EC = l$, and describe the circle R from M as a centre with the radius $ME' = \rho$, we know from the foregoing that the two vertices C and D of the rhombus will move so that they are always conjugate points of the circle R. It now follows from the above-mentioned property of the conjugate points of a circle that if one of the points C or D is guided in a circle, the other vertex D or C will describe a circular path for which M is a centre of similitude. For example, if the point C is guided in a circle K_1, say by means of a lever $O_1 C$

turning about the fixed centre O_1, the point D will move on the circle L_1 and this latter circle will become a straight line $D L_2$, when the circular path K_2 of the point C passes through the centre M of the levers. If the circular path of the point C becomes yet larger—for instance if it extends to K_3—the path of the point D will alter its curvature and become a circle L_3. Moreover, the two circular paths K_4 and L_4 of C and D must touch at the same point A of the original circle R, for if one of the conjugate points of a circle coincides with its circumference, the other conjugate must also coincide with it. If the circular guide of C is still greater—for example if it extends to K_5, the circular path of D will become smaller and will lie at L_5. This circle and K_5 will pass through the same points H and J of the original circle. Finally if the radius of the circular guide of C becomes infinite, *i.e.*, if the point C is guided along a straight line $C K_6$, the path L_6 of D will pass through the centre M. A further diminution of the circular path of D will cause the point C to describe a circle lying on the other side of the straight line K_6, *i.e.*, it will lie toward D, etc.

The relation between the radii of the two circular paths in which the vertices C and D move, is also readily found by calculation. Let $O C = r$ be the radius of the circle, Fig. 411, in which the point C is guided, and let a represent the distance $M O$ between the two fixed centres, then in any

FIG. 411.



position $CEDF$ of the rhombus, in which the axis of symmetry MCD makes any angle $CMO = \omega$ with the fixed base MO, we have:

$$MC = c = a \cos \omega + \sqrt{r^2 - a^2 \sin^2 \omega};$$

and since

$$cd = L^2 - l^2 = \rho^2,$$

we also have

$$cd = d(a \cos \omega + \sqrt{r^2 - a^2 \sin^2 \omega}) = \rho^2,$$

from which we find

$$(\rho^2 - da \cos \omega)^2 = d^2 (r^2 - a^2 \sin^2 \omega),$$

or,

$$d^2 (a^2 - r^2) - 2 d \rho^2 a \cos \omega = - \rho^4.$$

If we write this in the form

$$d^2 = - \frac{2 d \rho^2 a}{a^2 - r^2} \cos \omega = - \frac{\rho^4}{a^2 - r^2}$$

and place $\frac{\rho^2 a}{a^2 - r^2} = b$, we get by adding $b^2 = \frac{\rho^4 a^2}{(a^2 - r^2)^2}$ to both sides:

$$d^2 - 2 d b \cos \omega + b^2 = - \frac{\rho^4}{a^2 - r^2} + \frac{\rho^4 a^2}{(a^2 - r^2)^2} = \frac{\rho^4 r^2}{(a^2 - r^2)^2}.$$

But it is easy to see that if we make $MO_1 = b = \frac{\rho^2 a}{a^2 - r^2}$ the side DO_1 will likewise be determined by

$$\overline{DO_1}^2 = \overline{DM}^2 - 2 DM . MO_1 \cos \omega + \overline{MO_1}^2$$

$$= d^2 - 2 d b \cos \omega + b^2 = \frac{\rho^4 r^2}{(a^2 - r^2)^2};$$

consequently $D O_1 = \frac{\rho^2 r}{a^2 - r^2}$ and is independent of the angle ω. From this follows that O_1 is the centre of a circle on which the point D moves, and that this centre is at a distance from the axis M of the levers, equal to

$$M O_1 = b = \frac{\rho^2 a}{a^2 - r^2} = \frac{L^2 - l^2}{a^2 - r^2} a,$$

while the radius of the circle is equal to

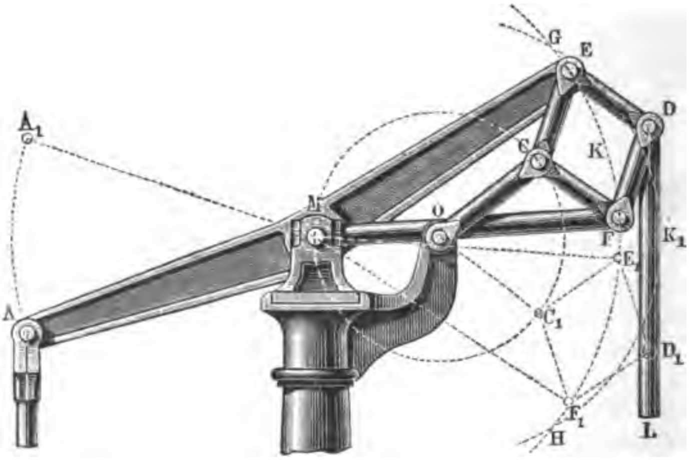
$$D O_1 = r_1 = \frac{\rho^2 r}{a^2 - r^2} = \frac{L^2 - l^2}{a^2 - r^2} r.$$

If in this expression we make $a = r$, we get $r_1 = \infty$, *i.e.*, the point D is guided in a straight line. If in the expressions found for b and r_1 we exchange b for a and r for r_1, the expressions will then give the radius and centre of the circular path of the point C when the path of D is established by r_1 and b. We then likewise get, when $r_1 = b$ the radius $r = \infty$, *i.e.*, C is guided in a straight line when the path of the vertex D passes through the axis M of the levers, a fact already mentioned.

It follows from these investigations that this mechanism can be employed to describe circular arcs of any radius whatever, thus justifying the name of universal compass. The instrument is particularly useful in drawing very flat circular arcs, whose centres are distant and therefore not accessible. On the other hand, only that case is of interest in machinery in which the *path of one of the vertices of the rhombus becomes a straight line;* this *always occurs when the other free vertex swings in a circle passing through the axis of the levers.*

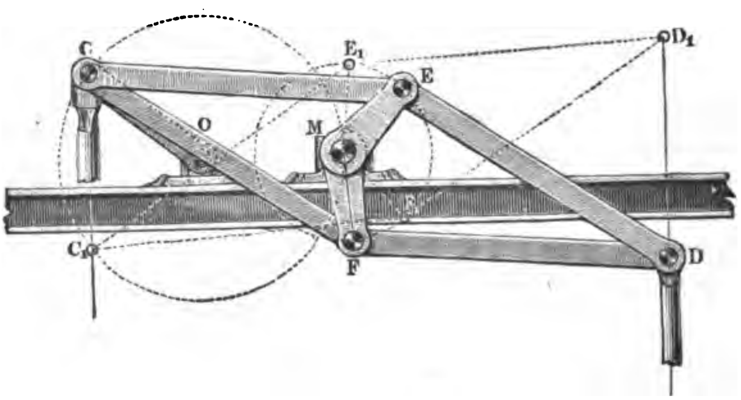
The arrangement of this straight-line motion is as follows: If the rod $D L$ is to be guided in the straight line $D D_1$, Fig. 412, it is only necessary to connect its end D with the vertex of the rhombus $D E C F$, whose opposite vertex C is guided by a lever $O C$ of such a length and position that the circle described by $O C$ around O, passes through the axis M of the two levers $M E$ and $M F$. One of the latter, say $M F$, may be formed of two simple links, while the other ME may, for the purpose of transmitting motion, be given the form of a beam EMA at the other end A of which the connecting-rod acts. Here, as in previously discussed straight-line motions, the links and levers may be arranged in pairs in order that the beam and the piston rod may have space to move between them. We see from the figure that the position of the system in which the points E, C and O lie in the same straight line corresponds to the upper limit of the motion. Now if, with the radius $OC + CE = r + l$, we describe the arc K_1 about O, this circle will cut the circle K described from M as a centre with a radius $ME = L$, in two points G and H, corresponding respectively to the extreme positions of E and F. Bearing this in mind it becomes easy to determine the dimensions L, l and r corresponding to a given stroke DD_1 of the piston rod.

FIG. 412.



In the arrangement hitherto discussed, the axis M was outside of the rhombus, but there is no objection to shortening the principal levers ME and MF sufficiently to bring the axis M within the rhombus, for the relations developed above are perfectly general and independent of the ratio of the lengths L and l. In this case we have a construction like Fig. 413, in which the rhombus $CEDF$ in a certain sense takes the place of the beam whose point of support M is the axis for the two equal levers ME and MF.

FIG. 413.



It is clear that from these straight-line motions we may obtain others by inversion. It is also clear that by suitable combination with the pantograph guides we may deduce other rectilinear motions from the given one. In this connection reference may be made to an article by *H. A. Huelsenberg* in the *Zeitschrift deutsch. Ing.*, 1877, p. 11.

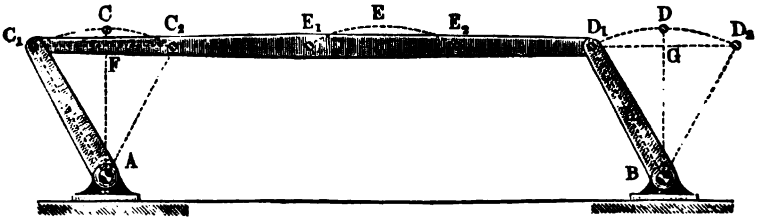
Finally we may remark that this ingenious and beautiful straight-line motion is the *only known system of links* which converts an oscillating circular motion into an *exact rectilinear one without the intervention of slide bars*, thus completely solving a problem which has engaged the most distinguished engineers since the time of *Watt*.

§ 107. **Rocking Arms.**—When power is to be transmitted by rods through long distances, the rods must be secured against bending by supports or guides placed at suitable points between their ends. For example, rollers may be used for this purpose, in which case the rod is composed of several parts rigidly connected; the pump rods of mines are usually of this kind, but these may be composed of separate pieces jointed together at their ends by pins, the joints being supported by *rocking arms.* Such arrangements were formerly employed for transmitting power from water wheels to machinery and are still used in the passages of mines.

A rocking arm is simply a bar oscillating about an axis with fixed bearings, the other end being jointed to one or two rods. The plane of motion of the arms is almost always vertical, the axis about which the arm swings being sometimes below and sometimes above the guided rod. Horizontal rockers, *i.e.*, such as swing about vertical axes, are seldom employed, or only when the direction of a rod in a horizontal plane is to be changed.

Since the end of an arm moves in a circular arc, two such equal arms as $A\ C$ and $B\ D$, Fig. 414, will not communicate to

FIG. 414.



the rod $C\ D$ a *straight-line motion*, but a parallel movement, *i. e.*, a motion in which all its positions CD, C_1D_1, C_2D_2, are parallel. Evidently in this case every point E of the rod describes an arc $E_1 E E_2$ equal to the arc described by the ends C and D of the arms. The rise CF of the arc described by the arm $A\ C$ of the length a is expressed by $e = CF = a(1 - \cos \omega)$, where ω is the angle of deviation $C_1 A\ C$ of the arm to each side of its normal position, in which the arm is perpendicular to the direction of the rod. If we suppose the same angle of deviation on each side and the stroke s to be given, we may find this angle ω from:

$$\sin \omega = \frac{s}{2a}.$$

This rise e also agrees with the lateral displacement of the rod CD, *i.e.*, with the distance between the extreme positions C_1D_1 and CD of the rod. If one end, say D, of the rod is guided in a straight line, as often occurs with pumps whose piston rods are moved by a swinging lever, the parallel motion will of course cease, and in order that the rod CD may deviate as little as possible from the rectilinear path of D, the arrangement must be so chosen that the straight line in which D is guided will bisect the rise of the arc C_1C_2.

In no case is it well to assume the length a of the arm too small, for the angle of deviation ω becomes smaller the greater a is in comparison with s. It is therefore customary to choose a between $2s$ and $3s$, which corresponds to an angle of deviation ω varying between $14^\circ\ 29'$ and $9^\circ\ 35'$ and a rise varying between $0.064\,s$ and $0.042\,s$.

Rocking arms or bobs furnish a simple means of changing the direction of the rods; two cases being distinguished, when the change of direction is slight, and when it is nearly a right angle. When the change is very slight the rods can both be jointed to the same pin of an ordinary rocking arm, as shown in Fig. 415, but this causes a considerable lateral deviation of the rods from the directions of their strokes. Consequently when the change of direction of the rods is considerable, a two-armed rocker or bob, Fig. 416, is employed, *i.e.*, a bell-crank lever whose arms, when the lever is in its middle position, are perpendicular to the directions of the rods, and therefore are inclined to each other at the same angle as the rods.

FIG. 415.

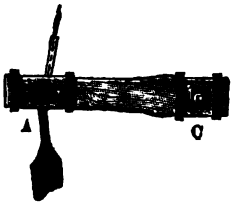
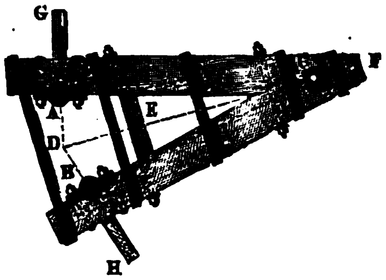


FIG. 416.



Let MG and MK, Fig. 417, be the directions of two rods making the angle $OMK = \alpha$ with each other, and let us suppose that the rods are both jointed to the pin M of a one-armed rocker whose centre C is so situated that the rocker in its middle position bisects the angle GMK. Now if ω again represents $ACM = BCM$, the half angle of oscillation of the arm, and a its length MC, we shall have for the stroke $BD = AH = s$ of the rods:

$$s = AB \cos ABD = 2a \sin \omega \cos \frac{\alpha}{2},$$

from which, for a given stroke s and given change of direction α, we obtain half the angle of oscillation, or

$$\sin \omega = \frac{s}{2a \cos \frac{\alpha}{2}}.$$

The greatest lateral displacement $f = EF = E_1F_1$ for this case is given by

$$f = CE - CF = a\left[1 - \cos\left(\omega + \frac{\alpha}{2}\right)\right].$$

FIG. 417.

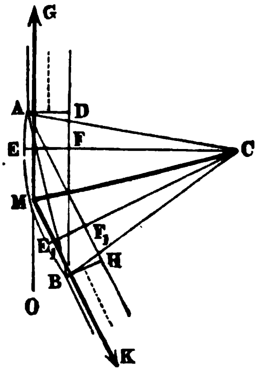
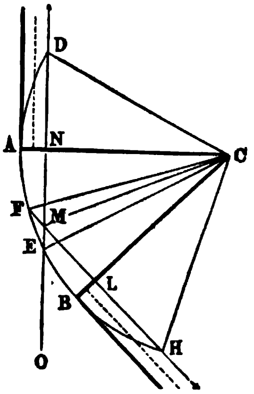


FIG. 418.



If the other ends of the rods were guided in straight lines, the directions ought to be so located as to coincide with the dotted lines which respectively bisect the lateral displacements EF and E_1F_1.

When the change of direction is small and the rocker has two arms, we obtain the following relations. Let MP and MQ, Fig. 418, be the directions of the two rods whose angle is bisected by MC. If from the axis C of the bob we let fall the normals CN and CL on to the rods, and regard these normals as the middle positions of the arms, *i.e.*, make $ND = NE = LF = LH = \frac{s}{2}$, we shall have CD, CE, CF and CH for the extreme positions of the arms. Then we again have:

$$\sin \omega = \frac{s}{2a},$$

and for the lateral motion,

$$f = AN = BL = a(1 - \cos \omega),$$

or approximately:

$$f = \frac{s^2}{8a},$$

which is independent of the change of direction $OMH = \alpha$ of the rods.

For the distance $MN = ML = d$ we have

$$d = a \cos \omega \tan \frac{\alpha}{2}.$$

When the two arms are of unequal length, the stroke as well as the lateral deviation is different for the two rods, and we have:

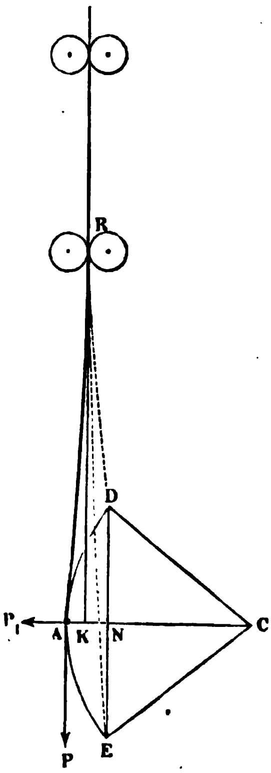
$$s_1 = 2a \sin \omega; \qquad s_2 = 2b \sin \omega,$$

$$f_1 = a(1 \cos \omega); \qquad f_2 = b(1 - \cos \omega),$$

$$\frac{s_1}{s_2} = \frac{f_1}{f_2} = \frac{a}{b},$$

EXAMPLE. When the change of direction of the rods is $\alpha = 30°$, the stroke of the rods $s = 1.5$ meters [59.06 ins.] and the length of an arm $a = 2s = 3$ meters [78.74 ins.], we have for a one-armed rocker :

FIG. 419.



$$\sin \omega = \frac{1.5}{6 \cos 15°} = 0.258 \; ; \quad \omega = 14° \, 56',$$

and for the lateral motion :

$$f = 3\,(1 - \cos 29° \, 56') = 0.400 \text{ meter } [15.75 \text{ ins.}].$$

On the other hand, for a two-armed rocker or bob, we have

$$\sin \omega = \frac{1.5}{6} = 0.25 \; ; \quad \omega = 14° \, 29' \; ;$$

$$f = 3\,(1 - \cos 14° \, 29') = 0.095 \text{ meter } [3.74 \text{ ins.}],$$

$$d = 3 \cos 14° \, 29' \tan 15° = 0.778 \text{ meter } [30.63 \text{ ins.}].$$

REMARK. When a rod is guided at one end in a circular arc, and at the other end in a straight line, it is necessary to joint the rod to the portion which is guided in a straight line, in order that the rod be subjected to tension and compression only, and not to bending stresses. The latter stresses occur when the rod in question is rigidly connected with the piece which is guided in a straight line. This case is represented in Fig. 419. Here the rod AR is subjected to a deflection $AK = \frac{f}{2}$, for it is not only acted upon by the pull or thrust P but also by a bending force P_1 at right angles to the former. The bending moment for the dangerous cross section at R is given by

$$M = P_1 \, . \, \overline{KR} - P \, . \, \overline{AK} = P_1 \, l_1 - P \frac{f}{2},$$

and the dimensions of the cross section are found by the rules given in Vol. I., § 276, for beams subjected to bending.

§ 108. Bell-crank Levers.—The dimensions of bell-crank levers are determined, in accordance with the rules laid down in the Strength of Materials, from the forces acting upon them. In the one-armed rockers, which serve simply to support and guide the rods, all the parts, the pins, rock-shaft and arm, are simply subject to the load of the rod resting upon them, the dimensions of the pins and of the shaft being determined in accordance with the rules given §§ 3 and 6 for journals and shafts which are subjected to bending. The magnitude of the force P transmitted by the rods when both of the latter act on the same pin, only influences the diameter of the latter by the shearing effect which it exerts. On the other hand, if the two ends of the rods act on different pins, as is always the case with bell-crank levers and bobs, these pins and all the connecting parts between them, arms and shafts for example, are subject to the action of this force and must be determined with reference to it.

Let a and b be the arms $A\,C$ and $B\,C$ of a bell-crank lever, Fig. 420, forming the angle β with each other, and let P and Q be the forces acting at A and B, then if we neglect the weight of the parts, the frictional and *all* other hurtful resistances, we have

$$Pa = Qb,$$

and for the pressure on the shaft,

$$R = \sqrt{P^2 + Q^2 + 2\,P\,Q \cos\beta} = P\sqrt{1 + \left(\frac{a}{b}\right)^2 + 2\frac{a}{b}\cos\beta}.$$

From P and Q we can then find the diameter of the pins, taking into account of course the manner in which the latter are strained by the forces. Namely, if the arm is provided with pins on each side, as in Fig. 421, the end of the rod must be forked and each pin must transmit a force $\frac{P}{2}$, hence the diameter of the pin must be found from

$$M = \frac{P}{2}\frac{l}{2} = \frac{W}{e}k = \frac{\pi d^3}{32}k,$$

or

$$d = \sqrt{\frac{32}{\pi}\frac{P}{4k}\frac{l}{d}} = 1.6\sqrt{P\frac{\lambda}{k}},$$

where λ again represents the ratio $\frac{l}{d}$, which in this case usually varies from 1 to 1.2, and k the maximum permissible stress, being $k = 6$ kilograms [8,500 lbs. per square inch] for wrought iron and $k = 10$ kilograms [14,200 lbs. per square inch] for steel. (Cast iron is in this case never used for pins.)

FIG. 420.

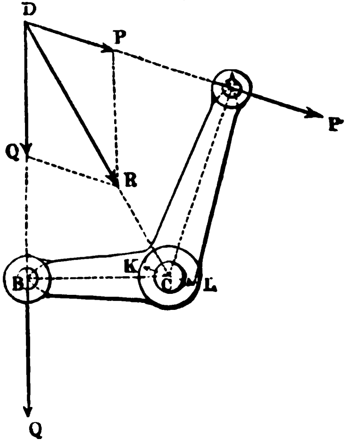
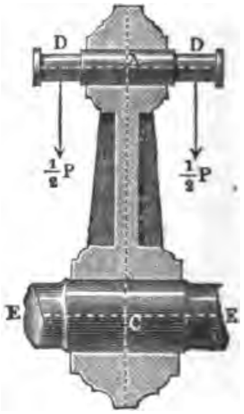
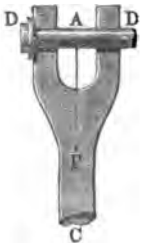


FIG. 421.



But experience shows that the use of a forked end of a rod turning on pins is not to be recommended, and that it is better to provide the arm with a forked end and rigidly fasten to it the pin DD so that the rod can turn about the central portion of the latter. The moment in this case is given by $\frac{Pl}{4}$, where l is the free length of the pin, so that we again have d given by

FIG. 422.



$$d = 1.6\sqrt{\frac{P\lambda}{k}}.$$

The same formula also serves to determine the diameter of each journal of the rock-shaft if we substitute for P the pressure $\frac{R}{2}$. But the diameter D at the centre of the rock-shaft is given by

$$\frac{RL}{4} = \frac{\pi}{32} D^3 k \quad \text{or} \quad D = 1.38\sqrt[3]{\frac{RL}{k}},$$

where L is the distance from centre to centre of the bearings.

The dimensions of the arms are determined from the formula for a beam fixed at one end and subjected at the other to a bending force P or Q, that is from

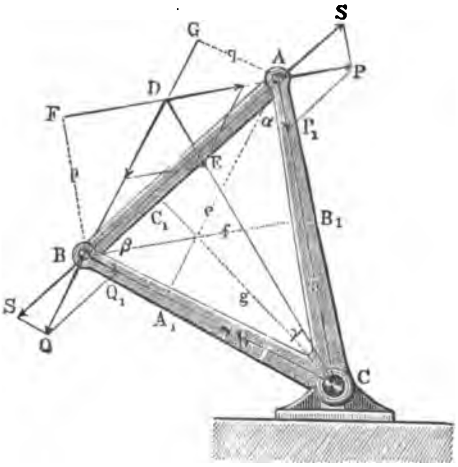
$$Pa = \frac{W}{e} k$$

(see Vol. I., § 235).

The length of the nave can be taken equal to 1.5 D, and the thickness of the metal around the eye of the nave equal to 0.4 D, where D is the diameter at the middle of the rock-shaft (see §§ 6 and 7).

In order that the arms of the bell-crank lever may not be subjected to bending, which is a very unfavorable form of strain and therefore requires a large amount of material, it is customary to unite the arms of the larger levers by diagonal pieces which act as *ties* or *struts* according as the forces acting on the rods tend to pull the arms apart or press them together.

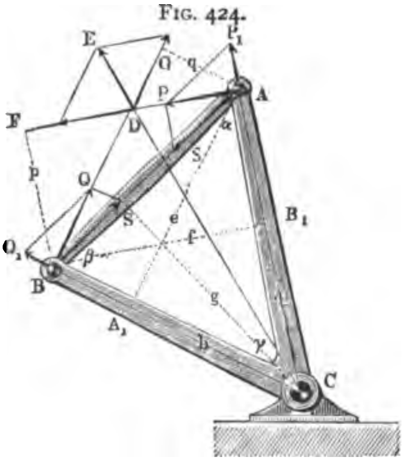
FIG. 423.



Let $CA = a$ and $CB = b$, Fig. 423 and Fig. 424, be the arms of a bob whose arms include the angle

$ACB = \gamma$, and let the ends of the rods be connected by a straight rod AB of the length c, then, for the middle position of the bob, in which the forces P and Q are perpendicular to the arms at the pins A and B, we shall have:

FIG. 424.



$$Pa = Qb.$$

These forces cause the thrusts or pulls P_1 Q_1 and S in the arms and connecting piece respectively. These may be readily found when we know the heights e, f and g of the triangle ABC, the distance $BF = p$ of the force P from B and $AG = q$ of the force Q from A, which quantities may be found from the drawing or easily calculated from the known parts of the triangle ABC. The equilibrium of the forces P, P_1 and S acting at A gives with reference to the centre of moments B:

$$Pp = P_1 f,$$

therefore

$$P_1 = P\frac{p}{f} = P \cot \alpha,$$

and in like manner the equilibrium of the forces Q, Q_1 and S with reference to the centre of moments A gives:

$$Q q = Q_1 e,$$

or

$$Q_1 = Q\frac{q}{e} = Q \cot \beta.$$

Moreover for the centre of moments C we have:

$$Pa = Sg = Qb,$$

or

$$S = P\frac{a}{g} = Q\frac{b}{g} = \frac{P}{\sin\alpha} = \frac{Q}{\sin\beta}.$$

Finally, for the pressure on the shaft we get:

$$R = \sqrt{P^2 + Q^2 - 2PQ\cos\gamma} = P\sqrt{1 + \left(\frac{a}{b}\right)^2 - 2\frac{a}{b}\cos\gamma}.$$

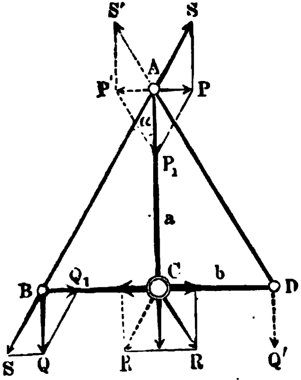
The connecting piece AB prevents the arms from being subjected to bending strains, for the forces P_1 and Q_1 act in the directions of the arms, subjecting the latter either to tension or compression. Similar remarks hold with respect to the connecting piece and the force S acting upon it. We also see from the figures that the arms AC and BC are subject to compression when the connecting piece AB is in tension and *vice versa*. In Fig. 423 the connecting piece AB acts as a *tie* and in Fig. 424 as a *strut*, and consequently if the rods act both in the forward and back stroke, the piece AB as well as the arms will be alternately stretched and compressed. These circumstances must be considered in choosing the material, for wrought iron is particularly suitable for pieces subjected to tension and cast iron for pieces subjected to compression. The determination of the cross sections F of the arms and of the connecting piece may in every case be easily found from

$$F = \frac{K}{k},$$

where K is the maximum force acting in the piece under consideration and k the maximum permissible stress in the material. In assuming k we must, if the piece is compressed, take into account the length of the latter, determining k in accordance with the rules established in Vol. I., § 274, Fifth Ger. Ed., for resistance to buckling.

It often happens in practice that the motion of a rod is communicated by a bob to two other rods simultaneously, which are parallel to each other; this arrangement is, for instance, employed in mines and in draining work for driving the piston rods of the two adjacent pumps. In such a case the bob is usually provided, as in Fig. 425, with three arms, $A\,C$, $B\,C$, $D\,C$, which are stiffened by the stays or struts $A\,B$ and $A\,D$. The lever arm $A\,C = a$ of the force is almost always at right angles to the lever $B\,D = 2\,b$. The stresses to which the various portions of the construction are subjected depend on whether the resistances at the points B and D act simultaneously and continuously during both the forward and back stroke of the rods, as is the case in *double-acting* pumps, or whether, as in *single-acting* pumps, the resistance is always in one direction and acts alternately at B and D. In the latter case when the force P acts in the direction $A\,P$ and the resistance Q at B acts in the direction $B\,Q$, according to what has preceded we shall have in $A\,C$ a thrust

FIG. 425.



$$P_1 = P\frac{a}{b},$$

in $B\,C$ a thrust

$$Q_1 = Q\frac{b}{a} = P,$$

and in $A\,B$ a pull

$$S = \frac{P}{\sin\alpha}.$$

The rod $A\,D$ and the arm $C\,D$ come into action only during the opposite motion, namely, when the force P acts in the direction $A\,P'$ and the resistance Q' acts at D. Then we have in $A\,C$ a thrust

$$P_1 = P\frac{a}{b},$$

in $D\,C$ a thrust

$$Q_1 = Q\frac{b}{a} = P,$$

and in $A\,D$ a pull

$$S = \frac{P}{\sin\alpha}.$$

In this case, therefore, the arms of the bob are subject to compression only, the arm $A\,C$ being always in compression, while $B\,C$ and $D\,C$ are so alternately; on the other hand, the stay rods are subject to tension only, and that alternately. If the resistances Q had an opposite direction, as in force pumps, it is easy to see that the arms would be in tension and the stay rods in compression. In every case the pressure on the rock-shaft is given by

$$R = \sqrt{P^2 + Q^2} = P\sqrt{1 + \left(\frac{a}{b}\right)^2}$$

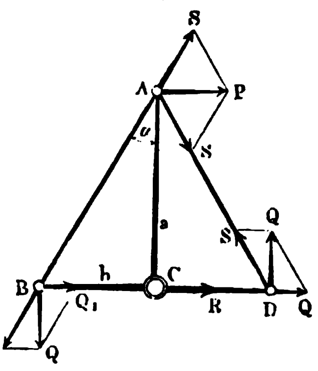
and its direction by $C\,R$ or $C\,R'$ according as P acts in one or the other direction; moreover, the direction of R is evidently always *parallel* to the stay rod $A\,D$ or $A\,B$ which is ***not under stress.***

In the other case supposed, Fig. 426, the resistances $Q\;Q$ act simultaneously at the extremities B and D of the lever, and are equal but opposite in direction, consequently the arm $A\,C$ of the force P will not be subjected to stress at all, but each of the stay rods will be strained by a force

$$S = \frac{P}{2\sin\alpha},$$

which in the forward and back stroke alternately stretches one of the stay rods and compresses the other. In like manner one arm is always stretched and the other compressed by a force

FIG. 426.



$$Q_1 = \frac{P}{2},$$

and we always have that arm compressed whose stay rod is stretched, and *vice versa*. The rock-shaft is in this case constantly subject to a pressure $R = 2\ Q_1 = P$, whose direction coincides with that of P.

If, as in Fig. 427, the arm $A\ C$ of the force be produced beyond C, we get a four-armed bob. If we cause rods to act at E as well as at A, it is clear that the motion can be effected by a *pull* P acting alternately at A and E whatever the direction of the resistances Q at B and D. Therefore, instead of

FIG. 427.

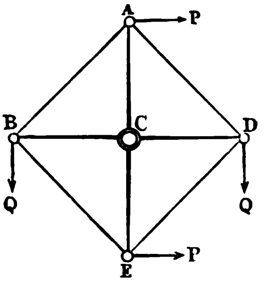


FIG. 428.

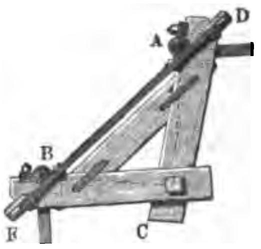


FIG. 429.

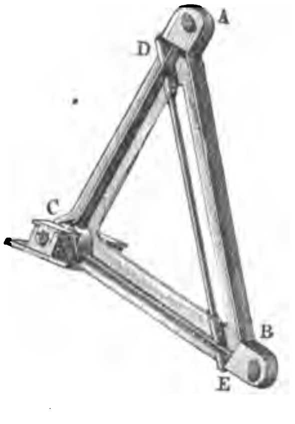
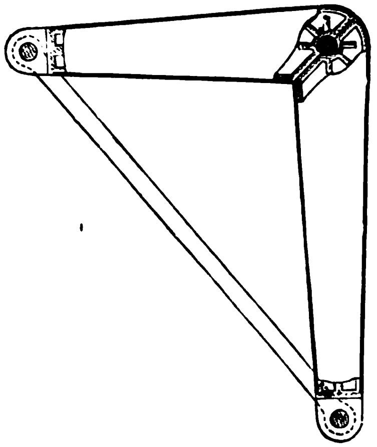


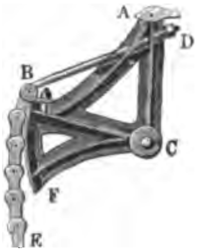
FIG. 430



transmitting the motion by a rigid *rod* acting at *A*, we can employ for this purpose two ropes or chains acting at *A* and *E*. The transmission of reciprocating motions through great distances by means of wire ropes has often been employed of late years, for this method offers advantages not possessed by rods. (See following chapter.)

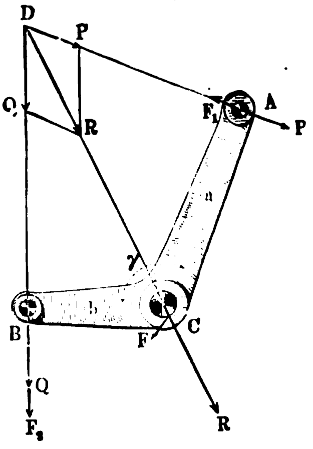
The construction of bobs is given in Figures 428 to 431. Fig. 428 represents a wooden bob and 429 a cast-iron one, while the bob* shown in Fig. 430 is mostly of wrought iron. When it is important to move a rod exactly in the direction of its axis, the corresponding arm is given the form of a sector, as in Fig. 431, the rod being connected with the latter by a chain. This construction is of course only possible when the rods are subjected to tension.

FIG. 431.



§ 109. Frictional Resistances.—The friction of the pins in the bearings on the rods and of the journals in the bearings of the rock-shaft increase the resistances acting on the bob and require an increase in the force *P* sufficient to overcome the resistance *Q*.

FIG. 432.



Let *a* and *b* again represent the lengths of the arms *A C* and *B C*, Fig. 432, of the force *P* and load *Q* respectively, d_1 and d_2 the diameters of the corresponding pins and *d* the diameter of the journals of the rock-shaft which is subjected to the pressure *R*. Now if φ is the coefficient of friction, we have for the work of friction at *A* corresponding to a slight angular motion ω:

---

* See *Rittinger* Erfahrungen, 1870.

$$W_1 = \varphi P \omega \frac{d_1}{2},$$

at B:

$$W_2 = \varphi Q \omega \frac{d_2}{2},$$

at C:

$$W = \varphi R \omega \frac{d}{2};$$

and the necessary force may be found from

$$P \omega a = Q \omega \left(b + \varphi \frac{d_2}{2}\right) + \varphi P \omega \frac{d_1}{2} + \varphi R \omega \frac{d}{2},$$

that is:

$$P = \frac{Q\left(b + \varphi \frac{d_2}{2}\right) + R \varphi \frac{d}{2}}{a - \varphi \frac{d_1}{2}}.$$

To be exact, it would be necessary to substitute in this formula:

$$R = \sqrt{(P - F_1)^2 + (Q + F_2)^2 - 2(P - F_1)(Q + F_2) \cos \gamma},$$

where F_1 and F_2 represent the friction of the pins, that is,

$$F_1 = \frac{W_1}{\omega a} = \varphi P \frac{d_1}{2a},$$

and

$$F_2 = \frac{W_2}{\omega b} = \varphi P \frac{d_2}{2b}.$$

But it suffices to employ for R the value:

$$R = \sqrt{P^2 + Q^2 - 2PQ \cos \gamma} = Q \sqrt{1 + \left(\frac{b}{a}\right)^2 - 2\frac{b}{a} \cos \gamma},$$

which gives

$$P = Q \frac{b + \varphi \frac{d_2}{2} + \varphi \frac{d}{2}\sqrt{1 + \left(\frac{b}{a}\right)^2 - 2\frac{b}{a}\cos\gamma}}{a - \varphi \frac{d_1}{2}}.$$

If there were no frictional resistances, the force necessary to overcome the load Q would be

$$P_0 = Q\frac{b}{a},$$

hence we have:

$$\frac{P_0}{P} = \eta = \frac{b}{a} \frac{a - \varphi \frac{d_1}{2}}{b + \varphi \frac{d_2}{2} + \varphi \frac{d}{2}\sqrt{1 + \left(\frac{b}{a}\right)^2 - 2\frac{b}{a}\cos\gamma}}.$$

The value $\frac{P_0}{P} = \eta$, or the ratio of the force P_0 theoretically necessary when friction is neglected to the force P actually exerted, is called the *efficiency* of the particular combination; on account of the hurtful resistances this ratio is of course always less than unity.

In the foregoing investigation the influence of the weight of the rods and bob has been neglected. In exact determinations this may be taken into account by finding the components of these weights borne by the journals. For example, if G represents the weight on the rock-shaft due to the bob and a portion of the rods, and δ the angle made by the pressure R with the force of gravity, it would be more exact to use

$$\varphi\sqrt{R^2 + G^2 + 2RG\cos\delta}$$

for the friction of the rock-shaft than φR. We see from this that the weight G may in some circumstances diminish the friction, namely, when the angle δ is obtuse, although generally the weight of the various parts increases the hurtful resistances, and consequently diminishes the efficiency.

EXAMPLE. When a wrought-iron bob has equal arms and changes the direction of a force Q of 10,000 kilograms [22,000 lbs.] through an angle of 60°, we have the following dimensions: the diameter of the pin on each arm, when $\lambda = \frac{l}{d} = 1.2$ and $k = 6$ kilograms [8,500 lbs. per sq. in.], is

$$d_1 = d_2 = 1.6\sqrt{10{,}000 \times \frac{1.2}{6}} = 71.5 = \text{about } 75 \text{ millimeters } [3 \text{ ins.}],$$

consequently the free length of the bolt is $l = 90$ millimeters [3.6 ins.].

The pressure R on the rock-shaft, when the weight of the parts is neglected, is

$$R = 2\,Q \sin 30^\circ = Q = 10{,}000 \text{ kilograms } [22{,}000 \text{ lbs.}],$$

hence the diameter of the shaft at the middle when the distance from centre to centre of the journals is $l = 0.5$ meter [19.69 ins.] is

$$D = 1.38\sqrt[3]{\frac{10{,}000 \times 500}{6}} = 130 \text{ millimeters } [5.2 \text{ ins.}].$$

Moreover the diameter of the shaft journals when $\lambda = 1.5$ is

$$d = 1.6\sqrt{5{,}000 \times \frac{1.5}{6}} = 56.6 = \text{about } 60 \text{ millimeters } [2.4 \text{ ins.}],$$

and the length $l = 90$ millimeters [3.6 ins.]. The tension in the stay rod is

$$S = \frac{Q}{\sin \beta} = \frac{10{,}000}{\sin 60^\circ} = 11{,}547 \text{ kilograms } [25{,}400 \text{ lbs.}],$$

therefore the requisite cross section is:

$$f = \frac{11{,}547}{6} = 1{,}925 \text{ square millimeters } [3 \text{ sq. ins.}].$$

The arms are subjected to the thrust $Q_1 = P_1 = Q \cot 60^\circ = 10{,}000 \times 0.5774 = 5{,}774$ kilograms [12,700 lbs.] and if we determine the permissible stress k in accordance with § 274, Vol. I., Fifth Ger. Ed., we get $\frac{5{,}774}{k}$ for the cross section.

Moreover if we assume $a = b = 1.5$ meters [59.06 ins.] and the coefficient of friction on journals $\varphi = 0.1$, we get for the requisite force

$$P = 10{,}000 \frac{1{,}500 + 0.1 \times 37.5 + 0.1 \times 30 \sqrt{1 + 1 - 2\cos 60^\circ}}{1{,}500 - 0.1 \times 37.5}$$

$$= 10{,}070 \text{ kilograms } [22{,}200 \text{ lbs.}].$$

The efficiency of the bob is therefore:

$$\eta = \frac{P_0}{P} = \frac{10{,}000}{10{,}070} = 0.993,$$

i.e., the friction causes a loss of work of only 0.7 of one per cent.

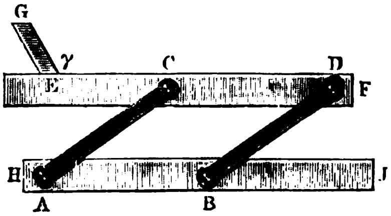
§ 110. Parallel Movements.

—It is often necessary in engineering work to move rod-shaped or other machine parts so that *a straight line connecting any two points of the body will be parallel in all positions.* This we may call a parallel movement, an example of which was given in § 107, where a long rod supported by two equal and parallel rocking arms, Fig. 414, was subjected to such a parallel movement. It is easy to see that this motion is also characterized by having all the points of the guided body moving in equal and parallel paths, which paths may be rectilinear or curvilinear. In the case mentioned, § 107, these paths are evidently circular arcs whose radii are given by the length of the rocking arms.

It is evident that a rigid body may be given a parallel movement when *two* of its points are guided in parallel and equal paths, provided the body cannot at the same time assume a rotation about an axis which is parallel to the straight line joining the two guided points. This does not prevent a body from receiving a parallel movement by guiding more than two of its points in such paths, and this often happens in actual constructions when it is desired to make the motion positive.

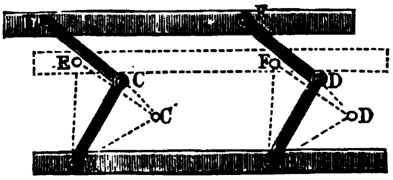
The simplest form of a parallel movement is that of a *parallel ruler*, Fig. 433, an instrument occasionally used in drawing, in which the two rulers EF and HJ are united by equal and parallel links AC and BD. The only condition of correctness is that $ABDC$ be an exact parallelogram, for it is clear that the edge EF will then always have the same direction, as the two points C and D must always move in equal and parallel arcs. It is evidently unnecessary that the edge EF should be parallel to AB or CD; on the contrary, a straight line EG, making with EF any angle γ, provided the angle is fixed, will likewise remain parallel to itself.

FIG. 433.



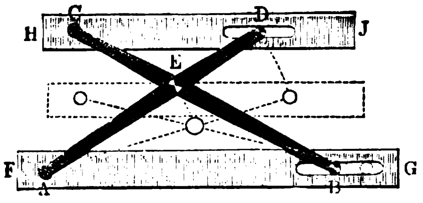
This mechanism is essentially the same as that described in § 107, and is extensively used for coupling driving wheels, and in this connection reference may be made to the chapter on "Crank Trains." The combination of two such parallel movements gives the *double parallel ruler*, Fig. 434, which after what has been said needs no further explanation.

FIG. 434.



The diagonal ruler, Fig. 435, is a rarely applied parallel movement depending on the property of a trapezoid $ABCD$, namely, that the diagonals form similar triangles with the parallel sides. If, therefore, two straight levers AD and BC, jointed at E, be united by jointing them to the bars FG and HJ at A and C, and if the ends D and B of the levers be so assumed that $AE : BE = DE : CE$, the bar HJ will move parallel to itself when the bar FG is held fast, the points D and B moving in the slits of the bars. An exact parallel movement is not, however, to be expected on account of the slits in which the pins D and B are guided. This arrangement has been ap-

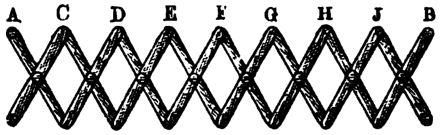
FIG. 435.



plied in parallel vises, but is objected to as wanting in durability.

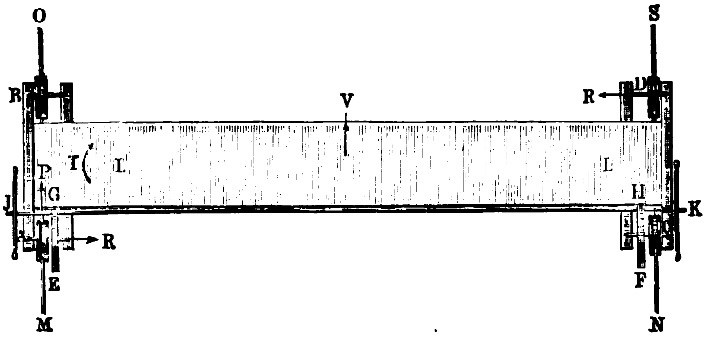
By modifying and repeating this diagonal ruler we get the *lazy tongs*, Fig. 436. It has been proposed to utilize this device as dividers for spacing purposes, for example, for spacing the rivet holes of boiler plates. If we suppose the pins *A*, *B*, *C*, . . . to be provided with points lying exactly in the axes of the pins, and place two points *A* and *B* at the ends of a straight line to be divided into equal parts, the points *C*, *D* . . . lying between *A* and *B* will furnish the points of division for as many parts as there are intervals between *A* and *B*, provided the separate links of the tongs are of equal length.

FIG. 436.



In traveling cranes, such as are employed in the erecting shops, the problem arises of guiding the bridge (which carries the truck) so that it will be parallel in all positions. For this purpose roller guides are always used, the bridge being provided at each end with two rollers or wheels *A*, *B* and *C*, *D*. Fig. 437, which run on overhead iron rails *M O* and *N S*. The wheels are provided with grooves which fit the rails and thus prevent all lateral displacement. The motion of the bridge

FIG. 437.



$L L$ is generally effected by providing the arbor of each of the rollers A and C with a toothed wheel, as E and F, which gear with the pinions G and H respectively. By turning these toothed wheels, the rollers A and C are turned and the bridge moved like a wagon. As the length $A C = l$ of the bridge is long in comparison with the distance $A B = C D = b$ between the wheels, it is always necessary to move the gears G and H simultaneously and with equal velocity, for if the shifting force P were to act on only one side, the bridge would *bind* and might easily be clamped fast by the binding action. The following will show when stoppage is unavoidable: Suppose that only one roller A is acted upon by the force P, which tends to move the bridge in the direction of the arrow V, now as this force does not pass through the centre of gravity of the bridge, it will tend to turn the latter in the direction of the arrow T, and, as a consequence, the rollers A and D will press against the inner edges of the rails, causing the latter to react with equal and opposite forces R. These reactions may be determined from

$$P l = R b \qquad \text{or} \qquad R = P \frac{l}{b} .$$

Therefore, in order that the bridge may move, the sliding friction between the flanges of the rollers and the sides of the rails must be overcome; the amount of this friction on each side is φR, where φ represents the coefficient of friction. Therefore to produce motion we must have:

$$P > 2 \varphi R, \qquad \text{or} \qquad P > 2 \varphi P \frac{l}{b} ,$$

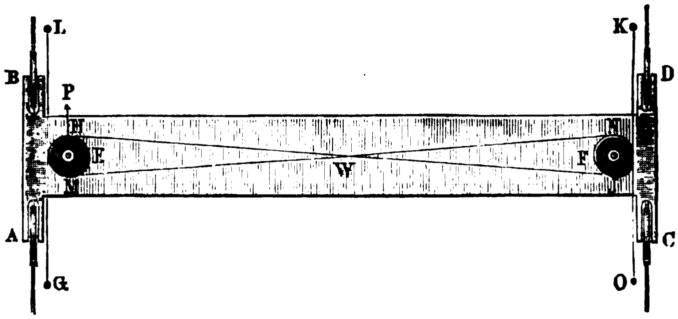
that is,

$$\frac{b}{l} > 2 \varphi,$$

But this condition is never fulfilled in this kind of construction, for $\frac{b}{l}$ is always smaller than 2φ $\left(\frac{b}{l}\right.$ varies ordinarily from $\frac{1}{6}$ to $\left.\frac{1}{10}\right)$, consequently the bridge cannot move, however great the force *P* be taken. Instead of moving, the bridge is clamped in a manner similar to the holdfast or dog described in § 95. For this reason it is necessary to let the moving force act on both sides simultaneously, in which case the tendency to turn the bridge disappears. This is effected by fixing the pinions *G* and *H* to a common shaft supported by the bridge. By turning this shaft, the two rollers *A* and *C* are moved uniformly.

In the long (15 meters [50 ft.] and more) carriages used in spinning machinery, another arrangement is employed whose object is to transmit the moving force, which acts at one end of the carriage, to the other end. For this purpose the carriage *W*, Fig. 438, is provided with wheels *A*, *B*, *C*, *D*, running on tracks. On the lower surface of the carriage is placed at each end a fixed spindle, two equal rope pulleys *E* and *F* turning loosely on each spindle. Around these two pairs of rope pulleys pass two ropes *G H J K* and *L M N O*, whose ends are fastened at *G* and *K*, and at *L* and *O*. One rope, *G H J K*, is guided by the lower pulleys and the other rope by the upper pulleys, the two crossing at the middle of the carriage. In order to understand the action of these crossed ropes, let us suppose, at the end *A B* of the carriage, a force *P* to act in the direction of the arrow *P*, then if the other end of the carriage *C D* does not move, there will be greater tension in the rope *G H J K* than existed originally, while the other rope

FIG. 438.

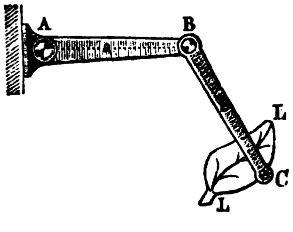


L M N O will lose tension. The consequence is that the spindles of the pulleys *F* are pressed in the direction *O K* because the tension on the rope *J K* exceeds that on *N O*. This very efficient rope guide, therefore, effects the motion of the carriage whatever the position of the driving force. For regular working, it is only necessary to tighten the ropes as they become slack under the influence of temperature and moisture.

Preceding investigations show that there is danger of clamping the carriage only when the length is comparatively great, and the distance between the wheels small. On the other hand, if the distance between the rails is small in comparison with the distance between the wheels, there will be no danger of clamping, it is only necessary that $\frac{b}{l} > 2\varphi$. If we take φ very large, say 0.25, we have $b > \frac{l}{2}$ as the condition to be fulfilled when there is motion, l again representing the distance between the rails, and b the distance between the wheels. For example, in the short truck carrying the hoisting apparatus of the traveling crane, it is sufficient to confine the transporting gear to a single roller or wheel.

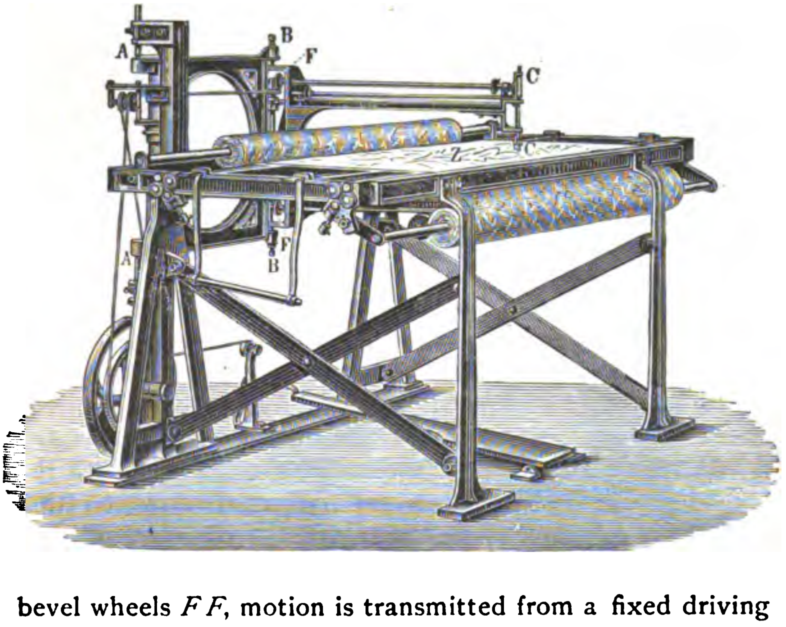
§ 111. **Parallel Movement by Levers.**—If on a fixed arbor *A*, Fig. 439, we suppose two equal levers *A B* to be placed, an arbor, passing through their ends *B* parallel to *A*, will remain parallel to *A* in any position of the system. During the motion the axis *B* must of course remain on the surface of a cylinder concentric to *A*. If we suppose this construction to be repeated by placing on the arbor *B* two equal arms *B C* lying in the same plane, the straight line passing through the ends *C* will also be always parallel to the axes *A* and *B*. This allows perfect freedom of motion to the straight line at *C*, for example, the point *C* can be guided along any plane curve *L C L*, as long

FIG. 439.



as the distance of C from A does not exceed the sum of the two arms $A B + B C$. There is here, strictly speaking, not a parallel movement of a body, but of straight lines, for this mechanism does not permit any line to constantly maintain its parallelism, except such as are parallel to the axis A, like B and C, while on the contrary the lines $A B$ and $B C$ can assume any direction whatever. An example of its application is found in a kind of *embroidering machine* where threads or cords are sewed on to curtains, Fig. 440. Here the needle C moves up and down, being guided by the cast-iron frame $B C$ turning about the centres $B B$, which in turn are attached to another frame $A B$ that moves about fixed centres $A A$. By means of cord pulleys whose axes coincide exactly with the straight lines $A A$ and $B B$, and by means of the little

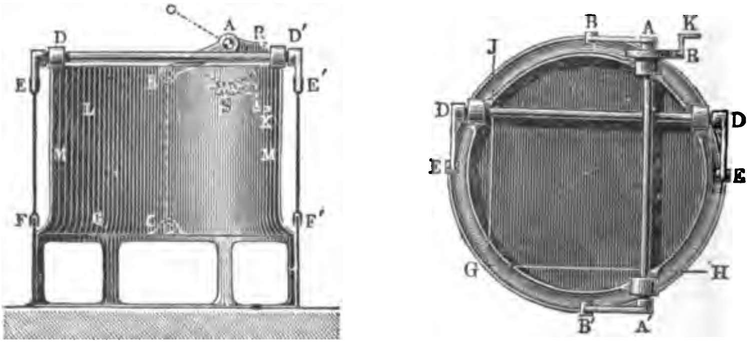
Fig. 440.



bevel wheels $F F$, motion is transmitted from a fixed driving shaft to the needle C in all its positions. Therefore if the guide of the needle be moved along the pattern drawn on the stretched piece Z, the stuff will be embroidered according to the outlines of the pattern. The material remains stationary and needs no motion except the occasional rolling up of the finished work on its roller.

A combination of two such lever mechanisms has been applied by *Redtenbacher*[*] to communicate a parallel movement to a cylindrical gate for turbines. The arrangement is shown in Fig. 441. Here *M M* is the vertical flume or pipe of the turbine carrying the horizontal shaft *A A'*. This shaft carries two equal levers *A B* and *A' B'* which are equal and lie in the same plane. The two pins *B* and *B'* are connected by two equal and parallel rods *B C*, *B' C'* with the ring-shaped gate *C F C' F'* surrounding the apertures of the pipe.

FIG. 441.



It is evident that in this arrangement the connecting line *C C'* must always remain parallel to the axis *A A'*, but this is by no means sufficient to guide the gate parallel to itself, for with the lever gear *A B C*, any point, as *G*, is free to move in a plane *G H* passing through *G* perpendicular to *A A'*. But if the gate is connected with the second similar system of levers *D E F* so arranged that *D D'* is likewise normal to the axis of the turbine pipe, *D E* equal and parallel to *D' E'* and *E F* equal and parallel to *E' F'*, the second system will constrain any point *G* to move in the plane *G J* passing through *G* perpendicular to *D D'*. It follows from this that the point *G* must remain on the line of intersection $G L$ of the two normal planes $A A'$ and $B B'$ just mentioned, the line of intersection being parallel to the axis of the pipe. As this holds for every point of the gate, it follows that the latter has a parallel movement. As it is best to allow the connecting rods of each shaft to take hold of the gate at points C, C' and F, F' which are diametrically opposite, this arrangement enables us to place the shafts $A A'$ and $D D'$ outside of the centre of the pipe, thus leaving space for the turbine shaft. By choosing the levers $A B$ and $D E$ sufficiently large, the shafts $A A'$ and $D D'$ could be placed entirely outside of the turbine pipe. It is evident from the figure that these shafts do not lie in the same plane but pass each other, and generally it is well to allow them to pass at right angles, but this is not a necessary condition, for they may make an oblique angle with each other provided they are placed normal to the rectilinear guides, *i.e.*, normal to the axis of the pipe. Nor is it necessary to make the levers $A B$ of the same length as the levers $D E$, provided that the levers belonging to each shaft are equal and parallel. Finally, the motion of the gate can be effected in any manner, for example, by a force acting at any point G by means of a screw or rack, but it is simplest to move one of the lever shafts $A A'$ by a worm wheel R and a worm S driven by a crank K.

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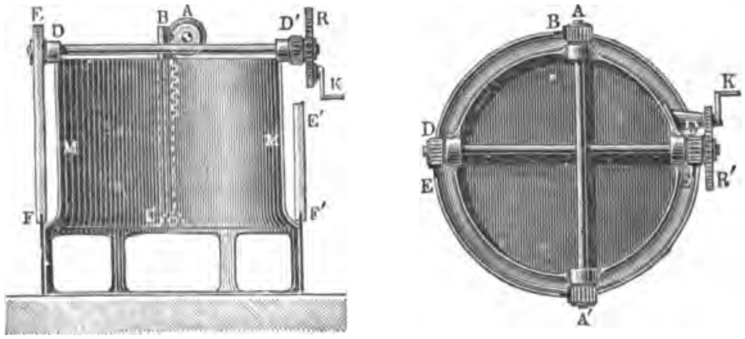
[*] See *Redtenbacher*, Die Bewegungsmechanismen, Pl. LXIII.

If we suppose the levers $A B, A' B', D E$ and $D' E'$ to be replaced by toothed gears or sectors and the connecting rods by racks, we shall have another parallel movement, Fig. 442,

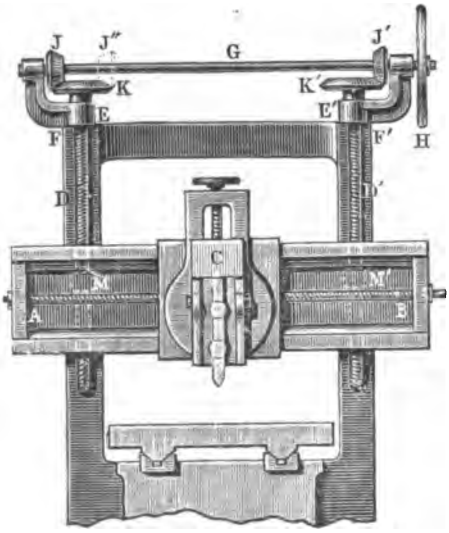
FIG. 442



likewise suggested by *Redtenbacher* for cylindrical gates. It is clear that the remarks concerning the relative position of the shafts $A\ A'$ and $D\ D'$, as well as those made with reference to the lever arms, apply here also, and that the wheels of one shaft may be of different radius from those of the other shaft. The racks may be jointed to the cylindrical gate by pins or rigidly connected with it by bolts.

§ 112. **Parallel Movement by Screws.**—In many machines screws are used to give a parallel movement to individual parts, for example, in the planers employed in machine shops. In rolling mills the two bearings of the upper roll are moved through equal distances by two vertical screws of the same pitch. This arrangement, as it is used in a planer, is shown by Fig. 443. Here the cross-head $A\ B$, on which the tool-box C slides horizontally, is hung from two vertical screws D and D' whose bearings are at E and E' on the uprights F and F'. Turning the horizontal shaft G by means of the hand-wheel H, the bevel wheels J, K and J', K' are turned equal distances at the same time. As both screws have the same pitch, a rotation of the hand-wheel H will cause each of the nuts M and M' rigidly connected with $A\ B$ to rise or fall the same distance, thus producing a parallel movement of the cross-head $A\ B$. If h is the pitch of the screws, z_1 the number of teeth in each of the wheels J, J', and z_2 the number in K and K', we shall

FIG. 443.



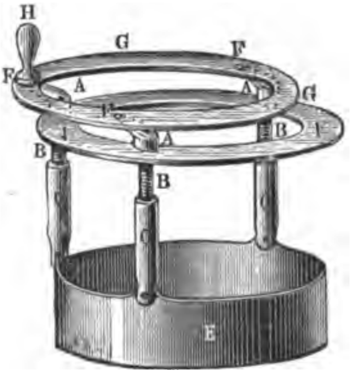
have for the vertical movement s of the cross-head when the hand-wheel H turns through the angle α:

$$s = \frac{\alpha}{2\pi}\frac{z_1}{z_2}h.$$

We see from this that when, as is usually the case, the pitch h of the screws is slight (say 10 to 20 millimeters [0.4 to 0.8 in.]) and the hand-wheel H large, this arrangement will permit a very accurate adjustment of the cross-head $A\,A'$. When the wheels J and J' are arranged as in the figure, the screws have opposite directions of rotation, and one of them must have a *right-handed* and the other a *left-handed* thread. On the contrary when the wheels are placed as at J' and J'', the screws will be turned in the same direction and their threads will be alike.

Screws have also been used by *Cadiat* for producing a parallel movement of the cylindrical gate of a turbine. For this purpose the ring-shaped gate E, Fig. 444, is connected at three or four points of its circumference with as many vertical screws $A\,B$ of the same pitch, whose upper bearings are fastened to the pipe of the turbine by means of the ring I, and whose long, sleeve-shaped nuts C are connected with the gate E. To give the latter a parallel movement it is only necessary to impart an equal rotation to all the spindles of the screws simultaneously. This is accomplished, without the help of wheels, by providing the screw spindles at their upper ends with cranks or levers $A\,F$ which are of exactly the same length and parallel to each other. The ends of these cranks are provided with projecting pins F connected by a ring G. It is clear that with this arrangement the rotation of one of the cranks,

FIG. 444.



say by means of the handle H, will cause an equal rotation of all the other cranks and their screw spindles, for the action of the ring G is equivalent to that of the parallel ruler described in § 110. The cranks A F consequently retain the parallelism originally given them in all positions. For further information concerning this mode of coupling cranks by a common link G, see chapter on "Crank Trains."

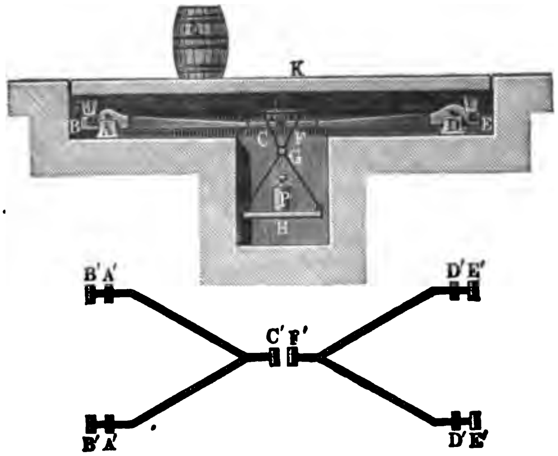
§ **113. Platform Scales.**—We may also class among parallel movements the devices employed for platform scales. These scales are used for quickly weighing large loads, and are, without exception, constructed so as to weigh by means of a smaller weight than the load. Instead of employing scales with equal arms, a large *platform* is used for the load, while the weights are generally placed in a suspended scale pan. In order that the position of the load on the platform may have no influence on the result, it is necessary that the motion of the platform be as nearly parallel as possible, and that each point be guided vertically. The necessity for this condition is easily seen from the following discussion. If we suppose the scales while the load and weights are in equilibrium, to receive a slight motion, the weight P moving through the vertical distance S and the centre of gravity of the load Q through the vertical distance s, we shall have, because of the equilibrium,

$$PS = Qs \text{ or } Q = P\frac{S}{s}.$$

Consequently if s were not the same for all points of the platform, different positions of the load on the platform would give different results for Q. However, as the motions of platforms are always very small quantities, an approximately correct parallel movement is usually considered as giving sufficient accuracy for practical purposes.

In all known platform scales the movement of the platform is effected by two levers, which support the platform in at least three points, the levers being so connected with each other that for a slight play of the latter, the points of support will describe vertical paths which are as nearly equal as possible. The simplest arrangement of this kind is shown in Fig. 445. Here the steel bearings of the platform *K* rest on the steel knife-edges *B* and *E*, which are attached to the shorter arms of the two levers *B C* and *E F* whose fixed fulcrums are at *A* and *D*. Here these two levers are exactly equal and have the same direction, so that the connecting lines *B A C* and *E D F* are horizontal when the platform is in a position of equilibrium. Now if both levers have a slight equal motion, the rise or fall at *J* will equal that at *L*, and will therefore be the same for all points of the platform. To prevent a lateral oscillation of the platform about the straight line *J L*, the knife-edges *B* and *E* are either made very long, or better still and more commonly they are made in two parts and placed at a considerable distance from each other, the levers being then forked-shaped, as shown in the plan at

FIG. 445.



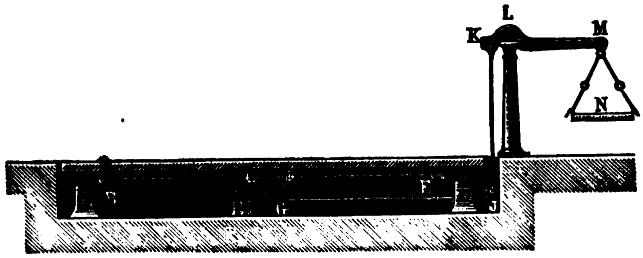
B' A' C' and *E' D' F'*. That the angle of oscillation of the two levers may always be the same, the scale pan *H* holding the weights can be suspended from the levers by equal links, and we see that the weight *P* to be placed on the scale pan *H* when the load *Q* is on the platform, is given by

$$P \,.\, A\,C = Q \,.\, A\,B, \quad \text{or} \quad P = Q\frac{A\,B}{A\,C} \doteq Q\frac{l_1}{l_2},$$

where l_1 is the lever arm $A\,B$ and l_2 the lever arm $A\,C$. The symmetry of the arrangement just described causes an *exact* parallel movement to be given to the platform.

For large platform scales, such as are used for weighing carts, etc., another arrangement, shown in Fig. 446, and due to *Schwilgue*, is employed. Here the short arm of the lever $H\,G\,J$ acts on the forks $B\,C$ and $E\,F$, its other end J being suspended by the tie rod $J\,K$ from the scale-beam $K\,L\,M$. Since in this arrangement the point G moves through a small circular arc, we cannot, strictly speaking, consider this arrangement as an exact parallel movement, but we may regard the resulting inaccuracies as unimportant. If the lever arms $A\,B = D\,E = a_1$, $A\,C = D\,F = a_2$, $H\,G = b_1$, $H\,J = b_2$ and $L\,K$ and $L\,M$ equal c_1 and c_2 respectively, we shall have for the ratio of P and Q the equation:

FIG. 446.



$$Q\,a_1\,b_1\,c_1 = P\,a_2\,b_2\,c_2,$$

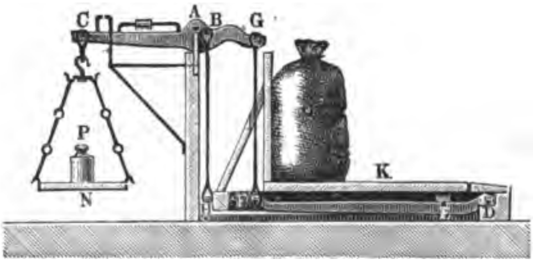
or

$$P = \frac{a_1\,b_1\,c_1}{a_2\,b_2\,c_2}\,Q.$$

If the scale-beam $K\,M$ has equal arms, *i.e.*, $c_1 = c_2$, and the ratio $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{10}$, we get the weight $P = \frac{1}{100}\,Q$, consequently each pound on the scale-pan N corresponds to a load of one hundred pounds on the platform.

It is not necessary to make the lever arms $A\,B$ and $D\,E$, which support the platform at B and E, of equal length provided they are connected so that the vertical movement of the points of support B and E are the same. Thus in *Quintenz's* scale, Fig. 447, the platform K is supported by the lever $D\,E$ at E, while at the point H it is suspended by the tie rod $H\,B$ from a second lever $A\,B$ whose other arm $A\,C$ carries the scale-pan N for the weights. The two levers are here connected by the coupling rod $G\,F$, which is so arranged that motion of the levers causes equal vertical movements in the two points B and E. If s represents a slight motion of the platform, that is of the points B and E, the points G and F will respectively move through the distances

FIG. 447.



$$s\frac{A\,G}{A\,B} \quad \text{and} \quad s\frac{D\,F}{D\,E}.$$

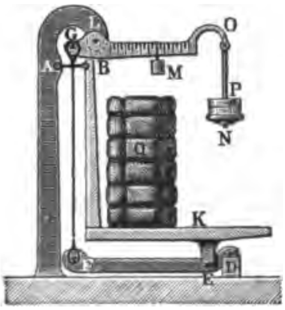
Since G and F are rigidly connected, these two distances must be equal, which gives us for the equation of condition of the scale:

$$A\,G : A\,B = D\,F : D\,E,$$

i.e., the *ratio* of the arms of the two levers must be equal.

There are a great number of different constructions of platform scales, but they all depend on the principles just discussed, and differ from each other principally in the arrangement of the parts. There will consequently be little difficulty in understanding these constructions, and we will therefore only describe one construction by *Millward*,* Fig. 448, distinguished for its simplicity. The platform K is supported by the two equal levers $A\,B$ and $D\,E$, which are horizontal when there is equilibrium and which permit perfect parallel movement. For the ratio of the weight P to the load Q we evidently have

FIG. 448.



$$P\frac{L\,O}{L\,G} = Q\frac{D\,E}{D\,F},$$

or

$$P\frac{b_2}{b_1} = Q\frac{a_1}{a_2}.$$

If from considerations of construction we have assumed $D\,E = a_1$, $D\,F = a_2$ and $L\,G = b_1$, this equation will give for $L\,O = b_2$:

$$b_2 = \frac{Q}{P}\frac{a_1}{a_2}b_1.$$

If we here make $\frac{Q}{P}$ equal, a constant, b_2, will give the distance of the point of suspension of the scale-pan N from L. On the other hand, if we assume a constant running weight M, the formula will enable us to determine the divisions on the lever $O\,L$ corresponding to the different loads Q.

---

* See Article by *Ruehlmann*, Mittheilungen des Gewerbvereins fuer Hanover, 1852, p. 344.

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